Associative deformation algebras on Weyl manifolds

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Abstract. One considers a pseudo-Riemannian manifold (M, g), a Weyl structure W on the conformal manifold $(M, \hat{g})$ and a 1-form π on M. One denotes by $\overset{W}{\nabla}$ the symmetric conformal Weyl connection, by $\overset{L}{\nabla}$ the π-semi-symmetric conformal connection. Let $\overset{L}{\overline{\nabla}}$, respectively $\overset{s}{\nabla}$ be the transposed connection, respectively the symmetric connection associated to $\overset{L}{\nabla}$. This paper continues the investigations initiated in [4]. It is illustrated the parallelism between the algebraic properties of some associative deformation algebras and the geometric properties of Weyl manifolds. It is proven that the algebra $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{W}{\nabla})$ is associative is equivalent to the fact that the connexions $\overset{L}{\overline{\nabla}}$ and $\overset{W}{\nabla}$ have the same curvature tensor field if $\overset{W}{R}_p$ is surjective and $W(g)$ is a closed 1-form. Also the algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{W}{\nabla})$ is associative if and only if the curvature tensors associated to $\overset{s}{\nabla}$ and $\overset{W}{\nabla}$ coincide, when $\overset{W}{R}_p$ is surjective and $W(g)$ is a closed 1-form.

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Introduction

Weyl geometry has applications in very active areas of pure and applied mathematics and has been studied by many authors ([1], [2], [5], [7], [8], [10]) from various points of view.

In this paper we continue the investigation initiated in [4] and study Weyl manifolds and properties of some associative deformation algebras associated to these manifolds. It is pointed out the parallelism between the algebraic properties of some deformation algebras and the geometrical properties of the Weyl manifolds. These tools used give new insights, of algebraic nature, on various topics like geodesic mappings, semi-symmetric connections etc.

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1 π-semi-symmetric conformal connections on Weyl manifolds

Let M be a connected paracompact, smooth manifold of dimension $n \geq 3$. Let $\mathcal{X}(M)$ be the Lie algebra of vector fields on M, T_pM the vector space of tangent vectors in a point $p \in M$, $\mathcal{T}^{(p,q)}(M)$ the $\mathcal{C}^\infty(M)$-module of tensor fields of type (p,q) on M, $\Lambda^p(M)$ the $\mathcal{C}^\infty(M)-$module of $p-$forms on M and $H^p(M)$ the $p-$th de Rham cohomology group of M.

Let g be a semi-Riemannian metric on M. A Weyl manifold is a triple $(M, \widehat{g}, W)$, where $\widehat{g} = \{e^u g \mid u \in \mathcal{C}^\infty(M)\}$ is the conformal class defined by g and $W : \widehat{g} \longrightarrow \Lambda^1(M)$ is a Weyl structure on the conformal manifold $(M, \widehat{g})$, hence

$$W(e^u g) = W(g) - du, \forall u \in \mathcal{C}^\infty(M). \tag{1.1}$$

A linear connection ∇ on M is compatible with the Weyl structure W if

$$\overset{W}{\nabla} g + W(g) \otimes g = 0. \tag{1.2}$$

There exists a unique torsion free linear connection $\overset{W}{\nabla}$, verifying (1.2), given by the formula:

$$\begin{aligned} 2g(\overset{W}{\nabla}_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + \\ &+ W(g)(X)\, g(Y,Z) + W(g)(Y)\, g(X,Z) - W(g)(Z)\, g(X,Y) + \\ &+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X),\ \forall X,Y,Z \in \mathcal{X}(M). \end{aligned} \tag{1.3}$$

$\overset{W}{\nabla}$ is called the Weyl conformal connection. This connection is invariant under a "gauge transformation" $g \longrightarrow e^u g$. So, the $1-$form $W(g)$ is required to change by (1.1).

Weyl introduced a $2-$form $\psi(W)$ on M by setting $\psi(W) = dW(g)$, $g \in \widehat{g}$, and called it the distance curvature. This is a gauge invariant. If $\psi(W) = 0$, then by (1.1), the cohomology class $[W(g)] \in H^1(M)$ of the closed form $W(g)$ does not depend on the choice of a metric in $\widehat{g}$. For simplicity, we write $ch(W) = [W(g)]$. The $2-$form $\psi(W)$ and the class $ch(W)$ are the obstructions for a Weyl structure to be a Riemannian structure. Indeed:

Proposition A [3]*Let $(M, \widehat{g}, W)$ be a Weyl manifold and $\overset{W}{\nabla}$ be the Weyl conformal connection. Then the following two conditions are equivalent:*

1) $\psi(W) = 0$ and $ch(W) = 0$;

2) There is a Riemann metric in $\widehat{g}$ such that $\overset{W}{\nabla} g = 0$.

