Relative topological dynamics

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Abstract. In this paper the notion of topological dynamics from the viewpoint of an observer is considered. Quasi-minimal-relative topological dynamics which is an approach to the notion of minimality for the non-invariant state spaces is studied. An extension of the notion of transitivity from the point of view of an observer is considered. Non-wandering sets for a relative topological dynamics is studied.

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1 Introduction

In the consideration of the nature an observer can modelize by an operator which evaluates each proposition by a number in the closed interval $[0,1]$ [1, 3]. We assume that X is the state space, i.e. "propositions" are members of X and μ as a function from X to $[0,1]$ is an observer of X. Moreover we assume that τ_μ is a relative topology, it means that: it is a collection of members of $[0,1]^X$, which are subsets of μ with the following properties [2]:
i) μ, $\chi_\phi \in \tau_\mu$, where χ is the characteristic function;
ii) If λ_1, $\lambda_2 \in \tau_\mu$, then $\lambda_1 \cap \lambda_2 \in \tau_\mu$;
iii) If $\{\lambda_i : i \in \Gamma\} \subset \tau_\mu$, then $\bigcup_{i\in\Gamma} \lambda_i \in \tau_\mu$.
τ_μ is a mathematical modelling of the notion of topology from the viewpoint of an observer μ.
A mapping $f : X \to X$ is called a relative continuous map or a $\mu-$continuous map if $f^{-1}(\eta) \cap \mu \in \tau_\mu$ for all $\eta \in \tau_\mu$, where $f^{-1}(\eta)(x) = \eta(f(x))$.
f is called a $\mu-$homeomorphism or a relative homeomorphism if f is a bijection and f and f^{-1} are $\mu-$continuous maps.

Example 1.1 Let X be the set of C^1 diffeomorphisms of $\mathbb{R}^n$, and let the observer μ defined by $\mu(g) = \begin{cases} 1 & if\ 0\ is\ a\ hyperpolic\ fixed\ point\ of\ g \\ 0 & otherwise \end{cases}$. We denote the stable manifold of the hyperbolic fixed point 0 of a diffeomorphism g by $W_g^s(0)$. For $m \in \{1,2,3,...,n\}$ we define λ^m by
$\lambda^m(g) = \begin{cases} \frac{1}{m} & if\ 0\ is\ a\ hyperbolic\ fixed\ point\ of\ g\ and\ dim(W_g^s)(0) = m \\ 0 & otherwise \end{cases}$. Let

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τ_μ be the relative topology generated by $\{\mu,\ \lambda^m,\ \chi_{\varnothing}\ :\ m \in \{1,2,3,...,n\}\}$, and $\psi : X \to X$ be defined by $\psi(g) = Dg(0)$. Then ψ is a $\mu-$continuous map.

We now would like to have a description of topological dynamics from the viewpoint of the observer μ. For this purpose we assume that T is the set of real numbers or the set of integer numbers, and (X, τ_μ) is a relative topological space. Moreover we assume that $\{\varphi^t : t \in T\}$ is a one parameter family of $\mu-$homeomorphisms from X to X, then we have the following definition.

Definition 1.1 With the above assumptions a triple $(X, \mu, \{\varphi^t : t \in T\})$ is called a relative topological dynamics if:
(i) φ^0 is the identity map on X;
(ii) $\varphi^s o \varphi^t = \varphi^{s+t}$ for all $s, t \in T$.

Example 1.2 Let X be the set of mappings x from $\mathbb{Z}$ to $\{0,1\}$, and $\mu : X \to [0,1]$ be defined by $\mu(x) = \begin{cases} 0 & if\ x(i) = 0\ for\ all\ i \\ \frac{1}{2} & otherwise \end{cases}$.
Moreover let T be the set of integer numbers and $\tau_\mu = \{\mu, \lambda_j, \chi_{\varnothing}\ :\ j \in \mathbb{N} \cup \{0\}\}$, where
$\lambda_j(x) = \begin{cases} \frac{1}{2^{j+1}} & if\ x(j) = 1\ or\ x(-j) = 1 \\ 0 & otherwise \end{cases}$. Then for $t \in T$, the mapping $\varphi^t : X \to X$ defined by $\varphi^t(x)(i) = x(i+t)$ is a $\mu-$continuous map. Moreover $(X, \mu, \{\varphi^t : t \in T\})$ is a relative topological dynamics. For given $\alpha \in [0,1]$ and $\lambda \in \tau_\mu$ the α-cut of λ is the set $\lambda^{-1}(\alpha, 1]$ and denoted by λ_α.

