On generalized recurrent Weyl spaces and Wong's conjecture

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Abstract. In [4], Y.-C Wong conjectured that the covariant derivative of the recurrence covector field of an affinely connected recurrent space A_n with a torsion-free connection will be symmetric if and only if the Ricci tensor of A_n is symmetric. In [6], it is proved that for a Recurrent Weyl space with a non-vanishing scalar curvature the covariant derivative of the recurrence vector field is symmetric if and only if the Weyl manifold is locally Riemannian. In [7], De and Guha introduced Generalized Recurrent Riemannian manifolds and in [8], Singh and Khan studied the nature of the recurrence vectors appearing in the definition of the Generalized recurrent Riemannian manifold. In the present work, Generalized Recurrent Weyl manifolds are defined and proved that for a Generalized Recurrent Weyl manifold with a non-vanishing constant scalar curvature the covariant derivatives of the recurrence vector fields are both symmetric if and only if the Weyl manifold is locally Riemannian. Moreover, some results about hypersurfaces of Generalized Recurrent Weyl manifolds are obtained.

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1 Introduction

Let A_n be an n-dimensional affinely connected space with torsion-free connection ∇ whose curvature tensor R is recurrent, i.e. there exists some non-zero 1-form ϕ such that $\nabla R = \phi \otimes R$.

In [1], Walker proved that for a Riemannian connection with $\nabla R = \phi \otimes R$ and $\phi \not\equiv 0$, the tensor $\nabla \phi$ is symmetric.

For a sub-flat or projectively flat without torsion, Wong [2] and Wong and Yano [3] have shown that if $\nabla R = \phi \otimes R$, $\phi \not\equiv 0$, the tensor $\nabla \phi$ will be symmetric if and only if the Ricci tensor is symmetric. Later, in reference [14] of his paper [4], Wong proved that for a 2-dimensional linear connection without torsion and with $\nabla R = \phi \otimes R$ $\phi \not\equiv 0$, $\nabla \phi$ is symmetric iff the Ricci tensor is symmetric. Following these results

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Wong proposed the following conjecture: For every linear connection with torsion zero and with $\nabla R = \phi \otimes R$ ($\phi \not\equiv 0$), the tensor $\nabla\phi$ is symmetric iff the Ricci tensor is (everywhere) symmetric. In his paper [5], Pandey gave an affirmative answer to Wong' s conjecture under the additional assumption that the curvature tensor is locally decomposable in the form $R^h_{ijk} = X^h_i Y_{jk}$ where R^h_{ijk} are the components of R and X^h_i and Y_{jk} are the components of the any non-zero tensors.

In [6], Canfes and Özdeger proved that the Wong' s conjecture is not true for a Weyl space unless it is locally Riemannian.

In [7], De and Guha introduced a type of non-flat Riemannian space whose curvature tensor R of type (1,3) satisfies the condition

$$(\nabla_U R)(X, Y, Z) = A(U)R(X, Y, Z) + B(U)(g(Y, Z)X - g(X, Z)Y)$$

where A and B ($B \neq 0$) are two 1-forms. In [8], Singh and Khan have shown that in a generalized recurrent Riemannian space of non-zero constant scalar curvature the the covariant derivative of the 1-form A is symmetric if and only if the the covariant derivative of the 1-form B is symmetric.

In the present work, as a generalizaion of the above mentioned results the generalized recurrent Weyl manifolds are defined and proved that for a generalized recurrent Weyl manifold with a non-vanishing constant scalar curvature the covariant derivatives of the recurrence vector fields are both symmetric if and only if the Weyl manifold is locally Riemannian.

2 Generalized Recurrent Weyl Spaces

As it is well-known, an n-dimensional differentiable manifold with torsion-free (symmetric) connection ∇ and a conformal metric tensor g satisfying the compability condition

$$\nabla g = 2T \otimes g \tag{2.1}$$

for some non-zero 1-form T is called a Weyl manifold which will denoted by $W_n(g, T)$ [9].

Under the renormalization

$$\tilde{g} = \lambda^2 g, \tag{2.2}$$

the vector field T is transformed into, by [9],

$$\tilde{T} = T + d \ln \lambda. \tag{2.3}$$

This transformation is called a gauge transformation. If the vector field T is locally zero or locally gradient, then W_n is locally Riemannian.

