

Qualitative analysis of the Tobin-Benhabib-Miyao dynamical system

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Abstract. The aim of this paper is to study the parametric dependence of the solutions in the Tobin-Benhabib-Miyao dynamical system. The critical points are explicitly computed and the evolution nearby the critical points is described. In particular, some conditions for the existence of critical points are given. Section 1 reviews the generalized Tobin economic flow as formulated by Benhabib and Miyao. In section 2 the critical points are given and the linearized system are discussed. In Section 3 it is shown that the trivial critical points are unstable and the orbits around them are studied. Section 4 studies some numerical examples for the nonlinear system.

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Key words: dynamic economic models, economic growth, bifurcation problems.

1 Tobin-Benhabib-Miyao dynamical system

The Tobin model [2, 5] regarding the role of money on economic growth has been extended by Benhabib and Miyao [1] to incorporate the role of expectation parameters, and to show that the variation of this parameter produces a Hopf Bifurcation in a three sector economy (see also [6, 7]).

The model is an ODEs system (TBM-system)

$$(1.1) \quad \begin{cases} \dot{k} &= sf(k) - (1-s)(\theta - q)m - nk \\ \dot{m} &= m(\theta - \bar{p} - n) \\ \dot{q} &= \mu(\bar{p} - q), \end{cases}$$

where

$$\bar{p} = \varepsilon(m - L(k, q)) + q.$$

The state space has the following variables:

- k = the capital labour ratio;

- m = the money stock per head;
- q = the expected rate of inflation;
- \bar{p} = the actual rate of inflation.

The real functions $f(k)$ and $L(k, q)$ are differentiable functions, usually defined as

$$(1.2) \quad f(k) = ak \quad , \quad L(k, q) = k^\alpha q^\beta$$

with

$$(1.3) \quad a > 0 = \text{Cst.} \ , \ \alpha + \beta = 1 \ , \ (\alpha \geq 0 \ , \ \beta \geq 0) \ ,$$

The Cauchy problem associated with system (1.1) is

$$(1.4) \quad \begin{cases} \dot{k} &= (as - n)k - (1 - s)(\theta - q)m \\ \dot{m} &= m(\theta - p - n) \\ \dot{q} &= \mu(p - q), \end{cases}$$

with initial conditions

$$(1.5) \quad \begin{cases} k(0) &= k_0 \\ m(0) &= m_0 \\ q(0) &= q_0, \end{cases}$$

and

$$(1.6) \quad p = \varepsilon \left(m - k^\alpha q^{1-\alpha} \right) + q.$$

The parameters $s, \theta, n, \mu, \varepsilon$ are

- s = saving ratio, with $0 \leq s \leq 1$,
- θ = rate of money expansion, with $|\theta| \leq 1$,
- n = population growth rate: $0 \leq n \leq 1$,
- μ = speed of adjustment of expectations $\mu \geq 0$,
- ε = speed of adjustment of price level: $\varepsilon \geq 0$.

It can be easily seen (1.4) that the TBM system cannot reduce to a linear system, in other words it is not possible to choose the parameters in a such a way that the nonlinearities can be canceled.

1.1 Critical points

From equations (1.4) follows that the equilibrium point $(k_*(\mu), m_*(\mu), q_*(\mu))$, at which $\dot{k} = 0 = \dot{m} = \dot{q}$, is the solution of the algebraic system

$$(1.7) \quad \begin{cases} (as - n)k - (1 - s)mn = 0, \\ \theta = q + n \\ k^\alpha q^\beta = m \end{cases}$$

that is

$$(1.8) \quad \begin{cases} Ak^\alpha - Bk = 0, \\ q = n - \theta \\ m = k^\alpha (n - \theta)^\beta \end{cases}$$

with

$$(1.9) \quad A \stackrel{\text{def}}{=} (s - 1)(n - \theta)^\beta n, \quad B \stackrel{\text{def}}{=} n - sa,$$

