On the composition of two Riemannian immersions

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Abstract. We study the properties of the composition of two Riemannian immersions of codimension one. A Riemannian immersion is an isometric immersion between two Riemannian manifolds. First, we establish the Gauss and Weingarten formulae of the composition in dependence on the second fundamental forms and the Weingarten maps (shape operators) of factors. Then, we apply these for particular Riemannian immersions (totally geodesic, totally umbilical, minimal) for obtaining information about their composition. The computations are done in local coordinates.

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1 Notations and preliminaries

Let M be a n-dimensional manifold isometrically immersed in a $(n+1)$ dimensional manifold $\overline{M}$ which is isometrically immersed in a $(n+2)$ dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$:

$$(M, g) \hookrightarrow (\overline{M}, \overline{g}) \hookrightarrow (\widetilde{M}, \widetilde{g}) \tag{1.1}$$

where $\overline{g}$ is the induced metric on $\overline{M}$ by the isometric immersion $\overline{i} : \overline{M} \hookrightarrow \widetilde{M}$ and g is the induced metric on M by the isometric immersion $i : M \hookrightarrow \overline{M}$. Thus, M is a submanifold isometrically immersed in $\widetilde{M}$ and $\widehat{g}$ is the induced metric on M by the isometric immersion $\overline{i} \circ i : M \hookrightarrow \widetilde{M}$:

$$(M, \widehat{g}) \hookrightarrow (\widetilde{M}, g). \tag{1.2}$$

Since the discussion is local, we may assume that M is imbedded in $\overline{M}$ and $\overline{M}$ is imbedded in $\widetilde{M}$ (and we may assume that $(\overline{i} \circ i)(P) = i(P) = P$).

In the following, the indices range is fixed in this way:
$\alpha, \beta, \gamma ... \in \{1, ..., n\}, i, j, k ... \in \{1, ..., n+1\}$, and $a, b, c ... \in \{1, ..., n+2\}$. We shall use the Einstein convention for summation.
Let P a point in M and $(u^1, ..., u^n) := (u^\alpha)$ the local coordinates around the P in M, such that $(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, ..., \frac{\partial}{\partial u^n}) := (\frac{\partial}{\partial u^\alpha})$ form a basis of the tangent space $T_P(M)$

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of M in P. The submanifold M in $\overline{M}$ can be represented locally by $x^i = x^i(u^\alpha)$, with $rank(\frac{\partial x^i}{\partial u^\alpha}) = n$, the submanifold M in $\widetilde{M}$ can be locally given by $y^a = y^a(u^\alpha)$, with $rank(\frac{\partial y^a}{\partial u^\alpha}) = n$, and the submanifold $\overline{M}$ in $\widetilde{M}$ can be represented locally by $y^a = y^a(x^i)$, with $rank(\frac{\partial y^a}{\partial x^i}) = n+1$.
In $T_P(\overline{M})$ we have the natural basis $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^{n+1}}) := (\frac{\partial}{\partial x^i})$. In $T_P(\widetilde{M})$ we have the natural basis $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, ..., \frac{\partial}{\partial y^{n+2}}) := (\frac{\partial}{\partial y^a})$. Then, the Riemannian metrics $\widetilde{g}$, $\overline{g}$, g and $\widehat{g}$ respectively are $\widetilde{g} = \widetilde{g}_{ab} dy^a dy^b$ on $\widetilde{M}$, $\overline{g} = \overline{g}_{ij} dx^i dx^j$ (for $\overline{M} \hookrightarrow (\widetilde{M}, \widetilde{g})$), $g = g_{\alpha\beta} du^\alpha du^\beta$ (for $M \hookrightarrow (\overline{M}, \overline{g})$),and $\widehat{g} = \widehat{g}_{\alpha\beta} du^\alpha du^\beta$ for $M \hookrightarrow (\widetilde{M}, \widetilde{g})$, respectively. If X is a vector field on M, we can identify X with $i_*(X)$ and with $(\overline{i} \circ i)_*(X)$. Then, for X_P and $Y_P \in T_P(M)$, we have $g(X_P, Y_P) = (i_* \overline{g})(X_P, Y_P)$ where $X_P = X^\alpha \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i} = X^\alpha \frac{\partial y^a}{\partial u^\alpha} \frac{\partial}{\partial y^a}$ and $Y_P = Y^\beta \frac{\partial x^j}{\partial u^\beta} \frac{\partial}{\partial x^j} = Y^\beta \frac{\partial y^b}{\partial u^\beta} \frac{\partial}{\partial y^b}$, therefore

$$g_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \overline{g}_{ij} = \frac{\partial y^a}{\partial u^\alpha} \frac{\partial y^b}{\partial u^\beta} \widetilde{g}_{ab} \tag{1.3}$$