The quantity A is called a satellite of weight p of the tensor g_{ij} if it admits a transformation of the form $\tilde{A} = \lambda^p A$ under the renormalization (2.2),[9].

The prolonged covariant derivative of a satellite A of g of weight p is defined by

$$\dot{\nabla} A = \nabla A - p(T \otimes A). \tag{2.4}$$

It is easy to see that prolonged covariant derivative preserves weight.

Let $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ denote the curvature tensor associated with the connection ∇.

The first and the second Bianchi identities for Weyl spaces are, by [6],

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2.5}$$

$$(\dot{\nabla}_X R)(Y,Z) + (\dot{\nabla}_Y R)(Z,X) + (\dot{\nabla}_Z R)(X,Y) = 0. \tag{2.6}$$

A non-flat Weyl space is called recurrent, by [6], if its curvature tensor R satisfies

$$(\dot{\nabla}_U R)(X,Y,Z) = \phi(U)R(X,Y,Z) \tag{2.7}$$

and it is denoted by $(RW)_n$, where ϕ is a nonzero 1-form of weight zero and $R(X,Y,Z)$ is the curvature tensor of type (1,3).

A non-flat Weyl space is called **generalized recurrent** whose curvature tensor R satisfies the condition

$$(\dot{\nabla}_U R)(X,Y,Z) = A(U)R(X,Y,Z) + B(U)(g(Y,Z)X - g(X,Z)Y) \tag{2.8}$$

where R is the curvature tensor of (1,3), A and B are two 1-forms of weights 0 and -2, respectively.

Such a space is denoted by $(GRW)_n$. It is easy to see that if the 1-form B is zero, then $(GRW)_n$ is $(RW)_n$.

Contracting (2.8) with respect to X we get

$$(\dot{\nabla}_U Ric)(Y,Z) = A(U)Ric(Y,Z) + (n-1)B(U)g(Y,Z). \tag{2.9}$$

Here Ric is the Ricci tensor of type (0,2).

Since the prolonged covariant derivative will be use throughout the paper it will be convenient to use expressions in the local coordinates.

By writing (2.9) in local coordinates, we have

$$\dot{\nabla}_l R_{ij} = A_l R_{ij} + (n-1)B_l g_{ij} \tag{2.10}$$

Changing the order of i and j in (2.10) and subtracting the equation so obtained from (2.10), we get

$$\dot{\nabla}_l R_{[ij]} = A_l R_{[ij]} \tag{2.11}$$

where bracket denotes antisymmetrization.

In fact, this is the relationship between recurrence vector A and complementary vector T.

Similary, changing the order of i and j in (2.10) and adding the equation so obtained to (2.10), we get

$$\dot{\nabla}_l R_{(ij)} = A_l R_{(ij)} + (n-1)B_l g_{ij} \tag{2.12}$$

where paranthesis in $R_{(ij)}$ denotes symmetrization.

Theorem 2.1 *In a generalized recurrent Weyl space, with non-zero constant scalar curvature r, the recurrence vector B is locally a gradient if and only if the vector $A-2T$ is locally a gradient.*

Proof:

Contracting (2.12) with g^{ij} we get

$$\dot{\nabla}_l r = A_l r + n(n-1)B_l \tag{2.13}$$

where r is the scalar curvature of weight -2.

Therefore by using (2.4) , we get

$$\nabla_l r + 2T_l r = rA_l + n(n-1)B_l. \tag{2.14}$$

If r is a constant and different from zero, then we have

$$2T_l r = A_l r + n(n-1)B_l. \tag{2.15}$$

Taking covariant derivative of (2.15) with respect to k, we get

$$2r\nabla_k T_l + 2T_l \nabla_k r = r\nabla_k A_l + A_l \nabla_k r + n(n-1)\nabla_k B_l. \tag{2.16}$$

Since r is a constant and different from zero, then we have

$$(-2\nabla_k T_l + \nabla_k A_l)r + n(n-1)\nabla_k B_l = 0. \tag{2.17}$$

Changing the order of the indices k and l in (2.17) we get

$$(-2\nabla_l T_k + \nabla_l A_k)r + n(n-1)\nabla_l B_k = 0, \tag{2.18}$$

and subtracting (2.18) from (2.17), we obtain

$$r(\nabla_{[k} C_{l]}) + n(n-1)\nabla_{[k} B_{l]} = 0. \tag{2.19}$$

where $C_l = A_l - 2T_l$. Since $r \neq 0$, C_l is locally a gradient if and only if B_l is locally a gradient □.