Theorem 1.1 *Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics from the viewpoint of the observer μ. Moreover assume that μ_α is an invariant set for φ^t, where $\alpha \in (0,1]$, and $t \in T$. Then $(\mu_\alpha, \{\varphi^t : t \in T\})$ is a topological dynamics, where the topology of μ_α is $\tau_\mu^\alpha = \{\lambda_\alpha : \lambda \in \tau_\mu\}$.*

Proof. The straightforward calculations show that τ_μ^α is a topology for μ_α. So we only prove the $\mu-$continuity of φ^t where $t \in T$.
If $\lambda_\alpha \in \tau_\mu^\alpha$ then $\varphi^{-t}(\lambda^{-1}((\alpha,1])) = \varphi^{-t}(\{x \in \mu_\alpha : \lambda(x) > \alpha\}) = \{y : \lambda(\varphi^t(y)) > \alpha$ and $\varphi^t(y) \in \mu_\alpha\} = (\varphi^{-t}(\lambda) \cap \mu)((\alpha, 1])$. So $\varphi^{-t}(\lambda^{-1}((\alpha,1])) \in \tau_\mu^\alpha$. □

2 Quasi-minimal relative topological dynamics

If $(X, \mu, \{\varphi^t : t \in T\})$ is a relative topological dynamics and $x \in X$ then the set $O(x) = \{\varphi^t(x) : t \in T\}$ is called the orbit of $x \in X$. If $\alpha \in [0,1]$ then $O_\mu^\alpha(x) = O(x) \cap \mu^{-1}((\alpha, 1])$ is the relative orbit of x or $\mu-$orbit of x with the level α.

Definition 2.1 A relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$ is called a quasi-minimal relative topological dynamics on μ_α or briefly "$\mu_\alpha-$quasi-minimal " if for all $x \in \mu_\alpha$,$O_\mu^\alpha(x)$ is a dense subset of μ_α, where the topology of μ_α is τ_μ^α.

It is important to mention this point that: in the above definition it is not necessary μ_α be an invariant set for φ^t.

Example 2.1 Let $\mathbb{R}$ be the set of all real numbers, also let $\mu : \mathbb{R} \longrightarrow [0,1]$ be defined by:

$$\mu(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, let τ_μ be the $\mu-$topology generated by the set $\{\lambda_i : i \in \mathbb{Z}\}$, where $\lambda_i : \mathbb{R} \longrightarrow [0,1]$ be defined by:

$$\lambda_i(x) = \begin{cases} 1/2 & \text{if } x \in (i, i+1) \\ 0 & \text{otherwise.} \end{cases}$$

Let $T = \mathbb{Z}$, $\varphi^t : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\varphi^t(x) = x + t$ where $t \in \mathbb{Z}$. Then $(\mathbb{R}, \mu, \{\varphi^t : t \in \mathbb{Z}\})$ is $\mu_\alpha-$quasi-minimal, where $\alpha \in (0,1)$.

Let $x \in \mu_\alpha$ and $\lambda_j^\alpha \in \tau_\mu^\alpha$, where $\alpha \in (0,1]$ and $j \in \mathbb{Z}$. Then there exists $t \in \mathbb{Z}$ such that $\varphi^t(x) \in \lambda_j^\alpha$. Because $x \in \mu_\alpha$ implies $x \notin \mathbb{Z}$. If $x \in (i, i+1)$ for some $i \in \mathbb{Z}$ then for $t = j - i$, $i + j - i < \varphi^t(x) = x + t < i + 1 + j - i$. So $\varphi^t(x) \in (j, j+1)$.

