A generalization of a growth model with endogenous fertility

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Abstract. In this paper, we study the equilibrium dynamics of an endogenous growth model with endogenous fertility choice. In this setting, we show that there is a unique equilibrium which is globally determinated. The validity of the neo-Malthusian relation between fertility and growth is then examined.

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1 Introduction

This paper focuses on the equilibrium dynamics of an endogenous growth model with physical capital in which fertility enters the utility function. Our model is an extension of the work of Yip and Zhang [11], who carried out a study to provide a reconciliation for the conflicting findings in the literature on the relationship between population growth and economic growth. Some works in fact suggested that high fertility suppressed per capita income growth, others instead expressed ambivalence about this neo-Malthusian relationship (see, e.g., Blanchet [3], Coale and Hoover [4], Kelley [6], Simon [9], Srinivasan [10]). In our endogenous growth framework, we follow the existing literature by allowing the number of children to enter the utility function (see, e.g., Barro and Becker [2], Palivos [7], Yip and Zhang [12]). The presence of fertility implies that the budget constraint is no longer convex, and so an extra condition has to be imposed to guarantee the sufficiency of the necessary conditions for an optimum. By assuming that the marginal product of capital minus the population growth rate is monotonically decreasing with respect to capital, and the elasticity of the marginal productivity of labor is less than one, we find that there is only one balanced growth path (BGP) equilibrium. Moreover, we examine the relationship between fertility growth and economic growth along a BGP equilibrium, and see that there exists a negative relationship between them only when all exogenous factors are controlled for. This suggests that the conflicting findings in the literature between population growth and economic growth may be originate from heterogeneity in unobserved variables across countries and over time in cross-country panel data sets.

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2 The model of Yip and Zhang and its generalization

We consider an economy populated by a continuum of identical infinitely lived agents who divide their available time between labor l and child-rearing $\phi(n)$, where n denotes the fertility or population growth rate and ϕ is a C^2 strictly increasing function with $\phi(0) = 0$. Absence of immigration and mortality implies that there is a one-to-one correspondence between the rate of population growth and the fertility rate. Note that time index has been dropped for notation simplicity. Normalizing the time endowment to unity yields the following time constraint

$$\phi(n) + l = 1. \tag{2.2.1}$$

Each agent of the economy derives utility from consumption c and fertility. We assume preferences to be characterized by a C^2 utility function of the form

$$u(c, n) = \ln(c) + v(n), \tag{2.2.2}$$

with v a function strictly increasing and strictly concave, with the property that $\lim_{n\to 0} v'(n) = +\infty$, and $\lim_{n\to+\infty} v'(n) = 0$. Observe that if

$$v(n) = (n^{1-\varepsilon} - 1)/(1 - \varepsilon),$$

with $\varepsilon > 0$ the elasticity of the marginal utility of fertility, then (2.2.2) reduces to the utility function postulated in Yip and Zhang [11]. The technology used by the agents is represented by a C^2 production function $f(k, \bar{k}l)$, where k is the individual agent's stock of capital, and $\bar{k}$ is the average economy-wide level of capital stock. $\bar{k}$ yields an externality such that in equilibrium, when $k = \bar{k}$, the production function is linear in accumulating stock of capital, as in Romer [8]. The production function f is assumed to be strictly increasing, strictly concave, linearly homogeneous in capital and fertility, and satisfying the Inada conditions. Thus, each input is essential for production. In addition, Euler's theorem on homogeneous functions implies that $kf_k(k, \bar{k}l) + lf_l(k, \bar{k}l) = f(k, \bar{k}l)$, and, consequently, that $f_{kl}(k, \bar{k}l) > 0$. For any given value of k, we assume that the opportunity cost of children in terms of output, $\phi'(n)f_l(k, \bar{k}(1 - \phi(n))) + k$, is strictly increasing in n, i.e.

