The Poincaré problem in the non–resonant case: an algebraic approach

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Abstract. In this paper we give an elementary algebraic proof of the fact that, if a polynomial vector field $\mathcal{X}$ of degree m possesses an invariant algebraic curve $f = 0$ of degree n then one has the following degree bound $n \leq m + 1$ in the generic case that $\mathcal{X}$ possesses all its infinite critical points simples. Additionally, a finite version of the above result is given for $m + 1$ critical points on an invariant straight line. By the way we just obtain the degree of an eventual polynomial inverse integrating under some conditions. Moreover, we get an application of the former theory to algebraic limit cycles in quadratic systems with invariant straight line.

M.S.C. 2000: 34C05; 34C14, 22E05.
Key words: polynomial differential equations, invariant algebraic curves, Poincaré problem.

1 Introduction and statement of the results

We consider here two-dimensional polynomial differential systems of the form

$$(1.1) \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y) = \sum_{i=0}^{m} P_i(x,y) \ , \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y) = \sum_{i=0}^{m} Q_i(x,y) \ ,$$

in which P, $Q \in \mathbb{C}[x,y]$ are coprime polynomials and $m = \max\{\deg P, \deg Q\}$ is called the *degree* of system (1.1). Here, $\mathbb{C}[x,y]$ denotes, as usual, the ring of the polynomials in two variables with complex coefficients and P_i and Q_i denote homogeneous polynomials of degree i. We shall associate to system (1.1) the vector field $\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$.

A point $(x_0, y_0) \in \mathbb{C}^2$ is called *finite critical point* of system (1.1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. Since P and Q are coprime, this implies that all such critical points are isolated. Extending system (1.1) to a differential equation in the complex projective plane $\mathbb{CP}^2$, a point $(X_0 : Y_0 : 0) \in \mathbb{CP}^2$ is termed *infinite critical point* of system (1.1) if $(X_0, Y_0) \in \mathbb{C}^2$ is a root of the homogeneous polynomial $yP_m(x,y) - xQ_m(x,y)$.

Differential Geometry - Dynamical Systems, Vol.8, 2006, pp. 54-68.
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Let $D\mathcal{X}$ be the jacobian matrix associated to vector field $\mathcal{X}$. The critical point (x_0, y_0) of (1.1) is classified according to its associated eigenvalues $\lambda, \mu \in \mathbb{C}$, i.e. the eigenvalues of the matrix $D\mathcal{X}(x_0, y_0)$. In particular, if $\lambda\mu = 0$ then the critical point is called *degenerate*. Otherwise it is termed *nondegenerate*. If $D\mathcal{X}(x_0, y_0)$ has exactly one eigenvalue equal to zero then the critical point (x_0, y_0) is called *elementary degenerate*. In addition, if the jacobian matrix $D\mathcal{X}(x_0, y_0)$ is not identically null and it possesses two zero eigenvalues we say that (x_0, y_0) is a *nilpotent* point. Finally, following Seidemberg [17], a critical point is called *simple* if $\lambda \neq \mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}^+$, where $\mathbb{Q}^+$ stands for the positive rational numbers.

If the irreducible polynomial $f \in \mathbb{C}[x, y]$ with $\deg f \geq 1$ satisfies the linear partial differential equation $\mathcal{X}f = Kf$, i.e.,

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf \ , \tag{1.2}$$

for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ with $\deg K \leq 1$, then $f = 0$ is called an *invariant algebraic curve* of system (1.1) and K is termed its *cofactor*. Several authors have remarked the importance of invariant algebraic curves to understand the dynamics of system (1.1).

Remark 1. It is clear from (1.2) that, given an invariant algebraic curve $f = 0$ with cofactor K, all the critical points of system (1.1) verify either $f(x_0, y_0) = 0$ or $K(x_0, y_0) = 0$ or both above conditions.

A function $V : U \subset \mathbb{R}^2 \to \mathbb{R}$ of class C^1 in U that satisfy the linear partial differential equation $\mathcal{X}V = V \mathrm{div}\mathcal{X}$ is called an *inverse integrating factor* of system (1.1) in U. Here $\mathrm{div}\mathcal{X}$ means the divergence of the associated vector field $\mathcal{X}$ to system (1.1). Observe that the set $\Sigma := \{(x, y) \in U : V(x, y) = 0\}$ is formed by orbits of system (1.1). Additionally, $1/V$ defines an integrating factor of (1.1) which allows to compute a first integral of (1.1) in $U \backslash \Sigma$.

Remark 2. In the particular case that system (1.1) is algebraically integrable, i.e. it possesses a rational first integral, it is easy to see that its associated inverse integrating factor V becomes a polynomial.

Already Poincaré in [16] studied the problem of bounding the degree of the invariant algebraic curves of system (1.1) in connection with the problem of the algebraic integrability of (1.1). More concretely Poincaré posed the following problem.

Poincaré's Problem: *Is it possible to give an upper bound N of the degree of an invariant algebraic curve of (1.1) in terms of the degree of (1.1)?*

In general, such a bound $N(m)$ does not exist. For instance, take the linear ($m = 1$) system $\dot{x} = px$, $\dot{y} = qy$ with $p, q \in \mathbb{N}$. This system possesses the rational first integral $H(x, y) = x^q y^{-p}$ and so invariant algebraic curves of arbitrary degree of the form $f(x, y) = x^q - cy^p = 0$ with c an arbitrary constant. This property is not exclusive of linear systems. In the next example of [14], the quadratic ($m = 2$) system

$$\dot{x} = y - \frac{2}{n} xy \ , \quad \dot{y} = -x + \frac{n+2}{n}(x^2 - y^2) \ ,$$

with $n \in \mathbb{N}$ has the rational first integral

$$H(x,y) = \left[\left(1 - \frac{n+2}{n} x\right)^2 - \frac{n+2}{n} y^2 \right] \left(1 - \frac{2}{n} x\right)^{-n-2} ,$$

and therefore possesses invariant algebraic curves of arbitrary degree $n + 2$.