Theorem 2.1 *Let $\alpha \in (0,1]$ and $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics, and let $(X, \mu, \{\varphi^t : t \in T\})$ be $\mu_\alpha-$quasi-minimal. Then the only closed subsets E of μ_α with $\varphi^t(E) = E$ are $\varnothing$ and μ_α where $t \in T$.*

Proof. Suppose $(X, \mu, \{\varphi^t : t \in T\})$ is $\mu_\alpha-$quasi-minimal, and $\varnothing \neq E$ is a closed subset of μ_α such that $\varphi^t(E) = E$. If $x \in E$ then $O_\mu^\alpha(x) \subset E$. So $\mu_\alpha = \overline{O_\mu^\alpha(x)} \subset E$. Hence $\mu_\alpha = E$. □

Theorem 2.2 *If $(X, \mu, \{\varphi^t : t \in T\})$ is $\mu_\alpha-$quasi-minimal, such that $\varphi^t(\mu_\alpha) \subset \mu_\alpha$ for all $t \in T$, then the following statements are equivalent.*
(i) The only closed subsets E of μ_α with $\varphi^t(E) = E$ are $\varnothing$ and μ_α where $t \in T$.
(ii) If $\varnothing \neq U \in \tau_\mu^\alpha$, then $\cup_{t \in T} (\varphi^t(U) \cap \mu_\alpha) = \mu_\alpha$.

Proof. $(i) \Longrightarrow (ii)$. If $\varnothing \neq U \in \tau_\mu^\alpha$, then there exists $\lambda \in \tau_\mu$ such that $U = \lambda_\alpha$. For all $t \in T$, $\varphi^{-t}(U) \cap \mu_\alpha = (\varphi^{-t}(\lambda) \cap \mu)_\alpha$. So $E = \mu_\alpha \setminus \cup_{t \in T}(\varphi^t(U) \cap \mu_\alpha)$ is a closed subset of μ_α. Moreover $\varphi^t(E) = E$. Since $E \neq \mu_\alpha$ we have $E = \varnothing$.
$(ii) \Longrightarrow (i)$. Let $x \in \mu_\alpha$ and let U be any non-empty open subset of μ_α. By (ii) $x \in \varphi^t(U) \cap \mu_\alpha$ for some $t \in T$. So $\varphi^{-t}(x) \in U$ and $O_\mu^\alpha(x)$ is dense in μ_α. Thus Theorem 2.1 implies to (i). □

Theorem 2.3 *Let $(X, \mu, \{\varphi^t : t \in T\})$ be $\mu_\alpha-$quasi-minimal and let $f : \mu_\alpha \longrightarrow \mathbb{R}$ be a $\mu-$continuous function such that, $fo\varphi^t = f$ for all $t \in T$. Then f is a constant function.*

Proof. Since $fo\varphi^t = f$ for all $t \in T$, then for $x \in \mu_\alpha$ we have $f(O_\mu^\alpha(x)) = \{f(x)\}$. Moreover $O_\mu^\alpha(x)$ is a dense subset of μ_α and f is $\mu-$continuous. So

$$f(\mu_\alpha) = f(\overline{O_\mu^\alpha(x)}) = \overline{f(O_\mu^\alpha(x))} = \{f(x)\}. \ \square$$

Example 2.2 Let $X = \mathbb{R}/\mathbb{Z}$ and let $\varphi^t : X \longrightarrow X$ be defined by $\varphi^t(x) = x + t\beta \pmod 1$ for $t \in \mathbb{Z}$. Moreover let

$$\mu(x) = \begin{cases} 0 & \text{if } x = 1/2, \\ 1 & \text{otherwise.} \end{cases}$$

and τ_μ be the relative topology generated by $\{\varnothing, \mu, \lambda_j : j \in \mathbb{N} \cup \{0\}\}$ where

$$\lambda_0(x) = \begin{cases} 1/2 & \text{if } x \in (1/2, 1), \\ 0 & \text{otherwise.} \end{cases}$$

and for $j \in \mathbb{N}$

$$\lambda_j(x) = \begin{cases} 1/2 & \text{if } x \in [1/2^{j+1}, 1/2^j), \\ 0 & \text{otherwise.} \end{cases}$$

