Spacelike Constant Mean Curvature surfaces of revolution in Minkowski 3-Space

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Abstract. This paper studies various ordinary differential equations that characterize spacelike constant mean curvature surfaces of revolution in Minkowski 3-space. Those differential equations are nonlinear and cannot be solved explicitly. Using numerical methods such as Runge-Kutta's or Euler's methods, we solve those differential equations and obtain examples of spacelike constant mean curvature surfaces of revolution in Minkowski 3-space.

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1 Introduction

Surfaces of revolution in Euclidean 3-space, namely Delaunay surfaces, unduloids, and nodoids have been studied for a long time and many examples of such surfaces have been discovered. On the other hand, Minkowski 3-space has more complicated geometric structures compared to Euclidean 3-space. In particular, Minkowski 3-space has distinguished axes of rotation, namely, *spacelike*, *timelike*, and *lightlike axes* (or *null axes*). Thus, we can consider three different kinds of rotations; rotations about spacelike, timelike, and lightlike axes. Especially, the rotations about spacelike axes (the so-called *Lorentz boosts*, which are regarded as *rotations between space and time directions*), and the rotation about timelike axis (this is similar to the conventional rotation in Euclidean 3-space) are the parity and time preserving linear transformations of Minkowski 3-space. Moreover, these linear transformations are isometries (the so-called *Lorentz transformations* in special relativity), i.e., they preserve the flat Lorentzian metric in Minkowski 3-space. The rotations about spacelike and timelike axes form a group called the special Lorentz group $\mathrm{SO}(2,1)$. As is well known in special relativity, the Lorentz transformation is a natural consequence of Albert Einstein's famous postulate of the constancy of the speed of light in flat (vacuum) Minkowski 4-spacetime.

Due to the complexity of the underlying geometry of Minkowski 3-space, one can expect much richer geometric and topological properties of surfaces of revolution

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in Minkowski 3-space. Such surfaces have been studied by a number of differential geometers. In [6], for example, L. McNertney classified spacelike and timelike surfaces of revolution about spacelike, timelike, and lightlike axes. In [4], J. Hano and K. Nomizu also classified constant mean curvature spacelike surfaces of revolution by studying profile curves. In the case of the revolution about the spacelike and timelike axes, the profile curve is obtained by rolling the focus of a quadratic curve along the axis of rotation, similarly to the Delaunay surfaces in Euclidean 3-space. They also studied equations for the profile curves when the axis of revolution is lightlike.

In this paper, we study various ordinary differential equations that characterize spacelike constant mean curvature surfaces of revolution in Minkowski 3-space. Those differential equations are nonlinear and cannot be solved explicitly. Using numerical methods such as *Runge-Kutta's* or *Euler's methods*, we solve those differential equations and obtain examples of spacelike constant mean curvature surfaces of revolution in Minkowski 3-space. Numerical computations and the graphics of curves and surfaces are done with the aid of the software *Maple* 9.5.

The second named author J. H Varnado is a junior undergraduate mathematics and physics major. The main part of this paper came out of his undergraduate research under the guide and supervison of the first named author during July, 2004 – December 2004.

2 Spacelike Surfaces in Minkowski 3-Space $\mathbb{E}^3_1$

Let $\mathbb{E}^3_1$ be Minkowski 3-space with linear coordinates ξ_0, ξ_1, ξ_2 and the standard flat Lorentzian metric $ds^2 = -(d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2$.

Definition 1 *Let $\langle \cdot, \cdot \rangle$ denote the inner product induced by the Lorentzian metric. Let v be a vector in $\mathbb{E}^3_1$. Then v is said to be*

1. spacelike *if $\langle v, v \rangle > 0$,*
2. null or lightlike *if $\langle v, v \rangle = 0$, and*
3. timelike *if $\langle v, v \rangle < 0$.*

Let M be a connected 2-manifold and $\varphi = (\varphi_1, \varphi_2, \varphi_3) : M \longrightarrow \mathbb{E}^3_1$ an immersion in $\mathbb{E}^3_1$. The immersion φ is said to be *spacelike* if the induced metric on M is *Riemannian*, i.e., positive definite. This inuduced metric I determines a conformal structure on M. More specifically,

Definition 2 *$\varphi : M \longrightarrow \mathbb{E}^3_1$ is said to be* conformal *if*

$$\langle \varphi_x, \varphi_y \rangle = 0, \tag{2.1}$$

$$|\varphi_x| = |\varphi_y| = e^{\frac{u}{2}}, \tag{2.2}$$

where (x, y) is a local coordinate system in M and $u : M \longrightarrow \mathbb{R}$ is a real-valued function from M.

Let $z = x + iy$ be a local complex coordinate of M. Then we can regard M as a Riemann surface with respect to this conformal structure.

