Parallelism of distributions and geodesics on $F(a_1, a_2, \ldots, a_n)-$structure Lagrangian manifolds

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Abstract. In this paper, authors have shown that if an almost product structure P on the tangent space of a $2n$ – dimensional Lagrangian manifold E is defined and the $F(a_1, a_2, \ldots, a_n)$ – structure on the vertical tangent space $T_v(E)$ is given, then it is possible to define the similar structure on the horizontal subspace $T_H(E)$ and also on the tangent space $T(E)$ of E. Linear connections on the Lagrangian $F(a_1, a_2, \ldots a_n)$ – structure manifold E are also discussed. Certain other interesting results like geodesics in E are also studied.

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1 Introduction

Let M be an n – dimensional and E be a $2n$ – dimensional differentiable manifold and let $\eta = (E, \pi, M)$ be the vector bundle with $\pi(E) = M$. Suppose U is a coordinate neighborhood in M with local coordinates $(x^1, x^2, \ldots x^n)$. The induced coordinates in $\pi^{-1}(U)$ are (x^i, y^α), $1 \leq i \leq n$, $1 \leq \alpha \leq n$ [6]. The canonical basis for tangent space $T_u(E)$ at $u \in \pi^{-1}(U)$ is $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha} \right\}$ or simply $\{\partial_i, \partial_\alpha\}$ where $\partial_i = \frac{\partial}{\partial x^i}$ etc. If (x^h, y^{α^1}) be coordinates of a point in the intersecting region $\pi^{-1}(U) \cap \pi^{-1}(U\prime)$, we can write

(1.1.1a) $$x^{i^1} = x^{i^1}(x^i)$$

(1.1.1b) $$y^{\alpha^1} = \frac{\partial x^{\alpha^1}}{\partial x^\alpha} y^\alpha$$

It is easy to prove for another canonical basis in the intersecting region.

(1.1.2a) $$\partial_{i^1} = \frac{\partial x^i}{\partial x^{i^1}} \partial_i$$

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$$\partial_{\alpha^1} = \frac{\partial y^\alpha}{\partial y^{\alpha^1}} \partial_\alpha \tag{1.1.2b}$$

We denote by $T(E)$ the tangent space of E spanned by $\{\partial_i, \partial_\alpha\}$ and its subspaces by $T_V(E)$ and $T_H(E)$ spanned by $\{\partial_\alpha\}$ and $\{\partial_i\}$ respectively. Obviously

$$T(E) = T_V(E) \oplus T_H(E) \tag{1.1.3}$$

and

$$dimT_V(E) = dimT_H(E) = n$$

Let us suppose that the Riemannian material structure on $T(E)$ is given by [4]

$$G = g_{ij}(x^i, y^\alpha) dx^i \otimes dx^j + g_{ab}(x^i, y^\alpha) \quad \delta y^a \otimes \delta y^b \tag{1.1.4}$$

where

$$g_{ij}(x^i, y^\alpha) = g_{ij}(x^i) \text{ and } g_{ab} = \frac{1}{2} \quad \partial a \partial b L(x^i, y^\alpha)$$

where $L(x^i, y^\alpha)$ the Lagrange function. We call such a manifold as Lagrangian manifold [4].

If $X \in T(E)$, we can write

$$X = \overline{X}^i \partial_i + X^\alpha \partial_\alpha \tag{1.1.5}$$

The automorphism $P : \chi(T(E)) \to \chi(T(E))$
defined by

$$PX = \overline{X}^i \partial_i + X^\alpha \partial_\alpha \tag{1.1.6}$$

is a natural almost product structure on $T(E)$ i.e. $P^2 = I$, I unit tensor field. If υ and h are the projection morphisms of $T(E)$ onto $T_V(E)$ and $T_H(E)$ respectively, then

$$P_0 h = \upsilon_0 P \tag{1.1.7}$$

2 The $F(a_1, a_2, \ldots a_n)$ – structure

If on the vertical space $T_V(E)$, there exists a non–null tensor field F_υ of type $(1,1)$ satisfying

$$a_n F_\upsilon^n + a_{n-1} F_\upsilon^{n-1} \ldots + a_2 F_\upsilon^2 + a_1 F_\upsilon = 0 \tag{2.2.1}$$

where $a_1, a_2 \ldots a_n$ are real or complex constants, we say that $T_V(E)$admits $F(a_1, a_2 \ldots a_n)$–structure [3]. In this case rank $(F_\upsilon) = r$ which is constant every where. Let us call F_υ as Lagrange vertical structure on $T_V(E)$

Theorem 2.1 *If Lagrange vertical structure F_υ is defined on the vertical space $T_V(E)$, it is possible to define similar structure on the horizontal subspace $T_H(E)$ with the help of the almost product structure of $T(E)$.*