Let π be a $1-$form on M. We denote by $\overset{L}{\nabla}$ the connection compatible with the Weyl structure W, which is π-semi-symmetric i.e. the torsion tensor is required to be $\overset{L}{T}(X,Y) = \pi(Y)X - \pi(X)Y,\ \forall X,Y \in \mathcal{X}(M)$, and

$$\begin{aligned} 2g(\overset{L}{\nabla}_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + \\ + W(g)(X)\, g(Y,Z) + W(g)(Y)\, g(X,Z) - \\ - W(g)(Z)\, g(X,Y) + 2\pi(Y)\, g(X,Z) - \\ - 2\pi(Z)\, g(X,Y) + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \end{aligned} \tag{1.4}$$

holds. The relation between these two connections is given by

$$\overset{L}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \pi(Y)X - g(X,Y)P, \tag{1.5}$$

where P is the dual vector field of π i.e. $g(Z,P) = \pi(Z), \ \forall Z \in \mathcal{X}(M)$.

We denote by $\overset{L}{\overline{\nabla}}$ the transposed connection of $\overset{L}{\nabla}$ i.e.

$$\overset{L}{\overline{\nabla}}_X Y = \overset{L}{\nabla}_Y X + [X,Y]. \tag{1.6}$$

The relations (1.5) and (1.6) lead to

$$\overset{L}{\overline{\nabla}}_X Y = \overset{W}{\nabla}_X Y + \pi(X)Y - g(X,Y)P. \tag{1.7}$$

Let us denote by $\overset{s}{\nabla}$ the symmetric connection associated to $\overset{L}{\nabla}$ i.e.
$\overset{s}{\nabla} = \frac{1}{2}(\overset{L}{\nabla} + \overset{L}{\overline{\nabla}})$. Hence

$$\overset{s}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \frac{1}{2}\pi(X)Y + \frac{1}{2}\pi(Y)X - g(X,Y)P. \tag{1.8}$$

Let A be a $(1,2)-$tensor field on M. The $\mathcal{C}^\infty(M)-$modul $\mathcal{X}(M)$ becomes a $\mathcal{C}^\infty(M)-$algebra if we consider the multiplication rule given by
$X \circ Y = A(X,Y), \ \forall X,Y \in \mathcal{X}(M)$. This algebra is denoted by $\mathcal{U}(M,A)$ and it is called the algebra associated to A. If ∇ and ∇' are two linear connections on M, then $\mathcal{U}(M,A)$ is called the deformation algebra defined by the pair (∇,∇') [4], [6]. The algebra $\mathcal{U}\left(M, \overset{L}{\nabla} - \overset{W}{\nabla}\right)$ is called the Weyl-Lyra algebra associated to the $1-$form π.

Two linear connections ∇ and ∇' on M are said to be projectively equivalent if their unparametrized geodesics coincide or equivalent if there exists a $1-$form ω so that the Weyl formula $\overset{s}{\nabla}'_X Y = \overset{s}{\nabla}_X Y + \omega(X)Y + + \omega(Y)X, \ \forall X,Y \in \mathcal{X}(M)$, holds, where $\overset{s}{\nabla}'$ and $\overset{s}{\nabla}$ are the torsion free connections associated to ∇' and ∇.

Theorem 1.1 *Let $(M,\widehat{g},W)$ be a Weyl manifold and π be a 1-form on M. Let $\overset{W}{\nabla}$ be the Weyl conformal connection, $\overset{L}{\nabla}$ be the π-semi-symmetric conformal connection and $\overset{L}{\overline{\nabla}}$ be the transposed connection of $\overset{L}{\nabla}$. Let $\overset{W}{R}, \overset{L}{R}$ and $\overset{L}{\overline{R}}$ be the curvature tensor fields associated to $\overset{W}{\nabla}, \overset{L}{\nabla}$ and $\overset{L}{\overline{\nabla}}$, respectively.*

Then the following assertions are equivalent:

i) $\pi = 0$.

ii) $\overset{L}{\overline{\nabla}} = \overset{W}{\nabla}$.