In order to get a real value for A , it must be

$$(1.10) \quad \boxed{n \geq \theta}.$$

Thus we have the following solutions of (1.8)₁:

1. $A = 0$ and $B \neq 0$: The first parameter vanishes when at least one of the following condition holds true

$$A = 0 \iff \begin{cases} s = 1 \\ n = \theta \vee n = 0. \end{cases}$$

There follows that, due to (1.10), as trivial critical points we can have:

$$(1.11) \quad \begin{cases} k_* = 0, m_* = 0, q_* = n - \theta, & (s = 1, n > \theta) \\ k_* = 0, m_* = 0, q_* = 0, & (n = \theta) \end{cases}$$

2. $A \neq 0$ and $B = 0$: It is

$$B = 0 \iff n = sa$$

and then as trivial critical point

$$(1.12) \quad \begin{cases} k_* = 0, m_* = 0, q_* = n - \theta, & (n = sa > \theta) \\ k_* = 0, m_* = 0, q_* = 0, & (n = sa = \theta) \end{cases}$$

3. $A \neq 0$ and $B \neq 0$: the solutions of (1.8)₁ are

$$k = \left(\frac{B}{A}\right)^{1/(\alpha-1)}, \quad k = 0$$

and taking into account (1.3),(1.9),

$$(1.13) \quad \begin{cases} k_* &= \left(\frac{A}{B}\right)^{1/\beta} \\ m_* &= \left(\frac{A}{B}\right)^{\alpha/\beta} (n-\theta)^\beta \\ q_* &= n-\theta \end{cases},$$

with

$$(1.14) \quad \frac{A}{B} = \frac{(s-1)(n-\theta)^\beta n}{n-sa}.$$

Another critical point, independent on the values of parameters, is obtained from system (1.7)

$$(1.15) \quad \begin{cases} k_* &= 0 \\ m_* &= 0 \\ q_* &= n-\theta \end{cases}$$

Directly from equation (1.7)₃, or from system (1.13) we can deduce a simple equation for the surface connecting the equilibrium points (including the trivial one)

$$(1.16) \quad m_* = k_*^\alpha q_*^{1-\alpha},$$

so that *the critical points belong to a family of paraboloids which only depends on the parameter α .*

2 Linearization

Let $\{k_*, m_*, q_*\}$, be the critical points (1.13) or the trivial system (1.11); denoting $x = (x_1, x_2, x_3) = (k - k_*, m - m_*, q - q_*)$, the linearization about the equilibrium point $(0, 0, 0)$ is

$$\dot{x} = \mathbf{A}x,$$

where $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}(s, \theta, n, \mu, \varepsilon)$ is the Jacobian matrix of the function

$$\left\{ (as - n)k - (1-s)(\theta - q)m, m(\theta - \bar{p} - n), \mu(\bar{p} - q) \right\},$$

i.e.,

$$\begin{pmatrix} as - n & -(1-s)(\theta - q) & (1-s)m \\ \varepsilon mL_1 & -\varepsilon m - n - q + \theta - \varepsilon(m - L) & m(\varepsilon L_3 - 1) \\ -\mu\varepsilon L_1 & \mu\varepsilon & -\mu\varepsilon L_3 \end{pmatrix}$$

(with $L_1 \stackrel{\text{def}}{=} \partial L / \partial x_1$, $L_3 \stackrel{\text{def}}{=} \partial L / \partial x_3$) and, according to (1.2),

$$\begin{pmatrix} as - n & -(1-s)(\theta - q) & (1-s)m \\ \alpha \varepsilon m k^{\alpha-1} q^\beta & -\varepsilon m - n - q + \theta - \varepsilon(m - k^\alpha q^\beta) & m(\beta \varepsilon k^\alpha q^{\beta-1} - 1) \\ -\alpha \mu \varepsilon k^{\alpha-1} q^\beta & \mu \varepsilon & -\mu \beta \varepsilon k^\alpha q^{\beta-1} \end{pmatrix}$$

