The iso-taxicab Gauss curvature

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Abstract. The Iso-Taxicab geometry is a non-Euclidean Geometry. It was defined in 1989 by K.O. Sowell. The aim of this paper is to introduce the Iso-Taxicab Gauss Curvature. We shall prove that in the case of $I-IV$ or $II-V$ orientation $K=0$. But, in the case of $III-VI$ orientation $K<0$.

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1 Introduction

The most popular, well known geometry is the Euclidean Geometry. Non-Euclidean geometries are usually difficult. In 1975, E. F. Krause introduced a new geometry, *taxicab geometry*, by using the metric

$$d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|$$

for $A=(a_1,a_2),\ B=(b_1,b_2)$ in Euclidean plane. The Taxicab Geometry is a non-Euclidean Geometry. It is, on the contrary, very easy to understand and has many beatiful applications in human life [3].

In 1989, using the similar argument, K. O. Sowell has introduced a new non-Euclidean geometry, *Iso-Taxicab geometry.* As it is mentioned by Sowell in [9] that in iso-taxicab geometry three distance functions arise depending upon the relative positions of the points A and B. At the origin three axes occur : the x-axis, the y-axis and the y'-axis. This latter axis, forms an angle of 60^0 with the x-axis and the y-axis. The three axes separate the plane into six regions called hextants. These hextants will be numbered $I-VI$ in a counterclockwise direction (Figure 1).

Points will still be named by ordered pairs $A=(a_1,a_2)$ of real numbers with respect to the x-axis and y-axis. At any point in the plane three lines may be drawn parallel to the axes. Depending on the position of the points, any two points may have $I-IV$, $II-V$ or $III-VI$ orientation to one another. If, for example, $A\in I$, $B\in IV$, then the line $[AB$ has $I-IV$ orientation. With this, the distance function for iso-taxicab geometry can be written as:

$$d_I(A,B) = \begin{cases} (i) & |a_1-b_1|+|a_2-b_2| \quad , \quad I-IV \ \text{ orientation} \\ (ii) & |a_2-b_2| \quad , \quad II-V \ \text{ orientation} \\ (iii) & |a_1-b_1| \quad , \quad III-VI \ \text{ orientation} \end{cases}$$

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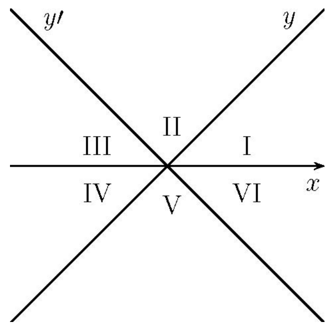

Figure 1:

If the two points lie on a line parallel to the x-axis, then formula (iii) is used; if the two points lie on a line parallel to the y-axis or to the y'-axis, then formula (ii) is used.

The definition of the inner-product and the norm in taxicab geometry is given in [2] and the definition of the inner-product in iso-taxicab geometry is given in [1]. We recall these definitions and then we will define the norm in iso-taxicab geometry.

If the two points lie on a line parallel to the x-axis, then formula (iii) is used; if the two points lie on a line parallel to the y-axis or to the y'-axis, then formula (ii) is used.

The similarities and differences of taxicab and iso-taxicab geometries are also given in [9]. Let's remind some of them :

(1). Because angle measure is not dependent upon the distance function, angles may be measured as they are in both Euclidean and Taxicab Geometry.

(2). Only Euclidean plane geometry has a rotation invariant metric; therefore one should expect properties involving angle measurement to be altered in taxicab geometries. Specifically, the side-angle-side axiom may be assumed in neither iso-taxicab geometry nor in taxicab geometry. Of course, any theorem which relies on the side-angle-side axiom is also invalid in each of these taxicab geometries. Among these are the angle-side-angle theorem and the side-side-side theorem.

(3). The taxicab circle is a square and $\pi_T = 4$. The iso-taxicab circle is hexagon. Because the circumference of iso-taxicab circle is six times the radius, $\pi_I = 3$.

(4). Using an equilateral triangle as a unit of area, we can find areas of figures in iso-taxicab geometry. It is easy to prove that the formula for the area of an iso-taxicab circle is $A = 2 \cdot \pi_I \cdot r^2$ "equi-tri" units, whereas the formula for the area of a taxicab circle is $A = \frac{1}{2} \cdot \pi_T \cdot r^2$ square units.

2 The Iso-taxicab Inner Product

Let's first give the 21 cases that α and β be in.

	1	2	3	4	5	6	7	8	9	10	11
α	I	II	III	IV	V	VI	II	III	III	IV	IV
β	I	II	III	IV	V	VI	I	I	II	I	II

	12	13	14	15	16	17	18	19	20	21
α	IV	V	V	V	V	VI	VI	VI	VI	VI
β	III	I	II	III	IV	I	II	III	IV	V

Diagram 1.

Definition 1 *Let $\alpha = (a_1, a_2),\ \beta = (b_1, b_2) \in R^2$. On the account of Diagram 1, we define the iso-taxicab inner-product by*

$$\langle \alpha, \beta \rangle_I = \begin{cases} a_1b_1 + a_2b_2 & , \quad 1,4,7,8,9,10,11,12,13,15,16,17,18,20,21 \\ a_2b_2 & , \quad 2,5,14 \\ a_1b_1 & , \quad 3,6,19 \end{cases}$$

3 The Iso-taxicab Gauss Curvature

In this final section, we define the Gauss Curvature of Iso-taxicab plane.