iii) The algebra $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{W}{\nabla})$ is commutative.

iv) The algebra $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{W}{\nabla})$ is associative.

v) $\overset{L}{\overline{R}} = \overset{W}{R}$, when the 1-form $W(g)$ is closed and the mapping $\overset{W}{R}_p: T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, $\forall p \in M$.

vi) $\overset{L}{\overline{\nabla}}$ *and* $\overset{W}{\nabla}$ *are projectively equivalent.*

vii) The algebra $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{L}{\nabla})$ *is commutative.*

viii) The algebra $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{L}{\nabla})$ *is associative.*

ix) All the elements of the $\mathcal{C}^\infty(M)$*-module* $\mathcal{T}^{(1,1)}(M)$ *are derivations in the algebra* $\mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{W}{\nabla})$.

Proof. i)⇔ii)⇔iii), i)⇒iv), i)⇒v), i)⇒vi), i)⇔vii), i) ⇒viii) are obvious.

iv)⇒i). Using (1.7), the associativity condition $X \circ (Y \circ Z) =$ $= (X \circ Y) \circ Z$, $\forall X, Y, Z \in \mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{W}{\nabla})$ becomes

$$[g(Y,Z)\pi(X) - g(X,Z)\pi(Y) - g(X,Y)\pi(Z)]P + g(X,Y)\pi(P)Z = 0. \tag{1.9}$$

In a local system of coordinates, one gets

$$(g_{jk}\pi_i - g_{ik}\pi_j - g_{ij}\pi_k)\pi^r + g_{ij}\pi^s\pi_s\delta^r_k = 0. \tag{1.10}$$

Considering $r = k$ in (1.10) and summimg, we obtain $\pi^s\pi_s = 0$. Contracting (1.10) with g^{jk}, one has $(n-2)\pi_i\pi^r = 0$. This implies that $\pi = 0$.

v)⇒i). Let us denote by $\overline{A} = \overset{L}{\overline{\nabla}} - \overset{W}{\nabla}$. From $\overset{L}{\overline{R}} = \overset{W}{R}$, one gets

$$\begin{aligned}(\overset{L}{\overline{\nabla}}_X \overset{L}{\overline{R}})(Y,Z,V) = (\overset{W}{\nabla}_X \overset{W}{R})(Y,Z,V) + \overline{A}(X, \overset{W}{R}(Y,Z)V) - \\ - \overset{W}{R}(\overline{A}(X,Y),Z)V - \overset{W}{R}(Y,\overline{A}(X,Z))V - \overset{W}{R}(Y,Z)\overline{A}(X,V).\end{aligned} \tag{1.11}$$

Using Bianchi identities in (1.11) and $\overset{L}{\overline{T}}(X,Y) = \pi(X)Y - \pi(Y)X$, one has

$$\begin{aligned}2\pi(X)(\overset{L}{\overline{R}}(Z,Y)V - \overset{W}{R}(Z,Y)V) + 2\pi(Y)(\overset{L}{\overline{R}}(X,Z)V - \\ - \overset{W}{R}(X,Z)V) + 2\pi(Z)(\overset{L}{\overline{R}}(Y,X)V - \overset{W}{R}(Y,X)V) = \\ = \overline{A}(X, \overset{W}{R}(Y,Z)V) + \overline{A}(Y, \overset{W}{R}(Z,X)V) + \overline{A}(Z, \overset{W}{R}(X,Y)V) - \\ - \overset{W}{R}(Y,Z)\overline{A}(X,V) - \overset{W}{R}(Z,X)\overline{A}(Y,V) - \overset{W}{R}(X,Y)\overline{A}(Z,V).\end{aligned} \tag{1.12}$$

If we denote by $\overline{A}^i_{jk}, g_{ij}, \pi_i, \overset{W}{R}{}^i_{jkl}$ the local components of $\overline{A}, g, \pi$ and $\overset{W}{R}$, respectively, in a local system of coordinates, we obtain

$$\overline{A}^r_{il}\overset{W}{R}{}^s_{rjk} + \overline{A}^r_{jl}\overset{W}{R}{}^s_{rki} + \overline{A}^r_{kl}\overset{W}{R}{}^s_{rij} = \overline{A}^s_{ir}\overset{W}{R}{}^r_{ljk} + \overline{A}^s_{jr}\overset{W}{R}{}^r_{lki} + \overline{A}^s_{kr}\overset{W}{R}{}^r_{lij}. \tag{1.13}$$