$$\phi''(n)f_l(k, \bar{k}(1 - \phi(n))) - \bar{k}(\phi'(n))^2 f_{ll}(k, \bar{k}(1 - \phi(n))) > 0, \tag{2.2.3}$$

and that the elasticity of the marginal productivity of labor evaluated at $(1, l)$ is less than one, i.e.

$$\varepsilon_l \equiv -lf_{ll}(1, l)/f_l(1, l) < 1. \tag{2.2.4}$$

Condition (2.2.3) is automatically fulfilled if there is a constant or increasing marginal cost of child-rearing, i.e. if $\phi''(n) \geq 0$. Similarly for condition (2.2.4) when f is Cobb-Douglas. The representative consumer maximizes the present value of utility

$$\int_0^\infty u(c, n)\, e^{-\rho t} dt,$$

subject to the initial condition $k(0) > 0$, the time constraint (2.2.1), and the budget constraint

$$c + \dot{k} + nk = f(k, \bar{k}l), \tag{2.2.5}$$

where $\rho > 0$ is the constant rate of time preference. To solve the consumer's problem we define the current-value Hamiltonian

$$H(c, n, k, \lambda) = u(c, n) + \lambda[f(k, \bar{k}(1 - \phi(n))) - nk - c],$$

where λ is the co-state variable associated with the constraint (2.2.5). Maximizing H, with respect to the control variables c and n, leads to the following conditions

$$H_c = 0 \quad \Rightarrow \quad c^{-1} = \lambda, \tag{2.2.6}$$

$$H_n = 0 \quad \Rightarrow \quad v'(n) = \lambda[\bar{k}\phi'(n) f_l(k, \bar{k}(1 - \phi(n))) + k]. \tag{2.2.7}$$

¿From the Pontryagin maximum principle, we also obtain the following differential equation

$$\dot{\lambda} = \rho\lambda - H_k \;\Rightarrow\; \dot{\lambda} = \rho\lambda - \lambda[f_k(k, \bar{k}(1 - \phi(n))) - n], \tag{2.2.8}$$

plus the transversality condition at infinity

$$\lim_{t\to\infty} e^{-\rho t}\lambda k = 0. \tag{2.2.9}$$

Equations (2.2.1), (2.2.6)−(2.2.9) constitute the set of necessary conditions for an optimum. They are in general not sufficient for a maximum, but they become sufficient if the maximized current-value Hamiltonian, i.e. the function H after we have substituted the control variables by (2.2.6) and (2.2.7), is concave in the state variable k (see Arrow and Kurz [1]). Solving (2.2.6) and (2.2.7) in terms of the state and co-state variables yields $c = c(\lambda)$ and $n = n(k, \lambda)$. The first of these two expressions is immediate. The second follows as an application of the Implicit function theorem to the curve

$$F(n, k, \lambda) \equiv v'(n) - \lambda\bar{k}\phi'(n) f_l(k, \bar{k}(1 - \phi(n))) - \lambda k = 0. \tag{2.2.10}$$

In view of assumption (2.2.3), for any given k and λ, we have

$$F_n = v''(n) - \lambda\bar{k}[\phi''(n) f_l(k, \bar{k}(1 - \phi(n))) - \bar{k}(\phi'(n))^2 f_{ll}(k, \bar{k}(1 - \phi(n)))] < 0.$$

Next, we define the maximized current-value Hamiltonian as

$$\begin{aligned} H^0(k, \lambda) \;&=\; \max_{c,n} H(c, n, k, \lambda) \;=\; H(c(\lambda), n(k, \lambda), k, \lambda), \\ &=\; \ln c(\lambda) + v(n(k, \lambda)) + \lambda[f(k, \bar{k}(1 - \phi(n(k, \lambda)))) - n(k, \lambda)k - c(\lambda)]. \end{aligned}$$

Assumption (2.2.3) ensures that the Hessian of the current-value Hamiltonian is negative definite with respect to c and n, which is a sufficient condition for a maximum of H. Differentiating H^0 with respect to k, and then substituting (2.2.6) and (2.2.7), we obtain

$$H_k^0(k, \lambda) = \lambda[f_k(k, \bar{k}(1 - \phi(n(k, \lambda)))) - n(k, \lambda)].$$

Hence, if the following condition holds

$$\partial[f_k(k, \bar{k}(1 - \phi(n))) - n]/\partial k < 0,$$

then $H^0(k, \lambda)$ is concave in its state variable k, for given costate variable λ.