The above theorem may be stated in a different manner as follows:

For a Generalized Recurrent Weyl manifold with a non-vanishing constant scalar curvature the tensor ∇B is symmetric if and only if $\nabla(A - 2T)$ is symmetric.

Therefore we have,

Corollary 2.1 *A Generalized Recurrent Weyl manifold with a non-vanis-hing constant scalar curvature is locally Riemannian if and only if the covariant derivatives of the recurrence vector fields are both symmetric.*

In addition, for a $(GRW)_n$ with a non-constant scalar curvature, we can state,

Theorem 2.2 *In a generalized recurrent Weyl space, with non-constant scalar curvature r, if the tensors ∇C and ∇B are are both symmetric (equivalently B and C are both locally gradient), then C and B are collinear, where $C = A - 2T$.*

Proof:

Suppose r is not constant, from (2.14) we get

$$\nabla_k \nabla_l r + 2r\nabla_k T_l + 2T_l \nabla_k r = r\nabla_k A_l + \nabla_k r A_l + n(n-1)\nabla_k B_l. \tag{2.20}$$

Interchanging the indices k and l in (2.20) and subtracting the equation so obtained from (2.20), we get

$$(\nabla_{[k}C_{l]})r + n(n-1)C_{[l}\nabla_{r]}r + \nabla_{[k}B_{l]} + n(n-1)(B_kC_l - B_lC_k) = 0 \tag{2.21}$$

where $C_l = A_l - 2T_l$.

¿From (2.21), it follows that

$$(\nabla_{[k}C_{l]})r + n(n-1)\nabla_{[k}B_{l]} + n(n-1)(B_kC_l - B_lC_k) = 0 \tag{2.22}$$

which completes the proof □.

If, in particular, the complementary vector field T is locally gradient or zero, i.e. W_n is locally Riemannian, then we obtain the corresponding results in [8].

If $B = 0$ the Weyl space under consideration becomes a recurrent Weyl space.

In [6], it is proved that the Ricci tensor of $(RW)_n$ can not be symmetric unless the space is locally Riemannian. Consequently, we obtain the following result concerning the Wong's conjecture [4]:

The Wong' s conjecture is not true for any recurrent Weyl manifold with a non-vanishing scalar curvature, unless it is locally Riemannian.

3 Generalized Projectively Recurrent Weyl Spaces

A Weyl space W_n is called **generalized projectively recurrent** if its projective curvature tensor W of type (1,3) satisfies the condition

$$(\dot{\nabla}_U W)(X,Y,Z) = A(U)W(X,Y,Z) + B(U)(g(Y,Z)X - g(X,Z)Y) \tag{3.1}$$

where A and B are two 1-forms of weights 0 and -2, respectively.

The projective curvature tensor is given by [10], in local coordinates,

$$\begin{aligned} W^i_{jkl} &= \frac{1}{n^2-1}\left[\delta^i_k\left(nR_{jl} + R_{lj}\right) - \delta^i_l\left(nR_{jk} + R_{kj}\right)\right] \\ &+ R^i_{jkl} + \frac{2}{n+1}\delta^i_j R_{[kl]} \end{aligned} \tag{3.2}$$

It is well-known that the projective curvature tensor satisfies the following identities.

$$W^i_{jkl} + W^i_{jlk} = 0 \tag{3.3}$$

$$W^i_{jkl} + W^i_{ljk} + W^i_{klj} = 0 \tag{3.4}$$

$$W^i_{ikl} = W^i_{jki} = 0 \tag{3.5}$$

$$\dot{\nabla}_i W^i_{jkl} = \frac{(n-2)}{(n-1)}\left[\dot{\nabla}_l R_{jk} - \dot{\nabla}_k R_{jl} + \frac{1}{(n+1)}\left(\dot{\nabla}_k R_{[jl]} - \dot{\nabla}_l R_{[jk]}\right)\right] \tag{3.6}$$