If β is irrational, then we show φ^t has a dense orbit $O^\alpha_\mu(0)$ in μ_α where $\alpha \in (0, 1]$. For $t \in \mathbb{Z}$ since $\{\varphi^{nt}(0) : n \in \mathbb{N}\}$ is an infinite set in $\mathbb{R}/\mathbb{Z}$ then for $x \in \mu_\alpha$ there is a sequence $t_n \to +\infty$ with $\varphi^{t_n}(0) = t_n\beta \ (mod\, 1) \to x$, where $t_n = nt \in \mathbb{Z}$. For all $\epsilon > 0$ we can find $t_{n_i} > t_{n_j}$ with $| \varphi^{t_{n_i}}(0) - x | < \epsilon/2$ and $| \varphi^{t_{n_j}}(0) - x | < \epsilon/2$. So $| \varphi^{t_{n_i}}(0) - \varphi^{t_{n_j}}(0) | = |t_{n_i}\beta - t_{n_j}\beta| = |\varphi^{t_{n_i} - t_{n_j}}(0)| < \epsilon$. Moreover, $\varphi^{t_{n_i}}(0) \neq \varphi^{t_{n_j}}(0)$. Because if $\varphi^{t_{n_i}}(0) = \varphi^{t_{n_j}}(0)$ then $t_{n_i}\beta = t_{n_j}\beta$ and this is contradiction with irrationality of, β. Hence $\varphi^{k(t_{n_i} - t_{n_j})}(0)$, $k \in \mathbb{Z}$ form an $\epsilon-$dense subset of μ_α. We show if β is irrational then $(X, \mu, \{\varphi^t : t \in \mathbb{Z}\})$ is $\mu_\alpha-$quasi-minimal where $\alpha \in (0, 1]$. Let $x \in \mu_\alpha$ and $(y - \epsilon, y + \epsilon)$ be a neighborhood of $y \in \mu_\alpha$, where $\epsilon > 0$. Since $\{\varphi^t : t \in \mathbb{Z}\}$ has a dense set in μ_α i.e. $\overline{O^\alpha_\mu(0)} = \mu_\alpha$ then there is a sub-sequence $t_{n_i} \in \mathbb{Z}$ with $\varphi^{t_{n_i}}(0) \to y - x$ as $i \to \infty$. So $\varphi^{t_{n_i}}(x) = x + t_{n_i}\beta \ \ (mod\ 1) \ \ = \varphi^{t_{n_i}}(0) + x \ \ (mod\ 1) \ \ \to y \ \ (mod\ 1)$.

3 The non-wandering sets from the observer viewpoint

In this section we assume that the topology generated by τ^α_μ on μ_α is a metrizable topology. In this case we say that τ^α_μ is a topology on the metric space X, and we have the next definition.

Definition 3.1 Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics on a metric space X and let $x \in \mu_\alpha$. For given $\alpha \in (0, 1]$ we define
$w_{\mu_\alpha}(x) = \{y \in \mu_\alpha |\ \exists \{t_n\} \subset\ T \textit{ such that } \varphi^{t_n}(x) \to y \textit{ when } t_n \to +\infty\}$.

Theorem 3.1 *Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics of a compact metric space X, and let μ_α be compact for an $\alpha \in (0, 1]$ and $\varphi^t(\mu_\alpha) = \mu_\alpha$. Then for every $x \in \mu_\alpha$.*
(i) $w_{\mu_\alpha}(x) \neq \varnothing$
(ii) $w_{\mu_\alpha}(x)$ is a closed subset of μ_α with respect to τ^α_μ.

