Holomorphisms on the tangent and cotangent bundles

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Abstract. Using the tangent bundle (TM, π, M), the almost complex structure, natural, F and the almost complex structure $\overset{*}{F}$ are obtained the propositions from the paragraphs 1 and 2.

M.S.C. 2000: 53C60.
Key words: infinitesimal F-automorphism; infinitesimal holomorphism.

1 Structures $(TM, F, D(F))$, $(TM, \overset{*}{F}, D(F))$

Let be the tangent bundle $\xi = (TM, \pi, M)$, where $M_n = (M, [A]; \mathbb{R}^n)$ is a C^∞-differential manifold, n-dimensional, paracompact. In these conditions we have the differential manifold $E_{2n} = [TM, [\mathcal{A}]; \mathbb{R}^{2n}]$, who is C^∞-differential, n-dimensional, paracompact. For a local chart $(\varphi, U) \in A$, in x on M, correspond a local chart $(\widetilde{\varphi}, \widetilde{U}) \in \mathcal{A}$, in u on TM, where $\widetilde{u} = \pi^{-1}(U)$ $(\pi(u) = x)$. If $x \in U$ has the local coordinates $(x^i(x))$ (we will write x^i) $i = \overline{1, n}$, then $u \in \widetilde{U}$ has the local coordinates (x^i, y^i) with the known transformations $(x^i, y^i) \to (x^{i'}, y^{i'})$, to the change of charts $(\widetilde{\varphi}, \widetilde{U}) \to (\widetilde{\varphi}', \widetilde{U}')$, in u on TM, who correspond to the change $(\varphi, U) \to (\varphi', U')$, in x on M $(\pi(u) = x)$.

Analogously, we can consider the C^∞-differential manifold, $E^*_{2n} = [T^*M, [\mathcal{A}^*]; \mathbb{R}^{2n}]$, where we consider a local chart, in u* on T*M $(\pi^*(u^*) = x)$, who has the coordinates (x^i, τ_i), with the known laws in a change of local charts, in u*, on T*M. The geometries of manifolds E_{2n} and E^*_{2n} are not dual, if we take into account from TTM and TT*M.

It is know, in these conditions, the theory of nonlinear connections N (in ξ), $\overset{*}{N}$ (in $\overset{*}{\xi}$) and of the projectors (h,v) ($h + v = I$, $h^2 = h$, $v^2 = v$, $hv = 0$, $vh = 0$).

Because M is paracompact exists, globally, linear connections D, on M. Fixing D, we can obtain N. But not any N has this form ([2]).

Having N fix, there exist an almost complex structure, natural, $F : \mathcal{X}(TM) \to \mathcal{X}(TM)$ $(F^2 = -I)$, defined by ([2]):

$$F(\delta_k) = -\partial_k; F(\partial_k) = \delta_k$$

Differential Geometry - Dynamical Systems, Vol.8, 2006, pp. 77-81.

where $\{\delta_k = \delta/\delta x^k; \partial_k = \partial/\partial y^k\}$ is the adapted basis in $u \in \pi^{-1}(U)$. We will write (TM, F). Or, $F(X^h) = -X^v; F(X^v) = X^h$, ($X^h$, X^v are the horizontal and vertical lifts of $X \in \mathcal{X}(M)$).Using this structure we can make a theory of d-connections, linear, on $E = TM$.
Having so a metric structure ($E = TM$, G) and using $\theta(X,Y) = G(FX,Y)$, we obtain an almost symplectic structure θ, who is much study by the romanian mathematicians. When (F, G) is an almost hermitian structure ($G(FX,FY) = G(X,Y)$, $\forall X,Y \in \mathcal{X}(TM)$), R. Miron had been developed a theory of the almost hermitian model of the generalized Lagrange space.

For F we associate the almost complex structure,$\overset{*}{F}$: $\Lambda_1(TM) \to \Lambda_1(TM)$

$$(\overset{*}{F}\,\omega)(X) = -\omega(F(X))$$

$\forall X \in T_uTM$, $\omega \in T^*_uTM$, $u \in TM$(we identify T*TM with $\Lambda_1(TM)$). $\overset{*}{F}$ will be called *the dual of F*. The authors study the relations between F and $\overset{*}{F}$ from many points of view.

Let be D a linear connection, on $E = TM$, with the property that $DF = 0$ (D is not necessary a linear d-connection. Particulary, if D preserve the horizontal distribution H and the vertical distribution V, who are supplementary, for fixed N, then we will obtain the adequate results. In this case $DF = 0$ is study in [2]).

Definition 1.1 If *D is a linear connection, on $E = TM$, with the property $DF = 0$, then D will be called an almost Finsler connection from first genus. If D is a linear d-connection, then it will be called Finsler connection or normal.*

We will note the structure, in this case, from (TM, F, $D(F)$). Result the next propositions:

Proposition 1.1 Let *be* (TM, F, $D(F)$). *We have:*

$$(D_X\ \overset{*}{F}\,\omega) = -(D_X\omega)\circ F, \tag{1.1}$$

$$(D_X\ \overset{*}{F}\,\omega)\circ F = D_X\omega,\ \forall X \in \mathcal{X}(TM), \tag{1.2}$$

Proposition 1.2 If D^2_{XY} *is the covariant derivative from second order, then, if we have $DF = 0$, result:*

$$D^2_{XY}\ \overset{*}{F}\,\omega = -(D^2_{XY}\omega)\circ F, \tag{1.3}$$

Proposition 1.3 Let *be* (TM, F, $D(F)$). *Then we have the identities:*

$$\overset{*}{F}\,\omega\circ R(X,Y) + D_{T(X,Y)}\ \overset{*}{F}\,\omega = -\omega\circ F\circ R(X,Y) + (D_{T(X,Y)}\omega)\circ F, \tag{1.4}$$

where R is the curvature tensor of D and T is the torsion Tensor of D.