Let $T^\perp(M)$ denote the vector bundle of all normal vectors to M in $\overline{M}$ and let $T^\perp(\overline{M})$ denote the vector bundle of all normal vectors to $\overline{M}$ in $\widetilde{M}$.Thus, at each $P \in M \subset \overline{M} \subset \widetilde{M}$ we have:

$$T_P(\overline{M}) = T_P(M) \oplus T_P(M)^\perp; \;\; T_P(\widetilde{M}) = T_P(\overline{M}) \oplus T_P(\overline{M})^\perp. \tag{1.4}$$

From (1.4) we have a direct sum decomposition

$$T_P(\widetilde{M}) = T_P(M) \oplus T_P(M)^\perp \oplus T_P(\overline{M})^\perp. \tag{1.5}$$

Since the submanifold M is of codimension one in $\overline{M}$ (and the submanifold $\overline{M}$ is of codimension one in $\widetilde{M}$) we can always choose an unit normal section $N_1 \in T_P(\overline{M})$, normal at M in P (and we can choose an unit normal section $N_2 \in T_P(\widetilde{M})$, normal at $\overline{M}$ in P, respectively). In $T_P(\overline{M})$ we can consider the Gauss frame $((\frac{\partial}{\partial u^\alpha}), N_1)$, where

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}; \;\; N_1 = N_1^j \frac{\partial}{\partial x^j}. \tag{1.6}$$

In $T_P \widetilde{M}$ we can consider the Gauss frame $((\frac{\partial}{\partial u^\alpha}), N_1, N_2)$ where

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial y^a}{\partial u^\alpha} \frac{\partial}{\partial y^a}; \;\; N_1 = N_1^a \frac{\partial}{\partial y^a}; \; N_2 = N_2^a \frac{\partial}{\partial y^a} \tag{1.7}$$

Conversely, the elements of the basis $(\frac{\partial}{\partial y^a})$ from $T_P(\widetilde{M})$ in the Gauss frame $((\frac{\partial}{\partial u^\alpha}), N_1, N_2)$ are:

$$\frac{\partial}{\partial y^a} = C_a^\gamma \frac{\partial}{\partial u^\gamma} + D_a^1 N_1 + D_a^2 N_2. \tag{1.8}$$

The elements of the basis $(\frac{\partial}{\partial x^i})$ at $T_P(\overline{M})$ in the basis $(\frac{\partial}{\partial y^a})$ at $T_P(\widetilde{M})$ are:

$$\frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}. \tag{1.9}$$

But $N_2 \in T_P(\overline{M})^{\perp}$, therefore we have $\widetilde{g}(\frac{\partial}{\partial x^i}, N_2) = 0$. From (1.8) and (1.9) it results:

$$\frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} C_a^{\gamma} \frac{\partial}{\partial u^{\gamma}} + \frac{\partial y^a}{\partial x^i} D_a^1 N_1. \tag{1.10}$$

2 The Gauss and Weingarten formulae

In the following, we denote by $\widetilde{\nabla}$ the Riemannian connection on $\widetilde{M}$, with respect to its Riemannian metric $\widetilde{g}$. Let $\overline{\nabla}$ (∇ and $\widehat{\nabla}$ respectively) be the Riemannian connection induced on the submanifold $\overline{M} \hookrightarrow \widetilde{M}$ (on $M \hookrightarrow \overline{M}$ and on $M \hookrightarrow \widetilde{M}$ respectively) with respect to its Riemannian metric $\widetilde{g}$ (g and $\widehat{g}$ respectively). We recall some necessary facts and formulae from the theory of submanifolds (see B.Y. Chen [2], K. Yano, M. Kon [3]).
For $M \hookrightarrow \overline{M}$, the Gauss and Weingarten formulae are:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \ \overline{\nabla}_X \xi = -A_{\xi}(X) + \nabla_X^{\perp} \xi, \tag{2.1}$$