On the other hand, until recent time, examples of families of polynomial systems with invariant algebraic curves of arbitrary degree and without rational first integral were not known. Therefore, another question that appeared in a natural way, was to prove the existence of an upper bound $N(m)$ of the degree of invariant algebraic curves of polynomial systems (1.1) without rational first integral. Today it is well known that the former bound does not exist thanks to two independent works of Moulin-Ollanier [15] and Christopher and Llibre [10]. The example of [10] is the following one. Consider the quadratic system $\dot{x} = x(1-x)$, $\dot{y} = -\lambda y + Ax^2 + Bxy + y^2$ where $\lambda = c - 1$, $A = ab(c-a)(c-b)/c^2$ and $B = a + b - 1 - 2ab/c$. If, for every positive integer k, we take $a = 1-k$, $b \geq a$ and $c \in \mathbb{R} \backslash \mathbb{Q}$, then the system possesses no rational first integral and moreover it has the invariant algebraic curve

$$f(x,y) := \left(y - \frac{ab}{c} x\right) F(a,b,c;x) + x(1-x)F'(a,b,c;x) = 0 \ ,$$

of degree k. Here $F(a,b,c;x)$ denotes the hypergeometric function. Later on, new examples are given by Chavarriga and Grau [6]. For instance the family $\dot{x} = 1$, $\dot{y} = 2n + 2xy + y^2$ with $n \in \mathbb{N}$ has a unique invariant algebraic curve of the form $f(x,y) := H_n(x)y + 2nH_{n-1}(x)$ where $H_n(x)$ means the Hermite polynomial of degree n. In addition the above system has no generalized Darboux first integral for any $n \in \mathbb{N}$.

In the previous examples it has been seen that a bound $N(m)$ of the degree of the invariant algebraic curves of system (1.1) only in terms of the degree m of system (1.1) cannot be expected in the general case. However, some particular cases exist for which it has been able to prove the existence of such $N(m)$. It is important to remark in this direction the results of Cerveau and Lins Neto [3] showing that $N(m) \leq m+2$ when all the singularities (finite or not) of the invariant algebraic curve are of nodal type. Moreover, Carnicer in [1], proves that if system (1.1) does not have *dicritical* critical points (finite or not) then the same inequality holds, that is, $N(m) \leq m + 2$. We recall here that a critical point is dicritical when there are infinitely many invariant curves passing through it.

Another good result on the subject is the one given by Chavarriga and Llibre [7] showing that if the invariant algebraic curve $f(x,y) = 0$ is nonsingular then $N(m) \leq m+1$ and additionally, in the extremal case $N(m) = m+1$ system (1.1) has a rational first integral of the form $H(x,y) = f(x,y)/L^{\deg f}(x,y)$ where $L = 0$ is an invariant straight line.

Let $f(x,y) = \sum_{i=s}^{n} f_i(x,y) = 0$ be an irreducible algebraic curve such that the origin $(0,0)$ is a singularity of it, i.e., $f(0,0) = \partial f/\partial x(0,0) = \partial f/\partial y(0,0) = 0$. Here f_i is a homogeneous polynomial of degree i and $s \in \mathbb{N}\backslash\{0\}$. As f_s is homogeneous, it can be factorized as $f_s(x,y) = \prod_{i=1}^{k} L_i^{m_i}(x,y)$ where $L_i(x,y) = a_i x + b_i y$ are called the *tangents* of the curve $f = 0$ at the origin, $a_i, b_i \in \mathbb{C}$, $L_i \neq L_j$ for $i \neq j$ and $\sum_{i=1}^{k} m_i = s$. The origin is called an *ordinary* singularity of the curve $f = 0$ if the multiplicities $m_i = 1$ for $i = 1, \ldots, k$.

In this paper we prove the next results.

Theorem 1 *Let us assume the generic case that $P_m(x,y)$ and $Q_m(x,y)$ be coprime and the origin be an ordinary singularity of the curve $\Delta(x,y) := yP_m(x,y) - xQ_m(x,y) = 0$. If system (1.1) possesses all its infinite critical points simples then the degree of any invariant algebraic curve of it is bounded by $m+1$.*

Theorem 2 *Let $V(x,y)$ be a polynomial inverse integrating factor of degree n for system (1.1) and $\tilde{V}$ its projectivization. Let $P_m(x,y)$ and $Q_m(x,y)$ be coprime polynomials. Let p be an infinite simple critical point of system (1.1) such that $\tilde{V}(p) = 0$. Then the degree of V is exactly $m+1$.*

Let us observe that condition $\tilde{V}(p) = 0$ is fulfilled if $\widetilde{\text{div}}\mathcal{X}(p) \neq 0$ without knowing $\tilde{V}$ explicitly. Here $\widetilde{\text{div}}\mathcal{X}$ is the projectivization of the divergence $\text{div}\mathcal{X}$. We note that Walcher in [18] gives also a proof of Theorem 1 and Theorem 2 showing moreover that the additional requirement $\tilde{V}(p) = 0$ is automatically satisfied. Anyway we remark that the approach introduced in [18] uses analytical techniques such as the Pincaré–Dulac normal form. Our proof is completely algebraic and is based on the extension of the differential equation $P(x,y)dy - Q(x,y)dx = 0$ to the complex projective plane $\mathbb{CP}^2$ and the results of Seidenberg [17] about the reduction of singularities.

Two projective curves $F(X,Y,Z) = 0$ and $G(X,Y,Z) = 0$ are equivalent if there is a projective transformation $\phi : \mathbb{CP}^2 \to \mathbb{CP}^2$ with $\phi(X : Y : Z) = (U : V : W)$ defined by $(U,V,W)^t = T(X,Y,Z)^t$ where $T \in \mathcal{M}_3(\mathbb{C})$ is a nonsingular matrix such that $G(X,Y,Z) = F(U,V,W)$. Equivalence of projective curves is clearly an equivalence relation. Moreover, under this equivalence, singular points are mapped to singular points. Of course, given two different projective straight lines, they are always equivalent.

Assume that system (1.1) has a finite invariant straight line $\ell(x,y) = ax+by+c = 0$ with $a^2 + b^2 \neq 0$ and a critical point $p = (x_0, y_0) \in \mathbb{C}^2$ on such line, that is verifying $\ell(x_0, y_0) = 0$. Extending system (1.1) to $\mathbb{CP}^2$ trough the change $x = X/Z$, $y = Y/Z$, we obtain a projective differential equation, see equation (2.4). Besides the invariant straight line at infinity $Z = 0$, equation (2.4) possesses the invariant straight line $\tilde{\ell}(X,Y,Z) = aX + bY + cZ = 0$. Such lines are equivalent trough a projective transformation $\phi : \mathbb{CP}^2 \to \mathbb{CP}^2$. Additionally $(x_0 : y_0 : 1) \in \mathbb{CP}^2$ is a critical point of (2.4) such that $\tilde{\ell}(x_0, y_0, 1) = 0$.