Proposition 1 *If the marginal product of capital minus the population growth rate, i.e.* $f_k(k, \bar{k}(1 - \phi(n))) - n$*, is monotonically decreasing with respect to capital, then the necessary conditions* (2.2.1)*,* (2.2.6) *–* (2.2.9) *are also sufficient for optimality.*

3 Equilibrium analysis

Since our objective is to study the transitional dynamics along balanced growth paths implied by the model, we first recall this concept. A balanced growth path (BGP) equilibrium is a collection of functions of time $\{c, k\}$, solving the optimal control problem, such that the growth rates of c and k, $\dot{c}/c$ and $\dot{k}/k$, are constant over time, and n is constant. To facilitate the analysis of balanced growth, we will reduce the dimension of our system to a more tractable one. If we define $x = ck^{-1}$, differentiating (2.2.6) with respect to time, and then combining with (2.2.8), we get

(3.3.1) $$\dot{x}/x = x - f(1, 1 - \phi(n)) + f_k(1, 1 - \phi(n)) - \rho.$$

In the above we have used the equilibrium condition $k = \bar{k}$, as well as the homogeneity of degree zero of the first partial derivatives of f. Next, rewriting (2.2.10) in terms of n and x gives the curve

(3.3.2) $$F(n, x) \equiv v'(n) - x^{-1}[\phi'(n) f_l(1, 1 - \phi(n)) + 1] = 0.$$

As $F_n(n, x) < 0$, the Implicit function theorem allows to express n as a function of x, i.e.

(3.3.3) $$n = \psi(x),$$

and $F(\psi(x), x) = 0$. Implicit differentiation of this identity yields

(3.3.4) $$\psi'(x) = -F_x(\psi(x), x)/F_n(\psi(x), x),$$

and so $\psi'(x) > 0$, since $F_x(\psi(x), x) > 0$. In conclusion, (3.3.1) writes as

(3.3.5) $$\dot{x}/x = x - f(1, 1 - \phi(\psi(x)) + f_k(1, 1 - \phi(\psi(x))) - \rho \equiv h(x).$$

Furthermore, we rewrite equation (2.2.9) as

(3.3.6) $$\lim_{t \to \infty} e^{-\rho t} x^{-1} = 0.$$

Equation (3.3.5), with the initial condition $k(0) > 0$ and the transversality condition (3.3.6), forms a first-order nonlinear system which describes the global dynamics of the economy. All solutions $x(t)$ of (3.3.5) converge to a steady state equilibrium point x^* such that

$$h(x^*) = 0.$$

Any such solution will satisfy the transversality condition (3.3.6), since $k(t)$ and $c(t)$ will eventually grow at the same constant rate, implying that $x(t) = c(t)k^{-1}(t)$ will eventually assume a constant value. It is clear that each steady state equilibrium point of (3.3.5) corresponds to a BGP equilibrium along which $\dot{x}/x = 0$. Note that we confine our analysis to interior BGPs only. We do this because $x = 0$ would imply that the level of consumption is zero, which does not make sense from the economic point of view. Next, using Euler's theorem, we find that the derivative of $h(x)$ is given by

$$h'(x) = 1 + \phi'(\psi(x))\,\psi'(x)f_l(1, 1 - \phi(\psi(x)))(1 - \varepsilon_l).$$