where X,Y are the vector fields tangent to M, ξ is a vector field orthogonal to M, $h : TM \times TM \to TM^{\perp}$ is the second fundamental form of the submanifold M in $\overline{M}$ (which is symmetric with respect to both arguments:
$h(X, Y) = h(Y, X)$), $A_{\xi} : TM \to TM$ is the Weingarten map (or shape operator) in the normal direction ξ and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}(M) \subset T(\overline{M})$. From the formulas above, it follows that:

$$g(A_{\xi}(X), Y) = \overline{g}(h(X, Y), \xi). \tag{2.2}$$

Because M is a hipersurfaces in $\overline{M}$, it follows that $\nabla_X^{\perp}\xi = 0$. The forms of the Gauss and Weingarten formulae for $M \hookrightarrow \overline{M}$, in the local components, are:

$$\overline{\nabla}_{\frac{\partial}{\partial u^{\beta}}} \frac{\partial}{\partial u^{\alpha}} = \nabla_{\frac{\partial}{\partial u^{\beta}}} \frac{\partial}{\partial u^{\alpha}} + h(\frac{\partial}{\partial u^{\alpha}}, \frac{\partial}{\partial u^{\beta}}); \ \overline{\nabla}_{\frac{\partial}{\partial u^{\beta}}} N_1 = -A_{N_1}(\frac{\partial}{\partial u^{\beta}}), \tag{2.3}$$

with

$$\nabla_{\frac{\partial}{\partial u^{\alpha}}} \frac{\partial}{\partial u^{\beta}} = \Gamma_{\alpha\beta}^{\gamma} \frac{\partial}{\partial u^{\gamma}}; \ \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \overline{\Gamma}_{ij}^k \frac{\partial}{\partial x^k}, \tag{2.4}$$

where $\overline{\Gamma}_{ij}^k$ are the Christoffel symbols of the Riemannian connection $\overline{\nabla}$ and $\Gamma_{\alpha\beta}^{\gamma}$ are the Christoffel symbols of the Riemannian connection ∇, respectively. More of that, in the local components, we have

$$h(\frac{\partial}{\partial u^{\alpha}}, \frac{\partial}{\partial u^{\beta}}) = h_{\alpha\beta} N_1; \ A_{N_1}(\frac{\partial}{\partial u^{\beta}}) = A_{\beta}^{\gamma} \frac{\partial}{\partial u^{\gamma}}; \ A_{\alpha}^{\beta} = g^{\beta\gamma} h_{\gamma\alpha}. \tag{2.5}$$

For $\overline{M} \hookrightarrow \widetilde{M}$, we have the same consideration like above.
Let X,Y be the vector fields tangent to $\overline{M}$, ζ is a vector field orthogonal to $\overline{M}$, $\overline{h} : T\overline{M} \times T\overline{M} \to T\overline{M}^{\perp}$ is the second fundamental form of the submanifold $\overline{M}$ in $\widetilde{M}$, $\overline{A}_{\zeta} : T\overline{M} \to T\overline{M}$ is the Weingarten map in the normal direction ζ and $\overline{\nabla}^{\perp}$ is the connection in the normal bundle $T^{\perp}(\overline{M}) \subset T(\widetilde{M})$.We have $\overline{\nabla}_X^{\perp}\zeta = 0$ since $\overline{M}$

is a hipersurface in $\widetilde{M}$. Thus, in the local coordinates, the forms of the Gauss and Weingarten formulae for $\overline{M} \hookrightarrow \widetilde{M}$, are:

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \overline{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \overline{h}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i});\ \ \widetilde{\nabla}_{\frac{\partial}{\partial x^j}} N_2 = -\overline{A}_{N_2}(\frac{\partial}{\partial x^j}). \tag{2.6}$$

In local coordinates, the second fundamental form and the Weingarten map have the forms:

$$\overline{h}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \overline{h}_{ij} N_2;\ \ \overline{A}_{N_2}(\frac{\partial}{\partial x^i}) = \overline{A}_i^k \frac{\partial}{\partial x^k};\ \ \overline{A}_i^k = \overline{g}^{kj}\overline{h}_{ji} \tag{2.7}$$