A projective transformation $\phi(X,Y,Z) = (U,V,W)$ applied on equation (2.4) gives a new equation (2.2) of the same degree with, in general, $N \not\equiv 0$. We will make a projective transformation such that the invariant straight line $\tilde{\ell} = 0$ is mapped to the invariant line $W = 0$. Additionally, it is not difficult to see that the nature of the critical points of (2.4) remains invariant under such projective transformation. This means that, if $(x_0, y_0) \in \mathbb{C}^2$ is a simple critical point of system (1.1) then, in local coordinates at the local chart at the point $p = (U_0 : V_0 : 0) = \phi(x_0, y_0, 1) \in \mathbb{CP}^2$, p remains a simple critical point.

The above comments shows that, in $\mathbb{CP}^2$ the straight lines are equivalent, so the invariant straight line at infinity $Z = 0$ is not privileged with respect to another invariant straight line. Hence we have the next result which is, in fact, a finite version of Theorem 1.

Theorem 3 *Let us assume that system (1.1) possesses an invariant straight line $\ell(x, y) = 0$. In the generic case that the system has $m + 1$ different simple critical points on $\ell = 0$ in $\mathbb{CP}^2$, then the degree of any invariant algebraic curve of it is bounded by $m + 1$.*

The paper is organized as follows: In the second section we give some preliminaries and background on projective differential equations and formal differential equations that we will need to prove our results. Section 3 is devoted to the proof of Theorems 1, 2 and 3. Finally, in section 4 we apply the proved theorems to algebraic limit cycles in quadratic systems with invariant straight line.

2 Preliminaries and background

We will state some well-known results on differential equations that we shall use later on to prove the main results.

2.1 Projective differential equations

Following Darboux [12], we will also extend the equation of the orbits

$$P(x, y)dy - Q(x, y)dx = 0 \tag{2.1}$$

of system (1.1) to the complex projective plane $\mathbb{CP}^2$ in order to control the behavior of the solutions of (1.1) and in particular its critical points at infinity. ¿From now we can generalize to the case in which P and Q are polynomials of degree m. Now we shortly describe the process of immersing into $\mathbb{CP}^2$ a system (1.1) defined in the affine plane as well as the inverse path, i.e., submersing a differential equation defined in the complex projective plane into the affine plane by means of the so-called *local coordinates*, see also for instance [5].

A differential equation of degree m defined in $\mathbb{CP}^2$ is given by $\mathcal{P}(X, Y, Z)dX + \mathcal{Q}(X, Y, Z)dY + \mathcal{R}(X, Y, Z)dZ = 0$ where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are homogeneous polynomials of degree $m+1$ verifying $X\mathcal{P} + Y\mathcal{Q} + Z\mathcal{R} \equiv 0$. A point $p_0 = (X_0, Y_0, Z_0) \in \mathbb{CP}^2$ is a critical point of the former differential equation if $\mathcal{P}(X_0, Y_0, Z_0) = \mathcal{Q}(X_0, Y_0, Z_0) = \mathcal{R}(X_0, Y_0, Z_0) = 0$.

It is well known that the above differential equation on $\mathbb{CP}^2$ is equivalent to the differential equation

$$(ZM - YN)dX + (XN - ZL)dY + (YL - XM)dZ = 0 \ , \tag{2.2}$$

where $L(X, Y, Z)$, $M(X, Y, Z)$ and $N(X, Y, Z)$ are homogeneous polynomials of degree m. Additionally, we remark that L, M and N are not uniquely determined by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. On the contrary, these polynomials can be replaced by L', M' and N' where $L' = L + X\Lambda$, $M' = M + Y\Lambda$ and $N' = N + Z\Lambda$ where $\Lambda(X, Y, X)$ is any homogeneous polynomial of degree $m-1$ and equation (2.2) remains invariant. In the particular and interesting case $N \equiv 0$, the critical points of (2.2) must satisfy the next system of equations

$$ZL = 0 \ , ZM = 0 \ , XM - YL = 0 \ . \tag{2.3}$$

The projective curve $F(X, Y, Z) = 0$ with F an homogeneous polynomial of degree n is an invariant algebraic curve of (2.2) if $\tilde{\mathcal{X}}F = \tilde{K}F$ for some homogeneous polynomial $\tilde{K}(X, Y, Z)$ of degree $m-1$ called *cofactor*. Here we have defined the vector field $\tilde{\mathcal{X}} := L\partial/\partial X + M\partial/\partial Y + N\partial/\partial Z$. In this context, the cofactor $\tilde{K}$ associated to projective invariant algebraic curve $F = 0$ is not uniquely defined due to the former commented invariance of (2.2). In fact, an easy application of Euler theorem on homogeneous functions shows that when we change (L, M, N) by (L', M', N') then the cofactor changes to $K' = K + n\Lambda$, see [5] for example.

The projective differential equation (2.2) becomes equation (2.1) when taking local coordinates in a chart. To do this, let us consider a point $p_0 \in \mathbb{CP}^2$ in homogeneous coordinates $p_0 = (X_0 : Y_0 : Z_0)$. Without loss of generality we may consider $Z_0 \neq 0$. We define the *local coordinates* in p as $x = X/Z$, $y = Y/Z$. Hence, in local coordinates we have $p_0 = (x_0, y_0)$ where $x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$. We say that equation (2.1) is the differential equation (2.2) at the local chart at p_0 where $P(x, y) := L(x, y, 1) - xN(x, y, 1)$ and $Q(x, y) := M(x, y, 1) - yN(x, y, 1)$. It is easy to show that if $F(X, Y, Z) = 0$ is an invariant algebraic curve of (2.2) with associated cofactor $\tilde{K}(X, Y, Z)$ then $f(x, y) = 0$ is an invariant algebraic curve of (2.1) with cofactor $K(x, y) = \tilde{K}(x, y, 1) - nN(x, y, 1)$. Moreover , if p is a critical point of (2.2) with $Z_0 \neq 0$ then (x_0, y_0) is a critical point of (2.1).