The relations (1.7) and (1.13) lead to

$$(g_{ri}\overset{W}{R}{}^r_{ljk} + g_{rj}\overset{W}{R}{}^r_{lki} + g_{rk}\overset{W}{R}{}^r_{lij})\pi^s = (g_{il}\overset{W}{R}{}^s_{rjk} + g_{jl}\overset{W}{R}{}^s_{rki} + g_{kl}\overset{W}{R}{}^s_{rij})\pi^r. \tag{1.14}$$

Let $2\varphi_i$ and $\overset{\circ}{R}{}^i_{jkl}$ be the local components of the 1-form $W(g)$ and $\overset{\circ}{R}$, the curvature tensor field of $\overset{\circ}{\nabla}$, the Levi-Civita connection associated to g. We have

$$\overset{W}{R}{}^{i}_{jkl} = \overset{\circ}{R}{}^{i}_{jkl} + \delta^i_j(\varphi_{kl} - \varphi_{lk}) + \delta^i_k \varphi_{jl} - \delta^i_l \varphi_{jk} - g_{jk}\varphi^i_l + g_{jl}\varphi^i_k, \tag{1.15}$$

where $\varphi_{jl} = \frac{\partial \varphi_j}{\partial x^l} - \left| \begin{matrix} r \\ jl \end{matrix} \right| \varphi_r + \varphi_j \varphi_l - \frac{1}{2} g_{jl}\varphi^r \varphi_r, \varphi^r = g^{ri}\varphi_i, \varphi^i_l = g^{ij}\varphi_{jl}$, $\left| \begin{matrix} r \\ jl \end{matrix} \right|$ are the Christoffel symbols. Since the 1-form $W(g)$ is closed, one has $\varphi_{ij} = \varphi_{ji}$. Therefore (1.15) implies

$$g_{ir} \overset{W}{R}{}^{r}_{ljk} + g_{rj} \overset{W}{R}{}^{r}_{lki} + g_{kr} \overset{W}{R}{}^{r}_{lij} = 0. \tag{1.16}$$

The formulae (1.14) and (1.16) lead to

$$(g_{il} \overset{W}{R}{}^{s}_{rjk} + g_{jl} \overset{W}{R}{}^{s}_{rki} + g_{kl} \overset{W}{R}{}^{s}_{rij})\pi^r = 0. \tag{1.17}$$

Contracting in (1.17) by g^{li}, one gets $(n-2) \overset{W}{R}{}^{s}_{rjk}\pi^r = 0$. Hence (1.15) becomes $\overset{W}{R}{}^{i}_{jkl}\pi_i = 0$. Therefore $\pi_p(\overset{W}{R_p}(X_p, Y_p)Z_p) = 0, \forall p \in M$. Since $\overset{W}{R_p}$: $T_pM \times T_pM \times T_pM \mapsto T_pM$ is surjective, $\forall p \in M$, we get $\pi_p = 0, \forall p \in M$, so $\pi = 0$.

vi)⇒i). $\overset{W}{\nabla}$ and $\overset{L}{\overline{\nabla}}$ being projectively equivalent, there exists the 1-form ω on M such that

$$\overset{s}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \omega(X)Y + \omega(Y)X. \tag{1.18}$$

We denote by A'^i_{jk} and ω_k the local components of $A' = \overset{s}{\nabla} - \overset{W}{\nabla}$ and ω, respectively. Hence $A'^i_{jk} = \delta^i_j\omega_k + \delta^i_k\omega_j$. This relation implies $A'_k = A'^i_{ik} =$
$= (n+1)\omega_k$. From (1.8) we have $A'_k = \frac{n-1}{2}\pi_k$. Therefore $\pi = \frac{2(n+1)}{n-1}\omega$ and

$$(n-1)g(A'(X,Y),Z) \ = \ (n+1)[\omega(X)g(Y,Z) + \omega(Y)g(X,Z) - 2\omega(Z)g(X,Y)].$$

From (1.18) and the previous relation we get

$$\omega(X)g(Y,Z) + \omega(Y)g(X,Z) = (n+1)\omega(Z)g(X,Y). \tag{1.19}$$

In local coordinates, we have $\omega_i g_{jk} + \omega_j g_{ik} = (n+1)\omega_k g_{ij}$. Contracting by g^{ij}, one obtains $(n-1)\omega_k = 0$. Then $\omega = 0$ and $\pi = 0$.