For the immersion $M \hookrightarrow \widetilde{M}$, if $X, Y \in T(M)$ and ρ is an unit vector field orthogonal to M in $\widetilde{M}$ (therefore we have $\rho \in T^\perp(M) \oplus T^\perp(\overline{M})$),
$\widehat{h} : TM \times TM \to T^\perp(M) \oplus T^\perp(\overline{M})$ is the second fundamental form of the submanifold M in $\widetilde{M}$, $\widehat{A}_\rho : TM \to TM$ is the Weingarten map in the normal direction ρ and $\widehat{\nabla}^\perp$ is the connection in the normal bundle $T^\perp(M) \subset T(\widetilde{M})$, in the local coordinates we have:

$$\widehat{h}(\frac{\partial}{\partial u^\alpha}; \frac{\partial}{\partial u^\beta}) = \widehat{h}^1_{\alpha\beta} N_1 + \widehat{h}^2_{\alpha\beta} N_2; \tag{2.8}$$

$$\widehat{A}_{N_1}(\frac{\partial}{\partial u^\alpha}) = \widehat{A}^\beta_{\alpha 1} \frac{\partial}{\partial u^\beta};\ \ \widehat{A}_{N_2}(\frac{\partial}{\partial u^\alpha}) = \widehat{A}^\beta_{\alpha 2} \frac{\partial}{\partial u^\beta}. \tag{2.9}$$

From $\widehat{g}(\widehat{A}_\rho(X), Y) = \widetilde{g}(\widehat{h}(X,Y), \rho)$ it results $\widehat{A}^\gamma_{\alpha 1} = \widehat{g}^{\gamma\beta}\widehat{h}^1_{\alpha\beta}$ and $\widehat{A}^\gamma_{\alpha 2} = \widehat{g}^{\gamma\beta}\widehat{h}^2_{\alpha\beta}$. Thus, in the local components, the Gauss and Weingarten formulae for $M \hookrightarrow \widetilde{M}$ are:

$$\widetilde{\nabla}_{\frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \widehat{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial u^\gamma} + \widehat{h}^1_{\alpha\beta} N_1 + \widehat{h}^2_{\alpha\beta} N_2; \tag{2.10}$$

$$\widetilde{\nabla}_{\frac{\partial}{\partial u^\alpha}} N_1 = -\widehat{A}^\beta_{\alpha 1} \frac{\partial}{\partial u^\beta} + \widehat{\nabla}^\perp_{\frac{\partial}{\partial u^\alpha}} N_1;\ \ \widetilde{\nabla}_{\frac{\partial}{\partial u^\alpha}} N_2 = -\widehat{A}^\beta_{\alpha 2} \frac{\partial}{\partial u^\beta} + \widehat{\nabla}^\perp_{\frac{\partial}{\partial u^\alpha}} N_2. \tag{2.11}$$

If $\widetilde{\Gamma}^c_{ab}$ (respectively $\widehat{\Gamma}^\gamma_{\alpha\beta}$) are the Christoffel symbols of the Riemannian connection $\widetilde{\nabla}$ (respectively $\widehat{\nabla}$), then we have $\widetilde{\nabla}_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^c} = \widetilde{\Gamma}^a_{cb} \frac{\partial}{\partial x^a}$ (respectively $\widehat{\nabla}_{\frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \widehat{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial u^\gamma}$). On the other hand, $\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \in T(M) \subset T(\overline{M}) \subset T(\widetilde{M})$. From (2.6) it follows that:

$$\widetilde{\nabla}_{\frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial u^\gamma} + h_{\alpha\beta} N_1 + \overline{h}_{\alpha\beta} N_2. \tag{2.12}$$

From (2.10) and (2.12) we have:

$$\Gamma^\gamma_{\alpha\beta} = \widehat{\Gamma}^\gamma_{\alpha\beta};\ \ h_{\alpha\beta} = \widehat{h}^1_{\alpha\beta};\ \ \overline{h}_{\alpha\beta} = \widehat{h}^2_{\alpha\beta}. \tag{2.13}$$

From (2.8) and (2.13) it follows that

$$\widehat{h}(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}) = h_{\alpha\beta} N_1 + \overline{h}_{\alpha\beta} N_2. \tag{2.14}$$