2.1 Critical points (1.13)

In correspondence with the critical point (1.13) we have: at $(k = k_*, m = m_*, q = q_*)$,

$$\begin{pmatrix} as - n & -(1-s)(\theta - q) & (1-s)m \\ \alpha \varepsilon m k^{\alpha-1} q^\beta & -\varepsilon m - n - q + \theta - \varepsilon(m - k^\alpha q^\beta) & m(\beta \varepsilon k^\alpha q^{\beta-1} - 1) \\ -\alpha \mu \varepsilon k^{\alpha-1} q^\beta & \mu \varepsilon & -\mu \beta \varepsilon k^\alpha q^{\beta-1} \end{pmatrix}_{(k_*, m_*, q_*)}$$

i.e., taking into account (1.13),(1.16), in the untrivial equilibrium we have

$$\begin{pmatrix} as - n & -(1-s)n & (1-s)m_* \\ \alpha \varepsilon m_* r_*^{\alpha-1} & -\varepsilon m_* & m_*[(1-\alpha)\varepsilon r_*^\alpha - 1] \\ -\alpha \mu \varepsilon r_*^{\alpha-1} & \mu \varepsilon & -\mu(1-\alpha)\varepsilon r_*^\alpha \end{pmatrix}$$

with

$$r_* \stackrel{\text{def}}{=} \frac{k_*}{q_*}.$$

i.e., taking into account (1.13),(1.14)

$$(2.1) \quad r_* = \left[\frac{(s-1)n}{n-sa} \right]^{1/\beta}.$$

There follows, that the complete set of parameters which characterize the orbits nearby the critical points are:

$$a, s, n, \varepsilon, \mu, \alpha, m_*, r_*$$

where r_* is given by (2.1) and m_* , according to (1.13),(1.14), is

$$m_* = \left[\frac{(s-1)n}{n-sa} \right]^{\alpha/\beta} (n - \theta).$$

In order to have some real solution for m_* and r_* we must assume

$$\frac{(s-1)n}{n-sa} > 0$$

which implies the following condition on the parameter a

$$(2.2) \quad \boxed{a > \frac{n}{s}}$$

Thus we have proven the

Theorem 1 *The TBM system (1.4) – (1.6) admits real critical points (1.13) – (1.14) only iff equation (2.2) is fulfilled. If $a \leq \frac{n}{s}$, there not exist untrivial critical points.*

Of course in the trivial case $\frac{(s-1)n}{n-sa} = 0$, the parameter a can be arbitrarily chosen.

The linearized system is

$$(2.3) \quad \begin{cases} \dot{k} &= (as - n)k - (1-s)nm + (1-s)m_*q \\ \dot{m} &= \alpha \varepsilon m_* r_*^{\alpha-1} k - \varepsilon m_* m + m_* [(1-\alpha)\varepsilon r_*^\alpha - 1]q \\ \dot{q} &= -\alpha \mu \varepsilon r_*^{\alpha-1} k + \mu \varepsilon m - \mu(1-\alpha)\varepsilon r_*^\alpha q \end{cases}$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda^3 + c_1 \lambda^2 - c_2 \lambda + c_3 = 0,$$

where $c_1 \stackrel{\text{def}}{=} \text{tr} \mathbf{A}$, $c_2 \stackrel{\text{def}}{=} \text{sum of principal minors of order two}$, and $c_3 = \det \mathbf{A}$ are

$$(2.4) \quad \begin{cases} c_1 &= -\varepsilon [m_* + (1-\alpha)\mu r_*^\alpha] - n + as, \\ c_2 &= (1-\alpha)\mu [\varepsilon(\varepsilon m_* + n) - as] r_*^\alpha - as m_* \\ &+ m_* [\mu + 2n + \alpha(\mu + n)(1-s)r_*^{\alpha-1}] \\ c_3 &= \varepsilon m_* \mu [(as - n)(1 + (1-\alpha)\varepsilon r_*^\alpha) - (\varepsilon m_* + n)\alpha(1-s)r_*^{\alpha-1}]. \end{cases}$$