The inverse process consists on extending a differential equation (2.1) defined in the affine plane to $\mathbb{CP}^2$. To get it, we make the change to homogeneous coordinates $x = X/Z$ and $y = Y/Z$. Substituting into (2.1) we obtain

$$L(YdZ - ZdY) + M(ZdX - XdZ) = 0 \tag{2.4}$$

which is a projective differential equation (2.2) with $N \equiv 0$. Here we have defined $L(X, Y, Z) := Z^m P(X/Z, Y/Z)$ and $M(X, Y, Z) := Z^m Q(X/Z, Y/Z)$ which are homogeneous polynomials of degree m. The set of points $(X : Y : 0) \in \mathbb{CP}^2$ is called the *line at infinity*. The critical points of (2.4) that belong to the line at infinity are called *infinite critical points*.

An invariant algebraic curve $f(x, y) = 0$ of degree n of system (2.1) with cofactor $K(x, y)$ defines an invariant algebraic curve $F(X, Y, Z) = 0$ of (2.2) with cofactor $\tilde{K}(X, Y, Z)$ where $F(X, Y, Z) = Z^n f(X/Z, Y/Z)$ and $\tilde{K}(X, Y, Z) = Z^{m-1}K(X/Z, Y/Z)$.

2.2 Formal differential equations

In this section we summarize some definitions and results about formal differential equations and their solutions, that we shall use later on. For more details and proofs about these results see Seidenberg [17].

We consider the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). We denote by $\mathbb{K}[[x,y]]$ the ring of formal power series. A *unit* is an invertible element of this ring. In particular, if $U(x,y) = \sum_{i,j=0}^{\infty} u_{ij} x^i y^j$ is a unit then $u_{00} \neq 0$.

Let $F(x,y)$ be an irreducible non–unit of $\mathbb{K}[[x,y]]$ such that $F(x,y) \not\equiv 0$. An *analytic branch centered at* $(0,0)$ is the equivalence class of F under the equivalence $F \sim G$ if $F = U \cdot G$ with U unit. We note that here the adjective *analytic* does not mean the convergence of the power series. On the other hand $F(0,0) = 0$ because $F(x,y)$ is non-unit.

Given a representative of an analytic branch $F(x,y)$ centered at the origin, there are power series $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, not both identically null, such that $F(x(t), y(t)) = 0$. Such a pair is called a *branch expansion* of the analytic branch. Note that $x(0) = 0$ and $y(0) = 0$. Given a branch expansion $x(t), y(t)$, there is an irreducible non–unit $F(x,y) \not\equiv 0$ in $\mathbb{K}[[x,y]]$, uniquely determined up to a unit factor, such that $F(x(t), y(t)) = 0$. $F(x,y) = 0$ is called the *equation of the branch*.

Consider the formal differential equation

$$P(x,y)dy - Q(x,y)dx = 0, \tag{2.5}$$

where $P(x,y), Q(x,y) \in \mathbb{K}[[x,y]]$. For a formal power series $F(x,y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j$ we define $\partial F(x,y)/\partial x$ as the formal power series $\sum_{i=1,j=0}^{\infty} i f_{ij} x^{i-1} y^j$. Analogously, we define $\partial F(x,y)/\partial y$.

By a *solution* of the formal differential equation (2.5) we mean an analytic branch $(x(t), y(t))$, centered at the origin satisfying equation (2.5). More explicitly, if $F(x,y) = 0$ is the equation of the branch of the solution $(x(t), y(t))$ one has

$$P(x,y)\frac{\partial F}{\partial x} + Q(x,y)\frac{\partial F}{\partial y} = K(x,y)F(x,y), \tag{2.6}$$

for some $K \in \mathbb{K}[[x,y]]$. Conversely, every irreducible $F \in \mathbb{K}[[x,y]]$ with $F \not\equiv 0$ satisfying (2.6) for some $K \in \mathbb{K}[[x,y]]$, yields a solution of equation (2.5).

A branch $x(t) = \sum_{i=1}^{\infty} x_i t^i$ and $y(t) = \sum_{i=1}^{\infty} y_i t^i$, with $x_i, y_i \in \mathbb{K}$, centered at $(0,0)$, is called *linear* if x_1 or y_1 is not zero. We shall use the following results also from [17].

Theorem 4 (Seidenberg) *Let the origin* $(0,0)$ *be a critical point of the formal system* $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, *where* $P, Q \in \mathbb{C}[[x,y]]$, *with associated eigenvalues* $\lambda, \mu \in \mathbb{C}$. *In the following the dots denote higher order terms.*

1. *Let* $(0,0)$ *be a nondegenerate critical point. Then consider the formal differential system*

$$\dot{x} = \lambda x + \cdots \ , \ \dot{y} = \mu y + \cdots \ , \tag{2.7}$$

where $\lambda\mu \neq 0$. *If* $\lambda \neq \mu$ *then every formal solution of (2.7) at the origin has a horizontal or vertical tangent. Moreover,*

(i) *If* $\lambda/\mu \notin \mathbb{Q}^+$ *then (2.7) has exactly two formal solutions at the origin* $F_i(x, y) = 0$ *with* $i = 1, 2$. *They are linear branches with horizontal and vertical tangent respectively, i.e.,* $F_1(x, y) = x + \cdots$, $F_2(x, y) = y + \cdots$.

(ii) *If* $\lambda/\mu \in \mathbb{Q}^+$ *then the following holds.*

(a) *If* $\lambda = \mu$ *then, for each direction there exists only one linear branch formal solution at the origin.*

(b) *If* $\lambda/\mu \neq 1$ *(with* $\lambda/\mu > 1$*) then there is one unique linear branch formal solution at the origin with horizontal tangent* $F(x, y) = y + \cdots$. *The other formal solutions at the origin, if they exists, have vertical tangent, i.e., are of the form* $F(x, y) = x^s + \cdots$ *with* $s \in \mathbb{N}\backslash 0$.

(b.1) *If* $\lambda/\mu \in \mathbb{N}$ *then either there are no formal solutions at the origin with vertical tangent or there are infinity linear branch formal solution at the origin with vertical tangent.*

(b.2) *If* $\lambda/\mu \notin \mathbb{N}$ *then there is one unique linear branch formal solution at the origin with vertical tangent* $F(x, y) = x + \cdots$.