Theorem 3.1 *A Weyl manifold which is either generalized projectively recurrent or recurrent is projectively recurrent.*

Proof:

Suppose W_n is generalized projectively recurrent, then expressing (3.1) in local coordinates we have

(3.7) $$\dot{\nabla}_r W^i_{jkl} = A_r W^i_{jkl} + B_r G^i_{jkl}$$

where $G^i_{jkl} = \delta^i_l g_{jk} - \delta^i_k g_{jl}$. Contracting the indices i and l in (3.7), we get

(3.8) $$\dot{\nabla}_r W^i_{jki} = A_r W^i_{jki} + B_r G^i_{jki}.$$

Since $W^i_{jki} = 0$, then we have

(3.9) $$(n-1)B_r g_{jk} = 0$$

therefore (3.9) implies that $B_r = 0$. So from (3.8) we get

(3.10) $$\dot{\nabla}_r W^i_{jkl} = A_r W^i_{jkl}$$

Now suppose that W_n is recurrent with recurrence vector ϕ_r , then by taking prolonged covariant derivative of (3.2) we have

(3.11) $$\begin{aligned} \dot{\nabla}_r W^i_{jkl} &= \frac{1}{n^2-1}\left[\delta^i_k \left(n\dot{\nabla}_r R_{jl} + \dot{\nabla}_r R_{lj}\right) - \delta^i_l \left(n\dot{\nabla}_r R_{jk} + \dot{\nabla}_r R_{kj}\right)\right] \\ &+ \dot{\nabla}_r R^i_{jkl} + \dot{\nabla}_r \frac{2}{n+1}\delta^i_j R_{[kl]} \end{aligned}$$

And using (2.7), we obtain

(3.12) $$\dot{\nabla}_r W^i_{jkl} = \phi_r W^i_{jkl}$$

□.

4 Hypersurfaces of Generalized Recurrent Weyl Spaces

Let $W_n(g_{ij}, T_k)$ be a hypersurface, with coordinates $u^i (i = 1, 2, \cdots, n)$ of a Weyl space $W_{n+1}(g_{ab}, T_c)$ with coordinates $x^a (a = 1, 2, \cdots, n+1)$. The metrics of W_n and W_{n+1} are connected by the relations

(4.1) $$g_{ij} = g_{ab} x^a_i x^b_j \quad (i, j = 1, 2, \cdots, n \ ; \ a, b = 1, 2, \cdots, n+1)$$

where x^a_i denotes the covariant derivative of x^a with respect to u^i.

It is easy to see that the prolonged covariant derivative of a satellite A, relative to W_n, and W_{n+1}, are related by

(4.2) $$\dot{\nabla}_k A = x^c_k \dot{\nabla}_c A \quad (k = 1, 2, \cdots, n \ ; \ c = 1, 2, \cdots, n+1)$$

Let n^a be the contravariant components of the vector field of W_{n+1} normal to W_n which is normalized by the condition

(4.3) $$g_{ab} n^a n^b = 1.$$

The moving frame $\{x^i_a, n_a\}$ in W_n, reciprocal to the moving frame $\{x^a_i, n^a\}$ is defined by the relations

$$n_a x^a_i = 0 \quad n^a x^i_a = 0 \quad x^a_i x^j_a = \delta^j_i. \tag{4.4}$$

Since the weight of x^a_i is $\{0\}$, the prolonged covariant derivative of x^a_i with respect to u^k is found as

$$\dot{\nabla}_k x^a_i = \nabla_k x^a i = \omega_{ik} n^a \tag{4.5}$$

where ω_{ik} is the second fundamental form.

The generalized Gauss and Mainardi-Codazzi equations are obtained in [6], respectively as

$$R_{pijk} = \Omega_{pijk} + \overline{R}_{dbce} x^d_p x^b_i x^c_j x^e_k \tag{4.6}$$

$$\dot{\nabla}_k \omega_{ij} - \dot{\nabla}_j \omega_{ik} + \overline{R}_{dbce} x^b_i x^c_j x^e_k n^d = 0, \tag{4.7}$$

where $\overline{R}_{dbce}$ is the covariant curvature tensor of W_{n+1} and $\Omega_{pijk} = \omega_{pj}\omega_{ik} - \omega_{pk}\omega_{ij}$.