Definition 2 *(The Iso-Taxicab Unit Circle)*
It is clear that the iso-taxicab unit circle is hextant. The formula for iso-taxicab unit circle can be given as

$$C_I \quad = \quad \begin{cases} |x| + |y| = 1 & , \quad I - IV \quad \textit{orientation} \\ |y| = 1 & , \quad II - V \quad \textit{orientation} \\ |x| = 1 & , \quad III - VI \quad \textit{orientation} \end{cases}$$

Considering the circle of radius r, we can define the following polar parametrization of Iso-taxicab plane.

Definition 3 *Let (u, v) be any point in R_I^2. We define the polar parametrization of (u, v) as*

$$\varphi(u,v) \quad = \quad \begin{cases} (u\cos v,\ u\sin v) & , \quad I - IV \quad \textit{orientation} \\ (u\cos v,\ u) & , \quad II - V \quad \textit{orientation} \\ (-u,\ u\sin v) & , \quad III - VI \quad \textit{orientation} \end{cases}$$

Using this parametrization, let's now compute the Gauss Curvature.

Theorem 1. *The Gauss Curvature K of Iso-Taxicab plane is 0 in the case of $I-IV$ orientation or $II-V$ orientation, and negative in the case of $III-VI$ orientation.*

Proof. Notice that we have three cases.
Case 1. ($I-IV$ orientation).

$$\varphi(u,v)=(u\cos v,\ u\sin v)$$

Thus

$$\begin{aligned}\varphi_u &= (\cos v,\ \sin v)\\ \varphi_v &= (-u\sin v,\ u\cos v)\end{aligned}$$

Obviously φ_u has $I-IV$ orientation and φ_v has $II-V$ or $III-VI$ orientation. Hence

$$\begin{aligned}||\varphi_u|| &= 1,\\ ||\varphi_v|| &= u\cos v \text{ or } ||\varphi_v||=u\sin v\\ \text{and} &< \varphi_u,\varphi_v >=0.\end{aligned}$$

Therefore, $\{\varphi_u,\varphi_v\}$ is an orthogonal basis. If

$$\begin{aligned}E_1 &= \varphi_u\\ E_2 &= \frac{1}{u\cos v}\varphi_v\end{aligned}$$

then $\{E_1,E_2\}$ becomes orthonormal. Let $\{\theta_1,\theta_2\}$ be the dual base to $\{E_1,E_2\}$. Thus,

$$\begin{aligned}\theta_1 &= du\\ \theta_2 &= u\cos v dv\end{aligned}$$

By using the first structural equations [5], we obtain connection 1-form as $w_{12}=\cos v dv$. Hence

$$\begin{aligned}d\theta_1 &= w_{12}\curlywedge\theta_2\\ d\theta_2 &= w_{21}\curlywedge\theta_1\end{aligned}$$

So

$$dw_{12}=0$$

We know that [5]

$$dw_{12}=-K\ \theta_1\curlywedge\theta_2$$

Using this fact, we get

$$K=0$$

as desired.

In the case of $III-VI$ orientation for φ_v, $||\varphi_v|| = u\sin v$, the proof is similar.

Case 2. $(II-V$ orientation).

$$\varphi(u,v) = (u\cos v,\ u)$$

Thus

$$\begin{aligned}\varphi_u &= (\cos v,\ 1)\\ \varphi_v &= (-u\sin v,\ 0)\end{aligned}$$

Here notice that φ_u has $II-V$ orientation and φ_v has $III-VI$ or $I-IV$ orientation. Suppose φ_v has $III-VI$ orientation (Case of $I-IV$ orientation is the same). Hence

$$\begin{aligned}||\varphi_u|| &= 1,\\ ||\varphi_v|| &= u\sin v \text{ and}\\ &< \varphi_u, \varphi_v >= 0.\end{aligned}$$

Therefore, $\{\varphi_u, \varphi_v\}$ is an orthogonal basis. Let $\{\theta_1, \theta_2\}$ be the dual base to $\{E_1, E_2\}$.Hence,

$$\begin{aligned}&\theta_1 = dv\\ &\theta_2 = -du + \cos v dv\end{aligned}$$

As a result, on account of the structural equations, we obtain

$$K = 0$$

as desired.

Case 3. $(III-VI$ orientation).

$$\varphi(u,v) = (-u,\ u\sin v)$$

Thus

$$\begin{aligned}\varphi_u &= (-1, \sin v)\\ \varphi_v &= (0, u\cos v)\end{aligned}$$

So, φ_u has $III-VI$ orientation and φ_v has $II-V$ or $I-IV$ orientation. In either case,

$$\begin{aligned}||\varphi_u|| &= 1,\\ ||\varphi_v|| &= u\cos v \text{ and}\\ &< \varphi_u, \varphi_v >= 0.\end{aligned}$$

Therefore, $\{\varphi_u, \varphi_v\}$ is an orthogonal basis. If

$$\begin{aligned}E_1 &= \varphi_u = (-1, \sin v)\\ E_2 &= \frac{1}{u\cos v}\varphi_v = (0, 1)\end{aligned}$$

then $\{E_1, E_2\}$ becomes orthonormal. If $\{\theta_1, \theta_2\}$ is the dual base to $\{E_1, E_2\}$, then

$$\theta_1 = -du$$
$$\theta_2 = \sin v du + dv$$

If w_{12} is the connection 1-form, using the first structural equations we get

$$\begin{aligned} w_{12} &= \cos v \sin v du + \cos v dv \\ dw_{12} &= (-\sin^2 v + \cos^2 v) dv \curlywedge du \end{aligned}$$

On the other hand [5],

$$dw_{12} = -K\ \theta_1 \curlywedge \theta_2$$

an hence

$$K = \sin^2 v - \cos^2 v.$$

Notice that for $v \in III$ or $v \in VI$, $\cos^2 v = 1$ and $\sin^2 v < 1$. As a result, we get $K < 0$, as claimed. □

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