Proof. (*i*) We choose a sequence $\{t_n\}$ such that $t_n \to +\infty$ and $y_n = \varphi^{t_n}(x) \cap \mu_\alpha$. Since μ_α is compact y_n has a convergent subsequence to $q \in \mu_\alpha$ thus $q \in w_{\mu_\alpha}(x)$. So $w_{\mu_\alpha}(x) \neq \varnothing$.
(*ii*) If $z \notin w_{\mu_\alpha}(x)$, then it has a neighborhood $N(z)$ such that $N(z) \cap \{\varphi^t(x) : t \geq M\} \cap \mu_\alpha = \varnothing$ for some $M > 0$. This implies that the points of $N(z)$ do not belong to $w_{\mu_\alpha}(x)$ and so $w_{\mu_\alpha}(x)$ is closed. □

Definition 3.2 Let $(X, \mu, \{\varphi^t : t \in T\})$ be a relative topological dynamics and $\alpha \in (0, 1]$. A point $p \in \mu_\alpha$ is called a $\mu_\alpha-$non-wandering if for each neighborhood U

of p in τ_μ^α, there exists $0 \neq t \in T$ such that $\varphi^t(U) \cap U \cap \mu_\alpha \neq \varnothing$. We denote the set of μ_α−non-wandering points of $(X, \mu, \{\varphi^t : t \in T\})$ by Ω_{μ_α}.

Theorem 3.2 *If $(X, \mu, \{\varphi^t : t \in T\})$ is a relative topological dynamics, then for $\alpha \in (0, 1]$,*
(i) Ω_{μ_α} is closed.
(ii) $\cup_{x\in\mu_\alpha} w_{\mu_\alpha}(x) \subset \Omega_{\mu_\alpha}$ (so $\Omega_{\mu_\alpha} \neq \varnothing$).
(iii) All μ−periodic points belong to Ω_{μ_α}.
(iv) If there is a $t_0 \in T : \varphi^{t_0}(\mu_\alpha) = \mu_\alpha$, then $\varphi^{t_0}(\Omega_{\mu_\alpha}) = \Omega_{\mu_\alpha}$.

Proof. (*i*) Let $y \notin \Omega_{\mu_\alpha}$. Then it has a neighborhood $V(y) : \varphi^t(V) \cap V \cap \mu_\alpha = \varnothing$ for all $t \in T$. This implies that the points of $V(y)$ do not belong to Ω_{μ_α}, and so Ω_{μ_α} is closed.
(*ii*) Let $x \in \mu_\alpha$ and $y \in w_{\mu_\alpha}(x)$. Then there exists $\{t_n\} \subset T : \varphi^{t_n}(x) \to y$ when $t_n \to +\infty$. Suppose V is a neighborhood of y, then there is $\{t_n\} : \varphi^{t_n}(x) \in V \cap \mu_\alpha$. For $t_{n_0} < t_{n_1}$, we have $\varphi^{t_{n_0}}(x), \varphi^{t_{n_1}}(x) \in V \cap \mu_\alpha$. If $z = \varphi^{t_{n_1}}(x)$, and $t_n = t_{n_1} - t_{n_0}$ then $\varphi^{-t_n}(z) = \varphi^{-t_n}(\varphi^{t_{n_1}}(x)) = \varphi^{t_{n_0}-t_{n_1}+t_{n_1}}(x)$. So $\varphi^{-t_n}(z) = \varphi^{t_{n_0}}(x) \in V \cap \mu_\alpha$. Thus $z \in \varphi^{t_n}(V) \cap \varphi^{t_n}(\mu_\alpha)$. Moreover $z \in V \cap \mu_\alpha$ and this implies $z \in \varphi^{t_n}(V) \cap V \cap \mu_\alpha$ *i.e.* $\varphi^{t_n}(V) \cap V \cap \mu_\alpha \neq \varnothing$. Hence $y \in \Omega_{\mu_\alpha}$.
(*iii*) Let $p \in \mu_\alpha$ be a μ−periodic point of $(X, \mu, \{\varphi^t : t \in T\})$. So there exists $0 \neq t_0 \in T$ such that $\varphi^{t_0}(p) = p$. If U is a neighborhood of p then $p \in \varphi^{t_0}(U) \cap U \cap \mu_\alpha$.
(*iv*) Let $x \in \Omega_{\mu_\alpha}$ and V be a neighborhood of $\varphi^{t_0}(x)$. Then $\varphi^{-t_0}(V)$ is a neighborhood of x. Since $x, \varphi^{t_0}(x) \in \mu_\alpha$ then there is $t_1 \in T$ with $\varnothing \neq \varphi^{t_1-t_0}(V) \cap \varphi^{-t_0}(V) \cap \mu_\alpha = \varphi^{t_1-t_0}(V) \cap \varphi^{-t_0}(V) \cap \varphi^{-t_0}(\mu_\alpha)$. Thus $\varphi^{t_1}(V) \cap V \cap \mu_\alpha \neq \varnothing$. So $\varphi^{t_0}(x) \in \Omega_{\mu_\alpha}$ *i.e.* $\varphi^{t_0}(\Omega_{\mu_\alpha}) \subset \Omega_{\mu_\alpha}$.
Let $x \in \Omega_{\mu_\alpha}$ and V be a neighborhood of $\varphi^{-t_0}(x)$. Then $\varphi^{t_0}(V)$ is a neighborhood of x. Since $x, \varphi^{-t_0}(x) \in \mu_\alpha$ then there is $t_2 \in T$ with $\varnothing \neq \varphi^{t_2+t_0}(V) \cap \varphi^{t_0}(V) \cap \mu_\alpha = \varphi^{t_2+t_0}(V) \cap \varphi^{t_0}(V) \cap \varphi^{t_0}(\mu_\alpha)$. Thus $\varphi^{t_2}(V) \cap V \cap \mu_\alpha \neq \varnothing$ i.e. $\varphi^{-t_0}(x) \in \Omega_{\mu_\alpha}$. Hence $x \in \varphi^{t_0}(\Omega_{\mu_\alpha})$ *i.e.*$\Omega_{\mu_\alpha} \subset \varphi^{t_0}(\Omega_{\mu_\alpha})$. Thus $\varphi^{t_0}(\Omega_{\mu_\alpha}) = \Omega_{\mu_\alpha}$. □