But $\widehat{g}_{\alpha\beta} = g_{\alpha\beta}$ so, we have $\widehat{g} = g$ and $\widehat{\nabla} \equiv \nabla$. Therefore

$$\widehat{A}_{N_1}(\frac{\partial}{\partial u^\alpha}) = \widehat{g}^{\gamma\beta}\widehat{h}^1_{\alpha\beta}\frac{\partial}{\partial u^\gamma} = A_{N_1}(\frac{\partial}{\partial u^\alpha}) \tag{2.15}$$

$$\widehat{A}_{N_2}(\frac{\partial}{\partial u^\alpha}) = \overline{A}_{N_2}(\frac{\partial}{\partial u^\alpha}) - \overline{g}(\overline{h}(\frac{\partial}{\partial u^\alpha}, N1), N_2)\cdot N_1. \tag{2.16}$$

Hence, if f_1 and $f_2 \in \mathcal{F}(\widetilde{M})$ and $X \in T_P M$ we have:

$$\widehat{A}_{(f_1 N_1 + f_2 N_2)}(X) = f_1 \cdot A_{N_1}(X) + f_2 \cdot \{\overline{A}_{N_2}(X) - \overline{g}(\overline{A}_{N_2}(N_1), X)N_1\}. \tag{2.17}$$

3 The main results

In this section we are concerning with the properties of the composition of two particular Riemannian immersions as, for instance, of two totally geodesic immersions or of two minimal immersions or of totally umbilical immersions. Finally, we put the results into a table. The definitions from the theory of submanifolds are in [2] or in [3]. If M is a n-dimensional manifold isometrically immersed in a $(n+p)$-dimensional manifold, let $(e_\alpha)_{\alpha=\overline{1,n}}$ be a local orthonormal basis in $T_P(M)$ and let $\{N_x\}_{x=\overline{1,p}}$ be a local orthonormal basis in $T_P^{\perp}(M)$. We denote by H the mean curvature vector field defined by

$$H = \frac{1}{n}\sum_{\alpha=1}^{n} h(e_\alpha, e_\alpha) \tag{3.1}$$

If $A_x := A_{N_x}$ then, from (3.1), we have

$$H = \frac{1}{n}\sum_{x=1}^{p} trace(A_x)\cdot N_x \tag{3.2}$$

Let M be a n-dimensional submanifold isometrically immersed in a $(n+1)$ dimensional submanifold $\overline{M}$ which is isometrically immersed in a $(n+2)$ dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$.

Lemma 3.1 *If H, $\overline{H}$ and respectively $\widehat{H}$ are the mean curvature vector fields of submanifold $M \hookrightarrow \overline{M}$, $\overline{M} \hookrightarrow \widetilde{M}$ and respectively $M \hookrightarrow \widetilde{M}$, then:*

$$\widehat{H} = H + \frac{n+1}{n}\overline{H} - \frac{1}{n}\overline{h}(N_1, N_1). \tag{3.3}$$

Proof: From (3.2) we have

$$\widehat{H} = \frac{1}{n}[trace(\widehat{A}_{N_1})\cdot N_1 + trace(\widehat{A}_{N_2})\cdot N_2] \tag{3.4}$$

From (2.16) and from $\overline{g}(N_1, \frac{\partial}{\partial u^\alpha}) = 0$ we have

$$trace(\widehat{A}_{N_2}) = \sum_{\alpha=1}^{n}\overline{g}(\overline{A}_{N_2}(\frac{\partial}{\partial u^\alpha}), \frac{\partial}{\partial u^\alpha}) \tag{3.5}$$

Therefore,

$$trace(\widehat{A}_{N_2}) = trace(\overline{A}_{N_2}) - \overline{g}(\overline{h}(N_1, N_1), N_2) \tag{3.6}$$

From (3.2) it results

$$H = \frac{1}{n} trace(A_{N_1}) \cdot N_1 \tag{3.7}$$

and from (3.6) we have

$$trace(\widehat{A}_{N_2}) \cdot N_2 = (n+1)\overline{H} - \overline{h}(N_1, N_1) \tag{3.8}$$

So, from (3.7), (3.8) we have (3.3). □

Theorem 3.1 *(B.Y Chen [2]) If (M, g) is totally geodesic in $(\overline{M}, \overline{g})$ and $(\overline{M}, \overline{g})$ is totally geodesic in $(\widetilde{M}, \widetilde{g})$, then (M, g) is totally geodesic in $(\widetilde{M}, \widetilde{g})$.*
Proof: If (M, g) (respectively $(\overline{M}, \overline{g})$) is totally geodesic in $(\overline{M}, \overline{g})$ (respectively in $(\widetilde{M}, \widetilde{g})$) so, we have $h_{\alpha\beta} = 0$ (respectively $\overline{h}_{\alpha\beta} = 0$). From (2.14) it follows that $\widehat{h}(\frac{\partial}{\partial u^\alpha}; \frac{\partial}{\partial u^\beta}) = 0$, therefore M is totally geodesic in $\widetilde{M}$. □