Hence, $h'(x) > 0$ in view of assumption (2.2.4). Moreover, equation (3.3.2) and the behavior of v' at the boundary imply that $\lim_{x\to 0} h(x) < 0$, $\lim_{x\to+\infty} h(x) = +\infty$. Thus, by continuity of $h(x)$, there exists a unique point x^* such that $h(x^*) = 0$, and hence a unique BGP such that $x(t) = x^*$ along this path. Since $h'(x^*) > 0$, we have that $\dot{x}(t) < 0$ for $x(t) < x^*$ and $\dot{x}(t) > 0$ for $x(t) > x^*$, i.e. x^* is unstable in the sense that any initial value $x(0) \neq x^*$ generates a trajectory that monotonically diverges from x^*. In other words, there exists only one initial value, $x(0) = x^*$, such that a trajectory that starts from $x(0)$ converges to x^*. Hence the unique BGP is determinate.

Proposition 2 *The model has a unique but unstable BGP equilibrium.*

4 Fertility and growth

In this section, we investigate the relationship between population growth and economic growth at the BGP equilibrium. Hereafter an asterisk denotes the BGP equilibrium value of a variable. From (3.3.2), the definition of the BGP equilibrium implies that x^* must be constant, so that c^* and k^* are growing at the same constant rate g^*. Combining (2.2.6) and (2.2.8), we find that

$$g^* = f_k(1, 1 - \phi(n^*)) - n^* - \rho, \tag{4.4.1}$$

and so differentiating (4.4.1) with respect to n^* yields

$$g^*_{n^*} = -[\phi'(n^*)f_{kl}(1, 1 - \phi(n^*)) + 1] < 0.$$

Proposition 3 *There exists a neo-Malthusian* (*inverse*) *relationship between population growth and economic growth when all exogenous factors are controlled for.*

We know examine the case when some of the exogenous factors are allowed to vary. For simplicity, we assume the production function f to be Cobb-Douglas, i.e. $f(k, \bar{k}l) =$

$Ak^{\alpha}l^{1-\alpha}\bar{k}^{1-\alpha}$, where $A > 0$ is a scale parameter and $0 < \alpha < 1$. Along the BGP equilibrium, the budget constraint (2.2.5) becomes

$$g^* = A(1-\phi(n^*))^{1-\alpha} - n^* - \psi^{-1}(n^*), \tag{4.4.2}$$

where ψ^{-1} denotes the inverse function of the map (3.3.3). It exists since ψ is a monotone function. We suppose that there is an improvement in technological progress A. Direct differentiation of (4.4.1) and (4.4.2) with respect to A gives respectively

$$g_A^* = \alpha(1-\phi(n^*))^{1-\alpha} - \left[A\alpha(1-\alpha)(1-\phi(n^*))^{-\alpha}\phi'(n^*) + 1\right] n_A^*, \tag{4.4.3}$$

and

$$g_A^* = (1-\phi(n^*))^{1-\alpha} - \left[A(1-\alpha)(1-\phi(n^*))^{-\alpha}\phi'(n^*) + 1 + 1/\psi'(x^*)\right] n_A^*. \tag{4.4.4}$$

Hence, combining (4.4.3) and (4.4.4), we get

$$n_A^* = (1-\alpha)(1-\phi(n^*))^{1-\alpha}/M(A, n^*, x^*) > 0, \tag{4.4.5}$$

where $M(A, n^*, x^*) = A(1-\alpha)^2(1-\phi(n^*))^{-\alpha}\phi'(n^*) + 1/\psi'(x^*)$. Replacing (4.4.5) in (4.4.3) yields

$$g_A^* = -(1-\phi(n^*))^{1-\alpha}[1-\alpha-\alpha/\psi'(x^*)]/M(A, n^*, x^*).$$

The ambiguity in sign of g_A^* implies that the relationship between n^* and g^* is generally indeterminate. Thus, given other exogenous variables that are not controlled for, the neo-Malthusian relationship between population growth and economic growth does not naturally emerge.

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