viii)⇒i). The relations (1.5) and (1.7) imply

$\overset{L}{\overline{\nabla}}_X Y = \overset{L}{\nabla}_X Y + \pi(X)Y - \pi(Y)X.$

Therefore the associativity condition $X \circ (Y \circ Z) = (X \circ Y) \circ Z$, $\forall X, Y, Z \in \mathcal{U}(M, \overset{L}{\overline{\nabla}} - \overset{L}{\nabla})$ becomes $\pi(Y)[\pi(X)Z - \pi(Z)X] = 0$. Hence $\pi = 0$.

ix)⇔ii). $F \in \mathcal{T}^{(1,1)}(M)$ is a derivation in the algebra $\mathcal{U}(M, \overline{A})$ i.e. $F(\overline{A}(X,Y)) = \overline{A}(F(X), Y) + \overline{A}(X, F(Y))$ if and only if $\overline{A} = \overset{L}{\overline{\nabla}} - \overset{W}{\nabla} = 0$.

2 Deformation algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{W}{\nabla})$

The goal of this section is to characterize the deformation algebra of the symmetric connection associated to a π-semi-symmetric connection. Our algebraic approach give new insights of geometrical nature.

Theorem 2.1 *Let $(M, \widehat{g}, W)$ be a Weyl manifold and π be a 1-form on M. Let $\overset{W}{\nabla}$ be the Weyl conformal connection, $\overset{L}{\nabla}$ be the π-semi-symmetric conformal connection and $\overset{s}{\nabla}$ be the symmetric connection associated to $\overset{L}{\nabla}$. Let $\overset{s}{R}$ and $\overset{W}{R}$ be the curvature tensor fields associated to $\overset{s}{\nabla}$ and $\overset{W}{\nabla}$, respectively.*

Then the following assertions are equivalent:

i) $\pi = 0$.

ii) $\overset{s}{\nabla} = \overset{W}{\nabla}$.

iii) The algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{W}{\nabla})$ is associative when $n \neq 5$.

iv) $\overset{s}{\nabla}$ and $\overset{W}{\nabla}$ are projectively equivalent.

v) $\overset{s}{\nabla}_X \overset{s}{R} = \overset{s}{\nabla}_X \overset{W}{R}, \forall X \in \mathcal{X}(M)$, when the 1-form $W(g)$ is closed, $n \neq 4$ and the mapping $\overset{W}{R}_p \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, $\forall p \in M$.

vi) $\overset{s}{R} = \overset{W}{R}$ *when the 1-form $W(g)$ is closed, $n \neq 4$ and the mapping* $\overset{W}{R}_p \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM$ *is surjective, $\forall p \in M$.*

vii) All the elements of the algebra $\mathcal{U}\left(M, \overset{s}{\nabla} - \overset{W}{\nabla}\right)$ are characteristic fields.

viii) All the elements of the $\mathcal{C}^\infty(M)$-module $\mathcal{T}^{(1,1)}(M)$ are derivations in the algebra $\mathcal{U}(M, \overset{s}{\nabla} - \overset{W}{\nabla})$.

Proof. i)⇔ii)⇒iii), i)⇒iv), i)⇒v), i)⇒vi), i)⇒vii), vi) ⇒v), viii)⇔ii) are obvious.

iii) ⇒ i) Using (1.8), the associativity condition $A'(A'(X, Y), Z)) = A'(X, A'(Y, Z))$, where $A' = \overset{s}{\nabla} - \overset{W}{\nabla}$, becomes

$$\begin{aligned} \pi(Y)\pi(Z)X - \pi(X)\pi(Z)Y - 4\pi(Z)g(X, Y)P + \\ +4\pi(X)g(Y, Z) - 2g(Y, Z)\pi(P)X + 2g(X, Y)\pi(P)Z = 0. \end{aligned} \tag{2.1}$$

In local coordinates one has

$$\pi_j\pi_k\delta^r_i - \pi_i\pi_j\delta^r_k - 4\pi_k g_{ij}\pi^r + 4\pi_i g_{jk}\pi^r - 2g_{jk}\pi^s\pi_s\delta^r_i + 2g_{ij}\pi_s\pi^s\delta^r_k = 0. \tag{2.1$'$}$$

Considering $r = i$ in (2.1)$'$ and summing, we get

$$2(n - 3)g_{jk}\pi_s\pi^s = (n - 5)\pi_j\pi_k. \tag{2.1$''$}$$

Contracting with g^{jk}, (2.1)$''$ becomes $\pi^s\pi_s = 0$. Therefore $(n - 5)\pi_j\pi_k = 0$. Since $n \neq 5$, we get $\pi = 0$.