2. Let $(0, 0)$ *be a logarithmic critical point. Then, the formal differential system* $\dot{x} = \lambda x + y + \cdots$, $\dot{y} = \lambda y + \cdots$, *where* $\lambda \neq 0$ *has a unique linear branch formal solution at the origin with horizontal tangent* $F(x, y) = y + \cdots$.

3. Let $(0, 0)$ *be a elementary degenerate critical point. Then, the formal differential system* $\dot{x} = x + \cdots$, $\dot{y} = \cdots$, *has exactly two formal solutions at the origin* $F_i(x, y) = 0$ *with* $i = 1, 2$. *They are linear branches with horizontal and vertical tangent respectively, i.e.,* $F_1(x, y) = x + \cdots$, $F_2(x, y) = y + \cdots$.

4. Let $(0, 0)$ *be a nilpotent critical point. Then, the formal differential system* $\dot{x} = y + \cdots$, $\dot{y} = \cdots$, *can have either one formal solution at the origin or two linear branch formal solutions at the origin or infinity formal solutions at the origin.*

Let us consider an irreducible algebraic curve $f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$ such that $f(x_0, y_0) = 0$. We translate the point (x_0, y_0) to the origin. In particular $f \in \mathbb{C}[[x, y]]$ with $f(0, 0) = 0$, hence f is not a unit element in $\mathbb{C}[[x, y]]$ and in this ring it is possible for f to be a reducible element. By using the Newton-Poiseux algorithm, see [2], one can see that there are ℓ irreducible elements $\phi_i(x, y) \in \mathbb{C}[[x, y]]$, with $i = 1, \ldots, \ell$ such that f factorizes as

$$f(x, y) = x^r U(x, y) \prod_{i=1}^{\ell} \phi_i(x, y) \ , \tag{2.8}$$

being $r \in \mathbb{N} \cup \{0\}$ and $U \in \mathbb{C}[[x, y]]$ a unit element. Later on, in [5], it was proved that the above decomposition (2.8) is square free, that is, there is no repeated element ϕ_i neither $r \geq 2$.

Let the origin $(0, 0)$ be a singular point of system (1.1) and let $f = 0$ be an irreducible invariant algebraic curve of that system such that $f(0, 0) = 0$. The curve $f(x, y) = \sum_{i=s}^{n} f_i(x, y) = 0$ with f_i real homogeneous polynomials and $s \geq 1$, defines a finite number of branches at the origin corresponding to its irreducible nonunit factors in $\mathbb{C}[[x,y]]$. We known that $f_s(x,y) = \prod_{i=1}^{s} L_i(x,y)$ where $L_i(x,y) = a_i x + b_i y$ are the tangents of the curve $f = 0$ at the origin.

Finally, it is easy to see that each of the irreducible elements appearing in the above formal decomposition (2.8) of f is a formal solution of (1.1). Moreover, the tangents at the origin of these branches are given by $f_s = 0$ as defined above.

Let $(x_0, y_0) \in \mathbb{C}^2$ be a critical point with eigenvalues $\lambda, \mu \in \mathbb{C}$. Denoting by $v_\lambda, v_\mu \in \mathbb{C}^2$ the corresponding eigenvectors, we will call $L_\lambda(x,y)$ and $L_\mu(x,y)$ the non-null homogeneous polynomials of degree one belonging to $\mathbb{C}[x,y]$ such that $\nabla L_\lambda \perp v_\lambda$ and $\nabla L_\mu \perp v_\mu$ respectively. Here $\nabla := (\partial/\partial x, \partial/\partial y)$ is the gradient operator and $\perp$ means orthogonality with respect to the standard Euclidean scalar product in $\mathbb{C}^2$. Taking into account all this background, in [5] the following result is proved, which describe the tangents and the value of the cofactor at some generic class of critical points.

Theorem 5 (Chavarriga, Giacomini & Grau) *Let $f(x,y) = 0$ with $f \in \mathbb{C}[x,y]$ be an irreducible invariant algebraic curve with associated cofactor $K(x,y)$ of a polynomial differential system. Let $(x_0, y_0) \in \mathbb{C}^2$ be a nondegenerate or elementary degenerate critical point of the system with different associated eigenvalues λ and μ verifying $f(x_0, y_0) = 0$. Then, the equation of the tangents of the curve $f = 0$ at (x_0, y_0) is $f_s(x,y) = L_\lambda^r(x,y) L_\mu^{s-r}(x,y)$ with $s, r \in \mathbb{N}$, $r \leq s$. Moreover $K(x_0, y_0) = r\mu + (s-r)\lambda$.*

Remark 3. Theorem 5 and Remark 1 gives the possible values of the cofactor K of an invariant algebraic curve $f = 0$ of system (1.1) at a nondegenerate or degenerate elementary critical point $(x_0, y_0) \in \mathbb{C}^2$ whose ratio of eigenvalues does not equal one. Of course, we can extend system (1.1) to $\mathbb{CP}^2$. Hence, if $p_0 = (X_0 : Y_0 : Z_0)$ is a critical point of the projective equation (2.4), we can take local coordinates at this point and Theorem 5 and Remark 1 can be applied. We remark that, for an infinite critical point $p_0 = (X_0 : Y_0 : 0)$ we will obtain by the above procedure conditions on the degree n of the curve $f = 0$ because the coefficients of the cofactor also depend on n.

3 Proof of the main results

3.1 Proof of Theorem 1

Let $P_m(x,y) = \sum_{i+j=m} a_{ij} x^i y^j$ and $Q_m(x,y) = \sum_{i+j=m} b_{ij} x^i y^j$ with $a_{ij}, b_{ij} \in \mathbb{C}$. Doing a linear change of coordinates in system (1.1), we may assume $b_{m0} \neq 0$ without loss of generality.

Let $p_i = (X_i : 1 : 0) \in \mathbb{CP}^2$ with $i = 1, 2, \ldots, m+1$, be the infinite critical points of system (1.1). Since by assumption the origin is an ordinary singularity of the curve $\Delta(x,y) := y P_m(x,y) - x Q_m(x,y) = 0$, this means that $\Delta(x,y) = \prod_{i=1}^{m+1} L_i(x,y)$ with $L_i(x,y) = \alpha_i x + \beta_i y$ where $\alpha_i, \beta_i \in \mathbb{C}$ and $L_i \neq L_j$ for $i \neq j$ with $i, j \in \{1, 2, \ldots, m+1\}$, then $X_i \neq X_j$ for $i \neq j$.