If

$$(2.5) \quad \begin{cases} c_i &> 0, & (i = 1, 2, 3) \\ c_1 c_2 &= c_3 \end{cases}$$

then we have the solutions $\lambda_1 < 0$, $\lambda_{2,3} = \alpha \pm i\beta$. By the Hopf Bifurcation Theorem, there exist periodic solutions around the equilibrium point (k_*, m_*, q_*) .

3 Linearization nearby the trivial critical point

3.1 Trivial point with $q_* \neq 0$

In this section we will compute the solution of the linearized TBM-system nearby the trivial critical point

$$k_* = 0, \quad m_* = 0, \quad q_* = n - \theta \neq 0$$

According to (1.11)₁-(1.12)₁-(1.15) we have to distinguish among three different situations giving rise to the same trivial equilibrium:

1. (1.11)₁: $s = 1$, $n > \theta$.

In the first case the coefficients (2.4) reduce to

$$\begin{cases} c_1 &= -n + a, \\ c_2 &= 0 \\ c_3 &= 0, \end{cases}$$

and system (2.3) becomes

$$\begin{cases} \dot{k} &= (a - n)k \\ \dot{m} &= 0 \\ \dot{q} &= \mu\varepsilon m \end{cases}$$

Its solution is

$$\begin{cases} k &= k_0 e^{(a-n)t} \\ m &= m_0 \\ q &= q_0 + \mu\varepsilon m_0 t \end{cases}$$

being k_0, m_0, q_0 the initial conditions.

2. (1.12)₁: $n = sa > \theta$.

The coefficients (2.4) reduce to

$$\begin{cases} c_1 &= 0, \\ c_2 &= 0 \\ c_3 &= 0, \end{cases}$$

and system (2.3) becomes

$$\begin{cases} \dot{k} &= 0 \\ \dot{m} &= 0 \\ \dot{q} &= \mu\varepsilon m \end{cases}$$

Its solution is

$$\begin{cases} k &= k_0 \\ m &= m_0 \\ q &= q_0 + \mu\varepsilon m_0 t \end{cases}$$

3. (1.15)

In this case, which is independent on the values of the parameters, the coefficients (2.4) are

$$\begin{cases} c_1 &= -n + as, \\ c_2 &= 0 \\ c_3 &= 0, \end{cases}$$

and system (2.3) becomes

$$\begin{cases} \dot{k} &= (as - n)k - (1 - s)nm \\ \dot{m} &= 0 \\ \dot{q} &= \mu\varepsilon m \end{cases}$$

Its solution is

$$\begin{cases} k &= k_0 e^{(a s - n)t} \\ m &= m_0 \\ q &= q_0 + \mu \varepsilon m_0 t \end{cases}$$

being k_0, m_0, q_0 the initial conditions. There follows that, starting from an initial position close to the critical point (1.13), due to (2.2), the capital labour k will increase while the money stock m will be constant and the expected rate of inflation q will be linearly depending on time. Thus we can conclude that *the equilibrium in (1.13) is unstable*.

3.2 Trivial point with $q_* = 0$

In this section we will discuss the solution of the linearized TBM-system nearby the trivial critical point

$$k_* = 0, m_* = 0, q_* = 0, \quad (n = \theta)$$

The linearized system (2.3) becomes

$$\begin{cases} \dot{k} &= (a s - n)k - (1 - s)n m \\ \dot{m} &= 0 \\ \dot{q} &= \mu \varepsilon m \end{cases}$$

Taking into account the initial conditions (1.5), and the conditions on parameters there follows two different solutions:

1. (1.11)₂: $n = \theta \neq s a$, the general solution, is

$$\begin{cases} k(t) &= \left[k_0 - \frac{(1-s)n m_0}{a s - n} \right] e^{(a s - n)t} + \frac{(1-s)n m_0}{a s - n} \\ m(t) &= m_0 \\ q(t) &= q_0 + \mu \varepsilon m_0 t \end{cases}$$

Assuming the initial condition close to the trivial critical point (1.11), and taking into account (2.2) *the equilibrium in the origin is unstable*, in fact while $m = Cst.$ both $k(t)$ and $q(t)$ increase asymptotically.