iv) ⇒i) The proof is analogous with iv) ⇒i) from theorem 1.1.

v)⇒ i). We have

$$\begin{aligned} (\overset{s}{\nabla}_X \overset{s}{R})(Y, Z, V) = (\overset{W}{\nabla}_X \overset{W}{R})(Y, Z, V) + A'(X, \overset{W}{R}(Y, Z)V) - \\ - \overset{W}{R}(A'(X, Y), Z)V) - \overset{W}{R}(Y, A'(X, Z)V) - \overset{W}{R}(Y, Z)A'(X, V). \end{aligned} \tag{2.2}$$

Using the Bianchi identities, (2.2) implies

$$A'^r_{ih} \overset{W}{R}{}^l_{rjk} + A'^r_{jh} \overset{W}{R}{}^l_{rki} + A^r_{kh} \overset{W}{R}{}^l_{rij} = A'^l_{ir} \overset{W}{R}{}^r_{hjk} + A'^l_{jr} \overset{W}{R}{}^r_{hki} + A'^l_{kr} \overset{W}{R}{}^r_{hij}. \tag{2.3}$$

From (1.8) and (2.3) one gets

$$2\pi^r(g_{ih}\overset{W}{R}{}^{l}_{rjk} + g_{jh}\overset{W}{R}{}^{l}_{rki} + g_{kh}\overset{W}{R}{}^{l}_{rij}) + \pi_r(\delta^l_i \overset{W}{R}{}^{r}_{hjk} + \delta^l_j \overset{W}{R}{}^{r}_{hki} + \delta^l_k \overset{W}{R}{}^{r}_{hij}) = 0. \tag{2.4}$$

Considering $l = i$ in (2.4) and summing, one has

$$2\pi^r(g_{kh}\overset{W}{S}{}_{rj} - g_{jh}\overset{W}{S}{}_{rk}) + (n-2)\overset{W}{R}{}^{r}_{hjk}\pi_r + 2\pi^r g_{sh}\overset{W}{R}{}^{s}_{rjk} = 0, \tag{2.5}$$

where $\overset{W}{S}{}_{ij}$ are the components of the Ricci tensor associated to $\overset{W}{\nabla}$. Using $\pi_r \overset{W}{R}{}^{r}_{hjk} = -\pi^r g_{sh}\overset{W}{R}{}^{s}_{rjk}$ in (2.5), we get

$$2\pi^r(g_{kh}\overset{W}{S}{}_{rj} - g_{jh}\overset{W}{S}{}_{rk}) + (n-4)\overset{W}{R}{}^{r}_{hjk}\pi_r = 0. \tag{2.6}$$

Contracting by g^{hk}, (2.6) becomes

$$2(n-1)\overset{W}{S}{}_{rj}\pi^r + (n-4)\overset{W}{R}{}^{r}_{hjk}\pi_r g^{kh} = 0. \tag{2.6$'$}$$

From $\overset{W}{S}{}_{rj}\pi^r = \overset{W}{R}{}^{r}_{hjk}\pi_r g^{kh}$ and (2.6)$'$ we get $(n-2)\overset{W}{S}{}_{rj}\pi^r = 0$. Therefore (2.6) implies $(n-4)\overset{W}{R}{}^{r}_{hjk}\pi_r = 0$. Since $n \neq 4$, we get $\overset{W}{R}{}^{r}_{hjk}\pi_r = 0$. The mapping $\overset{W}{R}_p \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, $\forall p \in M$. So, $\pi_p = 0$. Hence $\pi = 0$.

vii)⇒i) The vector field $X \in \mathcal{U}(M, \overset{s}{\nabla} - \overset{W}{\nabla})$ is a characteristic field if there exists $f \in \mathcal{C}^\infty(M)$ such that $A'(X,X) = fX$, where $A' = \overset{s}{\nabla} - \overset{W}{\nabla}$. All the elements of the algebra $\mathcal{U}(M, A')$ are characteristic fields if and only if there exists a 1-form θ on M such that $\overset{s}{\nabla}_X Y = \overset{W}{\nabla}_X Y + \theta(X)Y + \theta(Y)X$. Therefore $\overset{s}{\nabla}$ and $\overset{W}{\nabla}$ are projectively equivalent. So one gets i).

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