Let $f(x,y) = 0$ be an invariant algebraic curve of system (1.1) of degree n with associated cofactor $K(x,y)$. Extending system (1.1) to $\mathbb{CP}^2$ by using the change

$x = X/Z$, $y = Y/Z$ we obtain the projective differential equation (2.4), where

$$L(X,Y,Z) := Z^m P(X/Z, Y/Z) = \sum_{i=0}^{m} Z^{m-i} P_i(X,Z) ,$$
$$M(X,Y,Z) := Z^m Q(X/Z, Y/Z) = \sum_{i=0}^{m} Z^{m-i} Q_i(X,Z) ,$$

are homogeneous polynomials of degree m. The invariant algebraic curve $f(x,y) = 0$ of system (1.1) defines the invariant algebraic curve $F(X,Y,Z) := Z^n f(X/Z, Y/Z) = 0$ of equation (2.4) with cofactor $\tilde{K}(X,Y,Z) = Z^{m-1}K(X/Z,Y/Z)$. Now we take local coordinates $u = X/Y$, $v = Z/Y$ at the points p_i. In such coordinates, the points $p_i = (X_i, 0)$ and the differential equation (2.4) at the local chart at p_i becomes

$$\begin{aligned} \dot{u} &= L(u,1,v) - uM(u,1,v) , \\ \dot{v} &= -vM(u,1,v) . \end{aligned} \tag{3.1}$$

Denoting by $\hat{\mathcal{X}}$ the vector field associated to this system, its Jacobian matrix at p_i is

$$D\hat{\mathcal{X}}(p_i) = \begin{pmatrix} \delta'(X_i) & * \\ 0 & -M(p_i) \end{pmatrix}$$

where $*$ is some entry, $\delta(x) := \Delta(x,1)$ and the prime denotes derivative with respect to the variable x. Hence the eigenvalues associated to the critical points p_i of system (3.1) are $\lambda_i = \delta'(X_i)$ and $\mu_i = -M(p_i)$ for $i = 1, \ldots, m+1$.

We claim that $\mu_i \neq 0$ for $i = 1, \ldots, m+1$. This is because otherwise $M(p_i) = 0$ and, since p_i is a critical point of (3.1), it follows $L(p_i) = 0$ for $i = 1, \ldots, m+1$. So $P_m(X_i, 1) = Q_m(X_i, 1) = 0$ for $i = 1, \ldots, m+1$ in contradiction with the coprimality of the polynomials P_m and Q_m.

Of course, besides the invariant line $v = 0$, system (3.1) possesses the invariant algebraic curve $\hat{f}(u,v) := F(u,1,v) = 0$ with cofactor $\hat{K}(u,v) := \tilde{K}(u,1,v) - nM(u,1,v)$. The values of this cofactor at p_i are

$$\hat{K}(p_i) = \tilde{K}(X_i,1,0) - nM(X_i,1,0) \quad \text{for} \quad i = 1, \ldots, m+1 . \tag{3.2}$$

Since we have assumed p_i to be simple critical points,i.e., $\lambda_i \neq \mu_i \neq 0$ and $\lambda_i/\mu_i \notin \mathbb{Q}^+$, applying statements 1(i) and 3 of Seidemberg Theorem 4, it follows that if $\hat{f}(p_i) = 0$ then, after a linear change of coordinates $\hat{f}(u,v) = u + \cdots$. But now, taking into account Theorem 5 and Remark 1, we have

$$\hat{K}(p_i) = \epsilon_i \lambda_i \quad \text{for} \quad i = 1, \ldots, m+1 , \tag{3.3}$$

where $\epsilon_i \in \{0,1\}$. If we denote $K(x,y) = \sum_{j=0}^{m-1} K_j(x,y)$ with K_j homogeneous polynomial of degree j, and putting $K_{m-1}(x,y) = \sum_{i+j=m-1} k_{i,j} x^i y^j$ it is clear that $\tilde{K}(X_i,1,0) = \sum_{j=0}^{m-1} k_{j,m-1-j} X_i^j$. Hence, from (3.2) and (3.3) it follows

$$\sum_{j=0}^{m-1} k_{j,m-1-j} X_i^j = nM(X_i,1,0) + \epsilon_i \lambda_i ,$$

for $i = 1, \dots, m+1$. This is an overdetermined linear system of equations $A\kappa = b$ where the matrix coefficients A is given by

$$A = \begin{pmatrix} 1 & X_1 & X_1^2 & \cdots & X_1^{m-1} \\ 1 & X_2 & X_2^2 & \cdots & X_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{m+1} & X_{m+1}^2 & \cdots & X_{m+1}^{m-1} \end{pmatrix} \in \mathcal{M}_{m+1,m}(\mathbb{C}) \ ,$$

the column vector of unknowns $\kappa = (k_{0,m-1}, k_{1,m-2}, \dots, k_{m-1,0})^t \in \mathbb{C}^m$ and the column vector of independent terms $b = (b_1, \dots, b_m)^t \in \mathbb{C}^m$ where $b_i := nM(X_i, 1, 0) + \epsilon_i \lambda_i$ for $i = 1, \dots, m$. First of all observe that the rank of A is m because the determinant D_m formed with the first m rows and columns of A is a Vandermonde determinant of order m and so $D_m \neq 0$ because $X_i \neq X_j$ when $i \neq j$. In order to have a compatible linear system we must impose $\det(A|b) = 0$, being $A|b \in \mathcal{M}_{m+1}(\mathbb{C})$ the enlarged matrix of the linear system. To do this, recall that $M(X_i, 1, 0) = P_m(X_i, 1) = \sum_{j=0}^m b_{j,m-j} X_i^j$. Hence by properties of the determinants one has

$$\det(A|b) = \det \left(\begin{array}{c|c} & nb_{m0} X_1^m + \epsilon_1 \lambda_1 \\ A & \vdots \\ & nb_{m0} X_{m+1}^m + \epsilon_{m+1} \lambda_{m+1} \end{array} \right) .$$