2. (1.11)₂: $n = \theta = s a$: the general solution, is

$$\begin{cases} k(t) &= k_0 - (1 - s) n m_0 t \\ m(t) &= m_0 \\ q(t) &= q_0 + \mu \varepsilon m_0 t \end{cases}$$

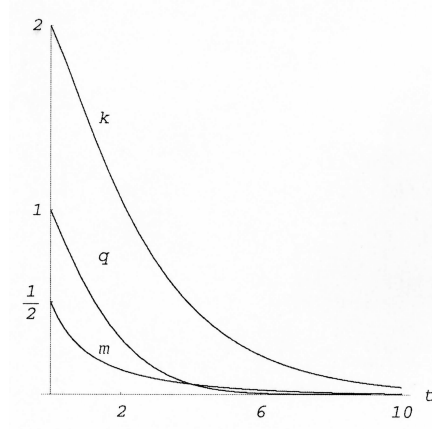


Figure 1: Numerical solution of the Cauchy problem for the nonlinear system (1.1) with conditions (4.1)

4 Nonlinear system

For the nonlinear system (1.1) by a numerical computation we can draw the time evolution of the functions $k(t)$, $m(t)$, $q(t)$ in correspondence with some special Cauchy problems and values of parameters.

For example if we assume $a = 1$, $\alpha = 1/2$, $\beta = 1/2$

$$(4.1) \quad \begin{cases} f(k) = k, \quad L(k, q) = \sqrt{kq} \\ s = 0.1, \quad \theta = 0.1, \quad n = 0.5, \quad \mu = 1, \quad \varepsilon = 0.5 \\ k_0 = 2, \quad m_0 = 0.5, \quad q_0 = 1 \end{cases}$$

we have the evolution of Fig. 1

With the conditions $a = 1$, $\alpha = 9/10$, $\beta = 1/10$

$$(4.2) \quad \begin{cases} f(k) = k, \quad L(k, q) = \sqrt[10]{k^9 q} \\ s = 0.1, \quad \theta = 0.1, \quad n = 0.5, \quad \mu = 1, \quad \varepsilon = 0.5 \\ k_0 = 2, \quad m_0 = 0.5, \quad q_0 = 1 \end{cases}$$

we have the evolution of Fig. 2. In other words, the solution decreases to zero.

Conclusion

We have discussed the Tobin-Benhabib-Miyao dynamical model by showing that in order to have some real critical points some conditions on parameters must be given. The critical trivial points are saddle points. With a numerical computation we have shown with some example that there exists some stable equilibrium points.

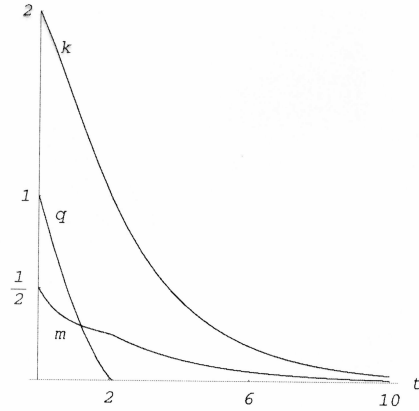


Figure 2: Numerical solution of the Cauchy problem for the nonlinear system (1.1) with conditions (4.2)

This suggest us that a bifurcation point on the parameters should exists (how it will be shown in a forthcoming paper).

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