Theorem 3.2 *If (M, g) is minimal in $(\overline{M}, \overline{g})$ and $(\overline{M}, \overline{g})$ is totally geodesic in $(\widetilde{M}, \widetilde{g})$, then (M, g) is minimal in $(\widetilde{M}, \widetilde{g})$.*
Proof: If $(\overline{M}, \overline{g})$ is totally geodesic in $(\widetilde{M}, \widetilde{g})$ then $\overline{h}_{ij} = 0$ and we have $\overline{H} = 0$. From (3.3) it results that $\widehat{H} = H$. But (M, g) is minimal in $(\overline{M}, \overline{g})$ so $H = 0$, therefore $\widehat{H} = 0$ and (M, g) is minimal in $(\widetilde{M}, \widetilde{g})$.□

Theorem 3.3 *If (M, g) is totally umbilical in $(\overline{M}, \overline{g})$ and $(\overline{M}, \overline{g})$ is totally geodesic in $(\widetilde{M}, \widetilde{g})$, then (M, g) is totally umbilical in $(\widetilde{M}, \widetilde{g})$.*
Proof: If (M, g) is totally umbilical in $(\overline{M}, \overline{g})$ then $A_{N_1}(\frac{\partial}{\partial u^\alpha}) = \lambda_1 \frac{\partial}{\partial u^\alpha}$. If $(\overline{M}, \overline{g})$ is totally geodesic in $(\widetilde{M}, \widetilde{g})$ then $\overline{h}_{ij}$=0 and from this it results $\overline{A}_{N_2} = 0$. Thus, from (2.17) we have

$$\widehat{A}_{(f_1 N_1 + f_2 N_2)}(\frac{\partial}{\partial u^\alpha}) = f_1 A_{N_1}(\frac{\partial}{\partial u^\alpha}) \tag{3.9}$$

(for any $f_1, f_2 \in \mathcal{F}(\widetilde{M})$). Therefore, it results $\widehat{A}_{(f_1 N_1 + f_2 N_2)}(X) = f_1 \lambda_1 X$, for all $X = X^\alpha \frac{\partial}{\partial u^\alpha} \in T_P(M)$. Hence, (M, g) is totally umbilical in $(\widetilde{M}, \widetilde{g})$. □

Theorem 3.4 *If (M, g) is totally umbilical in $(\overline{M}, \overline{g})$ and $(\overline{M}, \overline{g})$ is totally umbilical in $(\widetilde{M}, \widetilde{g})$, then (M, g) is totally umbilical in $(\widetilde{M}, \widetilde{g})$.*
Proof: If (M, g) is totally umbilical in $(\overline{M}, \overline{g})$ then $A_{N_1}(\frac{\partial}{\partial u^\alpha}) = \lambda_1 \frac{\partial}{\partial u^\alpha}$ and from $(\overline{M}, \overline{g})$ totally umbilical in $(\widetilde{M}, \widetilde{g})$ we have $\overline{A}_{N_2}(X) = \lambda_2 X$, where X is in $T_P\overline{M}$. From (2.17) it follows that:

$$\widehat{A}_{(f_1 N_1 + f_2 N_2)}(\frac{\partial}{\partial u^\alpha}) = (f_1\lambda_1 + f_2\lambda_2)\frac{\partial}{\partial u^\alpha} - f_2 \overline{g}(\overline{A}_{N_2} N_1, \frac{\partial}{\partial u^\alpha}) N_1 \tag{3.10}$$

But $\overline{A}_{N_2} N_1 = \lambda_2 N_1$ and $\overline{g}(\lambda_2 N_1, \frac{\partial}{\partial u^\alpha}) = 0$ and from these we have
$\widehat{A}_{(f_1 N_1 + f_2 N_2)}(X) = \lambda X$, where $X = X^\alpha \frac{\partial}{\partial u^\alpha} \in T_P(M)$ and $\lambda = f_1\lambda_1 + f_2\lambda_2$. Therefore, (M, g) is totally umbilical in $(\widetilde{M}, \widetilde{g})$. □