Defining the characteristic function $\alpha : \mathbb{N} \to \{0, 1\}$ as

$$\alpha(m) := \begin{cases} 1 & \text{if} \quad m \text{ is even}, \\ 0 & \text{if} \quad m \text{ is odd}, \end{cases}$$

we develop the former determinant by the last column so that

$$\det(A|b) = nb_{m0} D_{m+1} + \sum_{j=1}^{m+1} (-1)^{j+\alpha(m)} \epsilon_j \lambda_j \Omega_j \ , \tag{3.4}$$

where

$$D_{m+1} = \begin{vmatrix} 1 & X_1 & X_1^2 & \cdots & X_1^m \\ 1 & X_2 & X_2^2 & \cdots & X_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{m+1} & X_{m+1}^2 & \cdots & X_{m+1}^m \end{vmatrix} = \prod_{i>j} (X_i - X_j) \neq 0 \ , \tag{3.5}$$

is a Vandermonde determinant of order $m+1$ and Ω_j is the Vandermonde determinant of order m obtained after deleting row j and the last column of D_{m+1}

$$\Omega_j = \begin{vmatrix} 1 & X_1 & X_1^2 & \cdots & X_1^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{j-1} & X_{j-1}^2 & \cdots & X_{j-1}^{m-1} \\ 1 & X_{j+1} & X_{j+1}^2 & \cdots & X_{j+1}^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{m+1} & X_{m+1}^2 & \cdots & X_{m+1}^{m-1} \end{vmatrix} = \prod_{\substack{k>i \\ i,k \neq j}} (X_k - X_i) \neq 0 \ . \tag{3.6}$$

Since $\delta(x) = P_m(x,1) - xQ_m(x,1) = \sum_{i+j=m}(a_{ij} - xb_{ij})x^i$ and it verifies $\delta(X_i) = 0$ for $i = 1, \ldots, m+1$, we can factorize of the form $\delta(x) = -b_{m0}\prod_{i=1}^{m+1}(x - X_i)$ so that, as $\lambda_j = \delta'(X_j)$, we obtain

$$\lambda_j = -b_{m0}\prod_{\substack{i=1\\ i\neq j}}^{m+1}(X_j - X_i) \neq 0 \ . \tag{3.7}$$

Finally, from (3.6) and (3.7), it is easy to see that $\lambda_j\Omega_j = (-1)^{1+j+\alpha(m)}b_{m0}D_{m+1}$. Therefore, taking into account (3.4) we conclude that

$$\det(A|b) = b_{m0}D_{m+1}\left(n - \sum_{j=1}^{m+1}\epsilon_j\right) \ ,$$

and so $n = \sum_{j=1}^{m+1}\epsilon_j \leq m+1$ proving then Theorem 1.

3.2 Proof of Theorem 2

Let $\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$ be the associated vector field to system (1.1). As usual, let $P_m(x,y) = \sum_{i+j=m} a_{ij}x^iy^j$ and $Q_m(x,y) = \sum_{i+j=m} b_{ij}x^iy^j$ with $a_{ij}, b_{ij} \in \mathbb{C}$. Doing a linear change of coordinates we will assume that $p = (0:1:0) \in \mathbb{CP}^2$ and hence $a_{0m} = 0$ without loss of generality. In addition, since $b_{0m} \neq 0$ because otherwise P_m and Q_m are not coprime, rescaling the independent variable in system (1.1) we can obtain $b_{0m} = 1$.

The cofactor K associated to the inverse integrating factor $V(x,y)$ is, by definition, $K(x,y) = \mathrm{div}\mathcal{X}(x,y)$. Now, we extend system (1.1) to $\mathbb{CP}^2$ obtaining the projective differential equation (2.4), where $L(X,Y,Z) = Z^mP(X/Z,Y/Z)$ and $M(X,Y,Z) = Z^mQ(X/Z,Y/Z)$. The projectivization of $V = 0$ gives $F(X,Y,Z) := Z^nV(X/Z,Y/Z) = 0$ with cofactor $\tilde{K}(X,Y,Z) = Z^{m-1}\mathrm{div}\mathcal{X}(X/Z,Y/Z)$. Now we take local coordinates $u = X/Y$, $v = Z/Y$ at the point p so that, in such coordinates the point $p = (0,0)$ and the differential equation (2.4) at the local chart at p becomes $\dot{u} = L(u,1,v) - uM(u,1,v)$, $\dot{v} = -vM(u,1,v)$. Obviously that system has the invariant straight line $v = 0$ trough p. The Jacobian matrix $D\hat{\mathcal{X}}$ of the associated vector field $\hat{\mathcal{X}}$ to the former system at p is

$$D\hat{\mathcal{X}}(p) = \begin{pmatrix} a_{1,m-1} - 1 & * \\ 0 & -1 \end{pmatrix}$$

where $*$ is some entry. Hence the eigenvalues associated to the critical point p are $\lambda = a_{1,m-1} - 1$ and $\mu = -1$. Moreover in the local coordinates the system possesses the invariant algebraic curve $\hat{f}(u,v) := F(u,1,v) = 0$ with cofactor $\hat{K}(u,v) := \tilde{K}(u,1,v) - nM(u,1,v)$. The value of the cofactor at p is

$$\hat{K}(p) = \tilde{K}(0,1,0) - nM(0,1,0) = a_{1,m-1} + m - n \ . \tag{3.8}$$

By assumption p is a simple critical point ($\lambda \neq \mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}^+$) so that, applying statements 1(i) and 3 of Seidemberg Theorem 4, it follows that after a linear change of coordinates $\hat{f}(u,v) = u + \cdots$. Since $\hat{f}(p) = 0$ by hypothesis, it follows that $\hat{K}(p) = \lambda = a_{1,m-1} - 1$. Taking now into account (3.8) we deduce $n = m+1$ finishing the proof.