2 Infinitesimal automorphisms natural. Holomorphisms.

Let be the Nijenhuis tensor associated to F:

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]$$

Definition 2.1 Let *be the almost complex structure* $(TM,\, F)$*, natural. If* $\xi \in \mathcal{X}(TM)$ *and for the Lie derivative* $L_\xi F$*, we have:* $L_\xi F = 0$*, then* ξ *will be called infinitesimal F-automorphism, natural.*

Definition 2.2 Let *be the almost complex structure, natural,* $(TM,\, F)$*. An infinitesimal automorphism, natural, will be called holomorphic, natural, if the structure F, natural, is complex.*

Using these notions we study the necessary and sufficient conditions, that ξ, $F\xi$, ξ^h and ξ^v, its will be the infinitesimal automorphisms, natural, and their holomorphy. The Lie algebra is study. The base results from who is deduce the other results, are given in the next propositions.

Proposition 2.1 Let *be* $\xi = \xi^i \delta/\delta x^i + \xi^{n+r}\partial/\partial y^r$ *(in the adapted basis). Then* ξ *is an automorphism, natural, if in the adapted basis, we have,*

$$\xi^r \frac{\partial N_r^i}{\partial y^k} = \frac{\partial \xi^{n+i}}{\partial y^k} - \frac{\delta \xi^i}{\delta x^k}, \tag{2.-3}$$

$$\xi^r R_{kr}^i = \frac{\partial \xi^i}{\partial y^k} + \frac{\delta \xi^{n+i}}{\delta x^k} - \xi^{n+s}\frac{\partial N_k^i}{\partial y^s}, \tag{2.-2}$$

Proposition 2.2 If ξ, $F\xi$ *are infinitesimal automorphisms, then we have the structure equations ([5]):*

$$\begin{cases} \xi^{n+r}\overline{R}_{kr}^i & +\xi^r t_{kr}^i & = 0 \\ \xi^r \overline{R}_{kr}^i & +\xi^{n+r} t_{kr}^i & = 0 \end{cases} \tag{2.-1}$$

where t is the weak torsion of N, and $(\overline{R})$ *is the curvature of nonlinear connection N.*

1. A very important case is that, H is integrabil (and so $\overline{R} = 0$).
2. Another case is that, F, natural is integrabil (complex): $N_F = 0$ and so in the adapted basis, $\overline{R} = 0$, $t = 0$. This case is necessary for to study the holomorphy.
3. We study the relations between F, $\overset{*}{F}$, ω from point of view of holomorphy of ξ, starting from $L_\xi(\overset{*}{F}\,\omega)$. $L_\xi F = 0$.
4. If we have the metric structure $(TM,\, G)$ and F is the almost complex structure, natural, then we resume the study before, for $(TM,\, F,\, G)$ almost hermitian structure or hermitian structure, or almost kählerian, kählerian, respectively.Evidently, the study is bigger.

Application 2.1 *Let be* $V_n = (M, g, \nabla)$ *a pseudo-riemannian space and* $V_{2n} = (TM, G, D)$*, were G is pseudo-riemannian Sasaki lift (Sasaki - riemannian, respectively). Then using the results from the paper [1], we observe that the study of holomorphism is the trivial case, because, if F is integrable, then we have,* $T = t = 0$, $R^i_{jk}(x, y) = 0 \Leftrightarrow (r^i_{jkl} = 0)$*, that is* V_n *must be, locally, pseudo-euclidian.*

Proposition 2.3 If *we admit that* V_n *is not pseudo-euclidian, that is H, on TM, is not integrable, then:*

1. *The structure equations are:*

$$\left\{ \begin{array}{rll} \xi^{n+r} & +r^i_{jkr} & = 0 \\ \xi^r & +r^i_{jkr} & = 0 \end{array} \right. \tag{2.0}$$

2. *If* $\xi \in \mathcal{X}(TM)$ *satisfy 2.0, then the following statements are equivalent:*
   ξ*-infinitesimal automorphism, natural,*
   $F\xi$*-infinitesimal automorphism, natural*

Proposition 2.4 Let *be* $U = \xi^i(x)\partial/\partial x^i$*, Killing of space* V_n *and* ξ *the natural prolongation of U, to TM. If* ξ*,* $F\xi$ *are infinitesimal automorphisms, natural, then the structure equations are:*

$$\left\{ \begin{array}{rll} (\nabla_s \xi^r) & r^i_{jkr} & = 0 \\ \xi^r & r^i_{jkr} & = 0 \end{array} \right. \tag{2.1}$$

Consequence 2.1 We *have:*

$$\xi^r \nabla_s r^i_{jkr} = 0, \tag{2.2}$$

Proposition 2.5 If $V_n = (M, g, \nabla)$ *is symmetric in Cartan sense, then the structure equations are reduced to 2.2.*

In the case of metric cotangent bundle $(T^*M, \overset{*}{G})$, without the almost complex structure, introduced by R. Miron using the metric structure, the authors extend the study on the almost complex structure, natural, introduced by P. Stavre, but that independent from the metric structure, natural on T*M.

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