Proof. [*Proof*] Let us put

$$F_h = PF_\upsilon P \tag{2.2.2}$$

then F_h is a tensor field of type $(1,1)$ on $T_H(E)$.
Also

$$F_h^2 = (PF_\upsilon P)(PF_\upsilon P) = PF_\upsilon^2 P$$

as P is an almost product structure on $T(E)$.
Similarly $F_h^3 = PF_\upsilon^3 P$ and so on. Thus, we have

$$a_nF_h^n + a_{n-1}F_h^{n-1} + \ldots + a_2F_h^2 + a_1F_h$$

$$= P(a_nF_\upsilon^n + a_{n-1}F_\upsilon^{n-1} + \ldots + a_2F_\upsilon^2 + a_1F_\upsilon)P \tag{2.2.3}$$

$$= 0$$

by virtue of (2.2.1).
Thus, F_h gives $F(a_1, a_2, \ldots, a_n)$–structure on $T_H(E)$. □

Theorem 2.2 *If Lagrange vertical $F(a_1, a_2, \ldots, a_n)$ – structure F_υ of rank r be defined on $T_V(E)$, the similar type of structure can be defined on the enveloping space $T(E)$ with the help of projection morphism of $T(E)$.*

Proof. [*Proof*] Since Lagrange structure F_υ is defined on $T_V(E)$, the Lagrange horizontal structure F_h is induced on $T_H(E)$ by theorem (2.1). If υ and h are projection morphisms of $T_V(E)$ and $T_H(E)$ on $T(E)$, let us put

$$F = F_hh + F_\upsilon\upsilon \tag{2.2.4}$$

As $h\upsilon = \upsilon h = 0$ and $h^2 = h$, $\upsilon^2 = \upsilon$, we have

$$F^2 = F_h^2h + F_\upsilon^2\upsilon$$

Similarly $F^3 = F_h^3h + F_\upsilon^3\upsilon$ and so on.
Thus

$$\begin{aligned} & a_nF^n + a_{n-1}F^{n-1} + \ldots + a_2F^2 + a_1F \\ = \; & (a_nF_h^n + a_{n-1}F^{n-1} + \ldots + a_2F_h^2 + a_1F_h)h \\ & +(a_nF_\upsilon^n + a_{n-1}F_\upsilon^{n-1} + \ldots + a_2F_\upsilon^2 + a_1F_\upsilon)\upsilon \\ = \; & 0 \end{aligned}$$

by (2.2.1) and (2.2.3).
Hence

$$a_nF^n + a_{n-1}F^{n-1} + \ldots + a_2F^2 + a_1F = 0.$$

Since $rank(F_v) = rank(F_h) = r$, hence
$rank(F) = 2r$.
On $T(E)$ with $F(a_1, a_2, \ldots, a_n)$–structure of rank $2r$, let us define operators

$$\ell = -\frac{a_nF^{n-1} + a_{n-1}F^{n-2} + ... + a_2F}{a_1} \tag{2.2.5}$$

and

$$m = I + \frac{a_nF^{n-1} + a_{n-1}F^{n-2} + ... + a_2F}{a_1}$$

Then it is easy to show that
$\ell^2 = \ell$, $m^2 = m$, $\ell + m = I$, $\ell m = m\ell = 0$.
Hence the operators 'ℓ' and 'm' when applied to the tangent space are complementary projection operators [5]. □

3 Parallelism of distributions

Let E be $2n$–dimensional Lagrangian manifold. For $F(a_1, a_2 \ldots a_n)$–structure on $T(E)$, let L and M be the complementary distributions corresponding to complementary projection operators 'ℓ' and 'm'. Let $\bar{\nabla}$ and $\tilde{\nabla}$ be defined as follows

$$\bar{\nabla}_XY = \ell\nabla_X(\ell Y) + m(\nabla_X mY) \tag{3.3.1}$$

and

$$\tilde{\nabla}_XY = \ell\nabla_{\ell X}(\ell Y) + m\nabla_{mX}(mY) + \ell[mX, \ell Y] + m[\ell X, mY] \tag{3.3.2}$$

It can be shown easily that $\bar{\nabla}$ and $\tilde{\nabla}$ are linear connections on E.

Definition 3.1 *The distribution L is called ∇–parallel if for all $X \in L$, $Y \in T(E)$ the vector field $\nabla_YX \in L$.*

Definition 3.2 *The distribution L will be said ∇–half parallel if for all $X \in L$,$Y \in T(E)$,$(\Delta F)(X, Y) \in L$ where*

$$(\Delta F)(X, Y) = F\nabla_XY - F\nabla_YX - \nabla_{FX}Y + \nabla_Y(FX) \tag{3.3.3}$$

Definition 3.3 *We call the distribution L as ∇–anti half parallel if for all $X \in L$, $Y \in T(E)$, $(\Delta F)(X, Y) \in M$.*

We now prove the following theorems.