3.3 Proof of Theorem 3

After one rotation and one translation of coordinates, we can assume $\ell(x, y) = y = 0$. In such coordinates, system (1.1) reads for

$$\dot{x} = P(x, y) \ , \ \dot{y} = y\hat{Q}(x, y) \ . \tag{3.9}$$

Let $p_i = (x_i, 0) \in \mathbb{C}^2$ be different finite simple critical points of system (3.9) on the invariant line $y = 0$ for $i = 1, \dots, m$. Of course, in $\mathbb{CP}^2$, there is another critical point of (1.1) in the intersection point $p_{m+1} = (1 : 0 : 0)$ of the former invariant line and the line at infinity. After extending the differential system to $\mathbb{CP}^2$ by using the homogeneous coordinates coordinates $x = X/Z$, $y = Y/Z$, system (3.9) becomes

$$\begin{vmatrix} L & M & 0 \\ X & Y & Z \\ dX & dY & dZ \end{vmatrix} = 0 \ , \tag{3.10}$$

with the usual definition of the homogeneous polynomials $L(X, Y, Z)$ and $M(X, Y, Z)$. We shall take the local coordinates $u = X/Y$, $v = Z/Y$ and equation (3.10) becomes

$$\dot{u} = L(u, 1, v) - uM(u, 1, v) \ , \ \dot{v} = -vM(u, 1, v) \ . \tag{3.11}$$

Usually, this vector field has degree $m + 1$ and is *degenerated infinity* which means that all the infinity is filled up of critical points. But since $y = 0$ is an invariant straight line of (3.9) this implies $\deg M(u, 1, v) \le m - 1$ and therefore (3.11) has at most degree m with non-degenerated infinity. Moreover, in these local coordinates (u, v), the points p_i are infinity critical points of (3.11). In short we can apply now Theorem 1 to system (3.11) to conclude that all the possible invariant algebraic curves of (3.11) have degree at most $m+1$. But then the same bound holds for system (3.9) and finally the theorem is proved. ∎

4 An application: algebraic limit cycles in quadratic systems with an invariant straight line

It is well known, see [11] for example, that a quadratic system with an invariant straight line and a limit cycle is equivalent to

$$\dot{x} = \delta x - y + \ell x^2 + mxy + ny^2 \ , \ \dot{y} = x(1 + by) \ , \tag{4.1}$$

with $\delta \in [0, 2)$, $m \ge 0$ and $b = \pm 1$.

Proposition 6 *Let $\Delta_1 := (b\delta - m)^2 - 4\ell(b + n)$ and $\Delta_2 := m^2 + 4(b - \ell)n$. Assume that the two following set of generic conditions are verified.*

(i) *(i.1) Let either $\ell = 0$ or $\ell \ne 0$ and $1 - b/\ell \notin \mathbb{Q}^+$; (i.2) Let $n \ne 0$, $(b - 2\ell)\sqrt{n} \ne \pm m\sqrt{\ell}$ and*

$$\frac{4n(b - \ell) + m(m \pm \sqrt{\Delta_2})}{2bn} \notin \mathbb{Q}^+ \ .$$

(ii) *(ii.1) Let* $\ell \neq 0$ *and* $1-b/\ell \notin \mathbb{Q}^+$*; (ii.2) Let* $2\sqrt{\Delta_1} \neq b(\sqrt{\Delta_1} \pm m \mp b\delta)$*. Moreover, let either* $\mu \neq 0$ *and* $\lambda/\mu \notin \mathbb{Q}^+$ *or* $\lambda \neq 0$ *and* $\mu/\lambda \notin \mathbb{Q}^+$*. Here* $\lambda := \pm\sqrt{\Delta_1}/b$ *and* $\mu := (m - b\delta \pm \sqrt{\Delta_1})/(2\ell)$*.*

Then the degree of the invariant algebraic curves of (4.1) is bounded by 3. In particular, system (4.1) does not have algebraic limit cycles.

Proof. Assuming $\ell \neq 0$, the finite singular points of (4.1) on the invariant line $1+by=0$ are

$$p_{1,2} = \frac{1}{2b\ell}\left(m - b\delta \pm \sqrt{\Delta_1}, -\frac{1}{b}\right) \in \mathbb{C}^2$$

where $\Delta_1 := (b\delta - m)^2 - 4\ell(b+n)$ and we have supposed $\ell \neq 0$. The associated eigenvalues are

$$\lambda_{1,2} = \pm\frac{\sqrt{\Delta_1}}{b} \ , \ \mu_{1,2} = \frac{m - b\delta \pm \sqrt{\Delta_1}}{2\ell} \ .$$

Assuming $n \neq 0$, the infinite critical points of (4.1) are

$$p_3 = (1:0:0) \in \mathbb{CP}^2 \ , \ p_{4,5} = \left(1 : \frac{-m \pm \sqrt{\Delta_2}}{2n} : 0\right) \in \mathbb{CP}^2 \ ,$$

where $\Delta_2 := m^2 + 4(b-\ell)n$. Its associated eigenvalues are

$$\lambda_3 = b - \ell \ , \ \mu_3 = -\ell \ ,$$

and

$$\lambda_{4,5} = \frac{4(\ell - b)n + m(-m \mp \sqrt{\Delta_2})}{2n} \ , \ \mu_{4,5} = -b \ .$$

The set of conditions (i) come from imposing that p_i, with $i = 3, 4, 5$ be simple singular points, that is, either $\lambda_i \neq \mu_i \neq 0$ and $\lambda_i/\mu_i \notin \mathbb{Q}^+$ or $\mu_i \neq \lambda_i \neq 0$ and $\mu_i/\lambda_i \notin \mathbb{Q}^+$ for $i = 3, 4, 5$. Moreover, restrictions (ii) are equivalent to assume p_i, with $i = 1, 2, 3$ be simple singular points. Observe that p_3 is the intersection point in $\mathbb{CP}^2$ of the projective line $Z + bY = 0$ and the line at infinity $Z = 0$.

The degree of the invariant algebraic curves of (4.1) is bounded by 3 follows applying Theorem 1 in case (i) while in case (ii) we must use Theorem 3.

Finally, it is well known by a sequence of 3 papers of R.M. Evdokimenko that there are no algebraic limit cycles of third degree for quadratic systems, see a shorter proof in [4]. Additionally, Chin Yuan-shun in [9] proves that if a quadratic system has an algebraic limit cycle of degree 2, then after an affine change of variables the limit cycle becomes the circle $f(x,y) = x^2 + y^2 - 1 = 0$ and the system has the form $\dot{x} = -y(ax + by + c) - f(x,y)$, $\dot{y} = x(ax + by + c)$ with $a \neq 0$, $c^2 + 4(b+1) > 0$ and $c^2 > a^2 + b^2$. It is easy to see that this system is not affine–equivalent to system (4.1). So (4.1) has not an ellipse as limit cycle. ∎

Acknowledgements. Both authors are partially supported by a Spanish MEyC grant number BTM2005-06098-C02-02.

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