4 Transitivity from observer viewpoint

We begin this section with the definition of relative transitivity.

Definition 4.1 We say that a relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive if for all $\lambda, \eta \in \tau_\mu - \{\chi_\varnothing\}$ there is $t \in T$ such that $\varphi^t(\lambda) \cap \eta \neq \chi_\varnothing$.

This definition is a natural generalization of the notion of transitivity. In fact if (X, τ) is a topological space, then $\tau_{\chi_X} = \{\chi_U : U \in \tau\}$ is a relative topology for X. In this case the notion of "relative transitivity" is equivalent to "transitivity".

Example 4.1 The topological dynamics presented in Example 1.2 is relative transitive.

Theorem 4.1 *If τ_μ has a countable base then the following are equivalent for a relative topological dynamics $(X, \mu, \{\varphi^t : t \in T\})$.*
(i) $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive.
(ii) For some $\alpha \in (0, 1]$, $\{x \in \mu_\alpha : \overline{O_\mu^\alpha(x)} = \mu_\alpha\}$ is a dense G_δ.
(iii) There exists α such that $O_\mu^\alpha(x)$ is dense in μ_α for some $x \in \mu_\alpha$.

Proof. $(i) \Longrightarrow (ii)$. Since $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive then for all $\lambda, \eta \in \tau_\mu \setminus \chi_\varnothing$ there is $t \in T$ such that $\varphi^t(\lambda) \cap \eta \neq \chi_\varnothing$. Hence for some $\alpha \in (0,1]$, $\varphi^t(\lambda_\alpha) \cap \eta_\alpha \neq \varnothing$ where $\lambda_\alpha, \eta_\alpha \in \tau_\mu^\alpha \setminus \{\varnothing\}$. Let $\lambda_1^\alpha, \lambda_2^\alpha, ..., \lambda_n^\alpha, ...$ be a countable base for τ_μ^α. Then for $\lambda_n^\alpha \in \tau_\mu^\alpha$, $\cup_{t\in T} (\varphi^t(\lambda_n^\alpha) \cap \mu_\alpha)$ is dense in μ_α. Since for each λ_j^α, there exists $t \in T$ such that $\varphi^t(\lambda_n^\alpha) \cap \lambda_j^\alpha \neq \varnothing$, then $Y = \cap_{n=1}^\infty \cup_{t\in T} (\varphi^t(\lambda_n^\alpha) \cap \mu_\alpha)$ is a countable intersection of open, dense sets and is therefore non-empty by the Baire category theorem and dense in μ_α. Thus $O_\mu^\alpha(x) \cap \lambda_j^\alpha \neq \varnothing$, for every $x \in \mu_\alpha$ and $j = 1, 2, ...$ Hence $\{x \in \mu_\alpha : \overline{O_\mu^\alpha(x)} = \mu_\alpha\}$ is a dense G_δ.