Let W_n be a hypersurface of a Generalized Recurrent Weyl space W_{n+1} with recurrence vectors A_a and B_a which are not orthogonal to the hypersurface W_n. If we denote the tangential components of A_a and B_a, respectively, by A_k and B_k, then we have

$$A_k = x^a_k A_a, \qquad B_k = x^a_k B_a. \tag{4.8}$$

and

$$\dot{\nabla}_e \overline{R}^a_{bcd} = A_e \overline{R}^a_{bcd} + B_e G^a_{bcd} \tag{4.9}$$

where $G^a_{bcd} = \delta^a_d g_{bc} - \delta^a_c g_{bd}$. By taking prolonged covariant derivative of Gauss equation with respect to u^r, we get

$$\dot{\nabla}_r R_{ijkh} = \dot{\nabla}_r \Omega_{ijkh} + \dot{\nabla}_r(\overline{R}_{abcd} x^a_i x^b_j x^c_k x^d_h). \tag{4.10}$$

Furthermore,

$$\begin{aligned}\dot{\nabla}_r R_{ijkh} = \dot{\nabla}_r \Omega_{ijkh} &+ (\dot{\nabla}_e \overline{R}_{abcd}) x^a_i x^b_j x^c_k x^d_h x^e_r + \overline{R}_{abcd} x^b_j x^c_k x^d_h \omega_{ir} n^a \\ &+ \overline{R}_{abcd} x^a_i x^c_k x^d_h \omega_{jr} n^b + \overline{R}_{abcd} x^a_i x^b_j x^d_h \omega_{kr} n^c \\ &+ \overline{R}_{abcd} x^a_i x^b_j x^c_k \omega_{hr} n^d\end{aligned}$$

Since W_{n+1} is Generalized recurrent, from (2.8), we have

$$\begin{aligned}\dot{\nabla}_r R_{ijkh} &= \dot{\nabla}_r \Omega_{ijkh} + A_e \overline{R}_{abcd} x^a_i x^b_j x^c_k x^d_h x^e_r + B_e G_{abcd} x^a_i x^b_j x^c_k x^d_h x^e_r \\ &+ \overline{R}_{abcd} x^b_j x^c_k x^d_h \omega_{ir} n^a + \overline{R}_{abcd} x^a_i x^c_k x^d_h \omega_{jr} n^b \\ &+ \overline{R}_{abcd} x^a_i x^b_j x^d_h \omega_{kr} n^c + \overline{R}_{abcd} x^a_i x^b_j x^c_k \omega_{hr} n^d\end{aligned}$$

where $G_{abcd} = g_{ad} g_{bc} - g_{ac} g_{bd}$. Using (4.1), (4.8) and the Gauss equation , we have where $G_{ijkh} = g_{ih} g_{jk} - g_{ik} g_{jh}$.

$$\begin{aligned}\dot{\nabla}_r R_{ijkh} - A_r R_{ijkh} - B_r G_{ijkh} = &\ \dot{\nabla}_r \Omega_{ijkh} - A_r \Omega_{ijkh} + \overline{R}_{abcd} x^b_j x^c_k x^d_h \omega_{ir} n^a \\ &+ \overline{R}_{abcd} x^a_i x^c_k x^d_h \omega_{jr} n^b + \overline{R}_{abcd} x^a_i x^b_j x^d_h \omega_{kr} n^c \\ &+ \overline{R}_{abcd} x^a_i x^b_j x^c_k \omega_{hr} n^d\end{aligned} \tag{4.11a}$$

Theorem 4.1 *A hypersurface of a Generalized Recurrent Weyl space satisfies the relation*

$$\begin{aligned} A_r R_{ijkh} + A_h R_{ijrk} + A_k R_{ijhr} &= A_r \Omega_{ijkh} + A_h \Omega_{ijrk} + A_k \Omega_{ijhr} \\ &+ B_r G_{ijkh} + B_h G_{ijrk} + B_k G_{ijhr}. \end{aligned}$$

where G_{ijkh} is the Sylvestrian of g_{ij}.
Proof:

If we change the indices k, h, r cyclically in (4.11a), we obtain two more equations. Namely

$$\begin{aligned} (4.11b)\ \dot{\nabla}_k R_{ijhr} - A_k R_{ijhr} - B_k G_{ijhr} &= \dot{\nabla}_k \Omega_{ijhr} - A_k \Omega_{ijhr} + \overline{R}_{abcd} x^b_j x^c_r x^d_k \omega_{ih} n^a \\ &+ \overline{R}_{abcd} x^a_i x^c_r x^d_k \omega_{jh} n^b + \overline{R}_{abcd} x^a_i x^b_j x^d_k \omega_{hr} n^c \\ &+ \overline{R}_{abcd} x^a_i x^b_j x^c_r \omega_{kh} n^d \end{aligned}$$

$$\begin{aligned} (4.11c)\ \dot{\nabla}_h R_{ijrk} - A_h R_{ijrk} - B_h G_{ijrk} &= \dot{\nabla}_h \Omega_{ijrk} - A_h \Omega_{ijrk} + \overline{R}_{abcd} x^b_j x^c_h x^d_r \omega_{ik} n^a \\ &+ \overline{R}_{abcd} x^a_i x^c_h x^d_r \omega_{jk} n^b + \overline{R}_{abcd} x^a_i x^b_j x^d_r \omega_{hk} n^c \\ &+ \overline{R}_{abcd} x^a_i x^b_j x^c_h \omega_{rk} n^d \end{aligned}$$

adding (4.11a), (4.11b), (4.11c) and using second Bianchi identity we finally get

$$\begin{aligned} A_r R_{ijkh} + A_h R_{ijrk} + A_k R_{ijhr} &= A_r \Omega_{ijkh} + A_h \Omega_{ijrk} + A_k \Omega_{ijhr} \\ &+ B_r G_{ijkh} + B_h G_{ijrk} + B_k G_{ijhr}. \end{aligned}$$

□

Moreover, from second Bianchi identity we conclude,

Corollary 4.1 *If a hypersurface of Generalized Recurrent Weyl space is generalized recurrent with recurrence vectors A_r and B_r, then we have*

$$A_r \Omega_{ijkh} + A_h \Omega_{ijrk} + A_k \Omega_{ijhr} = 0.$$

Definition 4.1 *A Hypersurface of a Weyl space is called totally geodesic if $\omega_{ij} = 0$.*

By (4.11a), we can easily deduce that,

Theorem 4.2 *If a hypersurface of a Generalized Recurrent Weyl space is totally geodesic, then the hypersurface is Generalized Recurrent Weyl with recurrence vectors A_r and B_r.*

I°. f $B_a = 0$, we get the results obtained in [6].

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References

[1] A.G. Walker, *On Ruse's spaces of recurrent curvature*, London Math. Soc., 52 (1950), 36-54.

[2] Y. C. Wong, *Sub-flat affinely connected spaces*, Proceedings of International Math. Congress, Amsterdam, 1954.

[3] Y. C. Wong and K. Yano, *Projectively flat spaces with recurrent curvature*, Comm. Math. Helvetici 35 (1961), 223-232.

[4] Y. C. Wong, *Linear connections with zero torsion and recurrent curvature*, Trans. Amer. Math. Soc. 102 (1962), 471-506.

[5] P. N. Pandey, *On Wong's conjecture* (in Russian), Prog. Math., Alllahabad 18, 89-94.

[6] E. Ö. Canfes and A. Özdeğer, *Some applications of prolonged covariant differentiation in Weyl Geometry*, Journal of Geometry 60 (1997), 7-16.

[7] U.C. De and N. Guha, *On generalized recurrent manifolds*, J. Natl. Acad. Math. India 9, 1 (1991), 1-4.

[8] H. Singh and Q. Khan, *On generalized recurrent Riemannian manifolds,* Publ. Math. Debrecen 56/1-2 (2000), 87-95.

[9] V. Hlavaty, *Theorie d'immersion d'une W_m dans W_n*, Ann. Soc. Polon. Math. 21 (1949), 196-206.

[10] L. P. Eisenhart, *Non-Riemannian Geometry*, New York, 1927.

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