Theorem 3.4 *On the $F(a_1, a_2 \ldots a_n)$–structure manifold, the distributions L and M are $\bar{\nabla}$ as well as $\tilde{\nabla}$ parallel.*

Proof. [*Proof*] Since $\ell m = m\ell = 0$, hence from (3.3.1) and (3.3.2), we have

$$m\bar{\nabla}_X Y = m\nabla_X(mY)$$

If $Y \in L$, $mY = 0$ so $m\bar{\nabla}_X Y = 0$ Therefore
$\bar{\nabla}_X Y \in L$. Hence for $Y \in L$, $X \in T(E)$
$\Rightarrow \bar{\nabla}_X Y \in L$. So L is $\bar{\nabla}$–parallel.
Similarly for $X \in T(E)$, $Y \in L$

$$\tilde{\nabla}_X Y = m\nabla_{mx} mY + m[\ell X, mY] = 0 \quad \text{as} \quad mY = 0.$$

So $\tilde{\nabla}_X Y \in L$. Hence L is $\tilde{\nabla}$–parallel.
In a similar manner, $\bar{\nabla}$ and $\tilde{\nabla}$ parallelism of M can also be proved. □

Theorem 3.5 *On the $F(a_1, a_2 \ldots a_n)$–structure manifold, the distributions L and M are ∇–parallel if and only if ∇ and $\bar{\nabla}$ are equal.*

Proof. [*Proof*] If L, M are ∇–parallel then $\forall X, Y \in T(E) m\nabla_X(\ell Y) = 0$ and $\ell\nabla_X(mY) = 0$.

Therefore, since $\ell + m = I$,

$$\nabla_X(\ell Y) = \ell\nabla_X(\ell Y) \text{ and } \nabla_X mY = m\nabla_X(mY)$$

So

$$\nabla_X Y = \ell\nabla_X(\ell Y) + m\nabla_X(mY) = \bar{\nabla}_X Y$$

Hence $\nabla = \bar{\nabla}$.

The converse of the theorem can be proved easily. □

Theorem 3.6 *On the $F(a_1, a_2 \ldots a_n)$–structure manifold, E, the distribution M is $\bar{\nabla}$ –anti half parallel if for all $X \in M$, $Y \in T(E)$*

$$m\bar{\nabla}_Y(FX) = m\nabla_{FX} mY.$$

Proof. [*Proof*] Since $Fm = mF = 0$, hence in view of the equation (3.3.3) for connection $\bar{\nabla}$

$$m(\Delta F)(X, Y) = m\bar{\nabla}_Y FX - m\bar{\nabla}_{FX} Y \tag{3.3.4}$$

In view of the equation (3.3.1), we have

$$\bar{\nabla}_{FX} Y = \ell\nabla_{FX}(\ell Y) + m\nabla_{FX}(mY)$$
$$m\bar{\nabla}_{FX} Y = m\nabla_{FX}(mY) \quad \text{as} \quad \ell m = 0, m^2 = m$$
$$m(\Delta F)(X, Y) = m\bar{\nabla}_Y(FX) - m\nabla_{FX}(mY)$$

As $(\Delta F)(X, Y) \in L$ so $m(\Delta F)(X, Y) = 0$. Thus

$$m\bar{\nabla}_Y(FX) = m\nabla_{FX}(mY),$$

which proves the proposition. □

4 Geodesics on the Lagrangian manifold

Let γ be a curve in E with tangent T. Then γ is called geodesic with respect to connection ∇ if $\nabla_T T = 0$.

Theorem 4.1 *A curve γ will be geodesic with respect to connection $\bar{\nabla}$ if the vector fields*

$$\nabla_T T - \nabla_T(mT) \in M \quad \text{and} \quad \nabla_T(mT) \in L.$$

Proof. [*Proof*] Since γ is geodesic with respect to connection $\bar{\nabla}$, hence $\bar{\nabla}_T T = 0$. On making use of the equation (3.3.1), the above equation assumes the following form

$$\ell\nabla_T(\ell T) + m\nabla_T(mT) = 0.$$

Since $\ell + m = I$, we can write the above equation as

$$\ell\nabla_T(I - m)T + m\nabla_T(mT) = 0$$

or

$$\ell\nabla_T T - \ell\nabla_T(mT) + m\nabla_T(mT) = 0.$$

Therefore

$$\ell(\nabla_T T - \nabla_T(mT)) = 0$$

and $m\nabla_T(mT) = 0$. Hence $\nabla_T T - \nabla_T(mT) \in M$ and $\nabla_T(mT) \in L$, which proves the proposition. □

Theorem 4.2 *The $(1,1)$ tensor field 'ℓ' and 'm' are always covariantly constants with respect to connection $\bar{\nabla}$.*

Proof. [*Proof*] $\forall X, Y \in T(E)$, we have

$$(\bar{\nabla}_X \ell)(Y) = \bar{\nabla}_X(\ell Y) - \ell\bar{\nabla}_X Y. \tag{4.4.1}$$

Making use of (3.3.1), we get

$$(\bar{\nabla}_X \ell)(Y) = \ell\nabla_X(\ell^2 Y) + m\nabla_X(m\ell Y) - \ell\{\ell\nabla_X \ell Y + m\nabla_X mY\}$$

Since $\ell^2 = \ell$, $m^2 = m$, $\ell m = m\ell = 0$, we get

$$(\bar{\nabla}_X \ell)(Y) = \ell\nabla_X(\ell Y) - \ell\nabla_X \ell Y = 0.$$

So, 'ℓ' is covariantly constant. The fact that 'm' is covariantly constant can be proved analogously. □

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