$(ii) \Longrightarrow (iii)$. Since $\{x \in \mu_\alpha : \overline{O_\mu^\alpha(x)} = \mu_\alpha\}$ is a dense G_δ subset of μ_α then it is a non-empty set. So there is $O_\mu^\alpha(x)$ such that $O_\mu^\alpha(x)$ is dense in μ_α.
$(iii) \Longrightarrow (i)$. Suppose for $x_0 \in \mu_\alpha$, $\overline{O_\mu^\alpha(x_0)} = \mu_\alpha$ and $\lambda_\alpha, \eta_\alpha \in \tau_\mu^\alpha \setminus \{\varnothing\}$. We show there exists $t \in T$ such that $\varphi^t(\lambda_\alpha) \cap \eta_\alpha \neq \varnothing$. Since $\overline{O_\mu^\alpha(x_0)} = \mu_\alpha$ then there exist $t_i, t_{i+1} \in T$ such that $t_i < t_{i+1}$ and $\varphi^{t_i}(x_0) \in \lambda_\alpha$, $\varphi^{t_{i+1}}(x_0) \in \eta_\alpha$. Let $y = \varphi^{t_i}(x_0)$ and $t_n = t_{i+1} - t_i$. Then $\varphi^{t_n}(y) = \varphi^{t_{i+1}-t_i}(\varphi^{t_i}(x_0)) = \varphi^{t_{i+1}}(x_0) \in \eta_\alpha$, Moreover $\varphi^{t_n}(y) \in \varphi^{t_n}(\lambda_\alpha)$, hence $\varphi^{t_n}(y) \in \varphi^{t_n}(\lambda_\alpha) \cap \eta_\alpha$ i.e. $\varphi^{t_n}(\lambda_\alpha) \cap \eta_\alpha \neq \varnothing$. □

Theorem 4.2 *If $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive on a compact metric space X and if φ^t an isometry, for all $t \in T$, then $(X, \mu, \{\varphi^t : t \in T\})$ is $\mu_\alpha-$quasi-minimal for some $\alpha \in (0,1]$.*

Proof. Let d, be a metric on X such that for some $\alpha \in (0,1]$, $d(\varphi^t(x), \varphi^t(y)) = d(x, y)$ for all $t \in T$. Since $(X, \mu, \{\varphi^t : t \in T\})$ is relative transitive so there exists $x_0 \in \mu_\alpha$ *such that* $\overline{O_\mu^\alpha(x_0)} = \mu_\alpha$. If $x, y \in \mu_\alpha$ and $\epsilon > 0$, then there exist $t_n, t_m \in T$ such that $d(x, \varphi^{t_m}(x_0)) < \epsilon/2$, and $d(y, \varphi^{t_n}(x_0)) < \epsilon/2$. Hence

$$\begin{aligned} d(y, \varphi^{t_n - t_m}(x)) &\leq d(y, \varphi^{t_n}(x_0)) + d(\varphi^{t_n}(x_0), \varphi^{t_n - t_m}(x)) \\ &= d(y, \varphi^{t_n}(x_0)) + d(\varphi^{t_m}(x_0), x) \\ &< \epsilon. \end{aligned}$$

So $\overline{O_\mu^\alpha(x)} = \mu_\alpha$ i.e. $(X, \mu, \{\varphi^t : t \in T\})$ is $\mu_\alpha-$quasi-minimal. □

5 Conclusion

In the notion of quasi-minimal-relative topological dynamics it is not necessary the level sets be the invariant subsets of the dynamics. So it can be a good base for consideration of new dynamics with the non-invariant state spaces.

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