Theorem 3.5 *If (M,g) is totally geodesic in $(\overline{M},\overline{g})$ and $(\overline{M},\overline{g})$ is totally umbilical in $(\widetilde{M},\widetilde{g})$, then (M,g) is totally umbilical in $(\widetilde{M},\widetilde{g})$.*
Proof: If (M,g) is totally geodesic in $(\overline{M},\overline{g})$ then $A_{N_1}(\frac{\partial}{\partial u^\alpha})=0$. Since $(\overline{M},\overline{g})$ is totally umbilical in $(\widetilde{M},\widetilde{g})$, from (2.17) it follows that $\widehat{A}_{(f_1N_1+f_2N_2)}(X)=f_2\lambda_2 X$, where $X=X^\alpha\frac{\partial}{\partial u^\alpha}\in T_P(M)$. Therefore, (M,g) is totally umbilical in $(\widetilde{M},\widetilde{g})$.□

Theorem 3.6 *If (M,g) is minimal in $(\overline{M},\overline{g})$ and $(\overline{M},\overline{g})$ is minimal in $(\widetilde{M},\widetilde{g})$ then (M,g) is minimal in $(\widetilde{M},\widetilde{g})$ if and only if N_1 is an asimptotic section for $\overline{h}$ (i.e. $h(N_1,N_1)=0$).*
Proof: Since (M,g) is minimal in $(\overline{M},\overline{g})$ and $(\overline{M},\overline{g})$ is minimal in $(\widetilde{M},\widetilde{g})$ then $H=\overline{H}=0$. From (3.3) we have $\widehat{H}=-\frac{1}{n}h(N_1,N_1)$. Thus, $\widehat{H}=0$ if and only if $h(N_1,N_1)=0$. Therefore (M,g) is minimal in $(\widetilde{M},\widetilde{g})$ if and only if $h(N_1,N_1)=0$. □

I°. f (M,g) is minimal in $(\overline{M},\overline{g})$ and (M,g) is minimal in $(\widetilde{M},\widetilde{g})$, then $(\overline{M},\overline{g})$ is minimal in $(\widetilde{M},\widetilde{g})$ if and only if N_1 is an asimptotic section for $\overline{h}$. I°. f $(\overline{M},\overline{g})$ is minimal in $(\widetilde{M},\widetilde{g})$ and (M,g) is minimal in $(\widetilde{M},\widetilde{g})$ then (M,g) is minimal in $(\overline{M},\overline{g})$ if and only if N_1 is an asimptotic section for $\overline{h}$. I°. f (M,g) is totally geodesic in $(\overline{M},\overline{g})$ and $(\overline{M},\overline{g})$ is minimal in $(\widetilde{M},\widetilde{g})$, then (M,g) is minimal in $(\widetilde{M},\widetilde{g})$ if and only if N_1 is an asimptotic section for $\overline{h}$. I°. f (M,g) is minimal in $(\overline{M},\overline{g})$ then the mean curvature vector field $\widehat{H}$ of (M,g) in $(\widetilde{M},\widetilde{g})$ is normal to $\overline{M}$ in $\widetilde{M}$.
Indeed, if (M,g) is minimal in $(\overline{M},\overline{g})$ then, from (3.3), it follows that
$\widehat{H}=\frac{n+1}{n}\cdot\overline{H}-\frac{1}{n}\overline{h}(N_1,N_1)$. Hence, $\widehat{H}$ is normal to $\overline{M}$ in $\widetilde{M}$. □

Finally, from the theorems above, we can put the results into the table which has on the first column the properties of the immersion $M\hookrightarrow\overline{M}$, on the first line it has the properties of the immersion $\overline{M}\hookrightarrow\widetilde{M}$ and the other cells contain the properties of the composition of these immersions (thus, it results the properties of the immersion $M\hookrightarrow\widetilde{M}$). We denote by tg = totally geodesic submanifold, tu = totally umbilical submanifold and m = minimal submanifold and m^*=minimal submanifold if and only if $\overline{h}(N_1,N_1)=0$.

M in $\overline{M}\setminus\overline{M}$ in $\widetilde{M}$	tg	tu	m
tg	tg	tu	m^*
tu	tu	tu	□
m	m	□	m^*

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