Let $\varphi : M \longrightarrow \mathbb{E}^3_1$ be a conformal spacelike immersion. Then the induced metric is given by

$$I = e^u(dx^2 + dy^2) = e^u dz d\bar{z}.$$

In terms of the complex coordinates $z, \bar{z}$, the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are computed to be

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Proposition 1 *The conformality conditions* (2.1) *and* (2.2) *are equivalent to*

$$\langle \varphi_z, \varphi_z \rangle = \langle \varphi_{\bar{z}}, \varphi_{\bar{z}} \rangle = 0, \tag{2.3}$$

$$\langle \varphi_z, \varphi_{\bar{z}} \rangle = \frac{1}{2}e^u. \tag{2.4}$$

Note that if N is the unit normal vector field of $\varphi : M \longrightarrow \mathbb{E}^3_1$, then $\langle N, N \rangle = -1$, $\langle N, \varphi_x \rangle = \langle N, \varphi_y \rangle = 0$.

Definition 3 *Let $v = (v_0, v_1, v_2)$ and $w = (w_0, w_1, w_2)$ be two (spacelike or timelike) vectors in $\mathbb{E}^3_1$. Then the* Lorentzian cross product *is defined to be*

$$v \times w = \begin{vmatrix} -e_0 & e_1 & e_2 \\ v_0 & v_1 & v_2 \\ w_0 & w_1 & w_2 \end{vmatrix}, \tag{2.5}$$

where $e_0 = (1, 0, 0), e_1 = (0, 1, 0), e_2 = (0, 0, 1)$.

Remark 1 *One can easily see that $v \times w$ is, in fact, a normal vector to both v and w.*

Proposition 2 *Let $\varphi : M \longrightarrow \mathbb{E}^3_1$ be an immersed (spacelike or timelike) surface in $\mathbb{E}^3_1$. Then*

$$||\varphi_x \times \varphi_y||^2 = \langle \varphi_x, \varphi_y \rangle - ||\varphi||^2 ||\varphi_y||^2. \tag{2.6}$$

Proof. Straightforward by a direct computation. □

Let N denote a unit normal vector field of a spacelike immersion $\varphi : M \longrightarrow \mathbb{E}^3_1$. Let us define the following quantities to do some differential geometric computations:

$$E = \langle \varphi_u, \varphi_u \rangle, F = \langle \varphi_u, \varphi_v \rangle, G = \langle \varphi_v, \varphi_v \rangle$$
$$l = \langle \varphi_{uu}, N \rangle, m = \langle \varphi_{uv}, N \rangle, n = \langle \varphi_{vv}, N \rangle.$$

The equation (2.6) is also written as

$$||\varphi_x \times \varphi_y||^2 = F^2 - EG. \tag{2.7}$$

Remark 2 *Since* φ *is a spacelike surface, both tangent vectors* φ_x *and* φ_y *are spacelike vectors. So, the normal vector* $\varphi_x \times \varphi_y$ *is a timelike vector, i.e.,* $F^2 - EG < 0$. *Hence, the norm* $||\varphi_x \times \varphi_y||$ *is defined to be the* proper time

$$||\varphi_x \times \varphi_y|| := \sqrt{-||\varphi_x \times \varphi_y||^2} = \sqrt{EG - F^2} \tag{2.8}$$

and accordingly, the unit normal vector field N *of* φ *is given by*

$$N = \frac{\varphi_x \times \varphi_y}{\sqrt{EG - F^2}}. \tag{2.9}$$

In physics, massless particles move along null curves and massive particles move along timelike curves. The proper time is the actual time elapsed on a physical clock carried along the timelike curve.

The following formula is well-known in differential geometry of surfaces in Euclidean 3-space. The same formula still holds for (spacelike or timelike) surfaces in Minkowski 3-space $\mathbb{E}^3_1$.

Proposition 3 *The mean curvature* H *of a (spacelike or timelike) immersion* $\varphi : M \longrightarrow \mathbb{E}^3_1$ *is computed to be*

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)}. \tag{2.10}$$

Proof. The formula (2.10) is proved exactly the same way as in the Euclidean case such as, for instance, the elementary proof given in [7]. □

3 Spacelike Surfaces of Revolution in $\mathbb{E}^3_1$

In this section, we characterize spacelike surfaces of revolution about spacelike axes in Minkowski 3-space.

Proposition 4 *There are three distinguished types of rotations in* $\mathbb{E}^3_1$*; rotations about spacelike axes* ξ_1, ξ_2, *rotations about timelike axis* ξ_0, *rotations about null axes* $\xi_0 \pm \xi_1$, $\xi_0 \pm \xi_2$. *These rotations are characterized by the following rotational matrices:*

1. *The matrix corresponds to the rotation about* ξ_1 *is*

$$\begin{pmatrix} \cosh y & \sinh y & 0 \\ \sinh y & \cosh y & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

2. *The matrix corresponds to the rotation about* ξ_2 *is*

$$\begin{pmatrix} \cosh y & 0 & \sinh y \\ 0 & 1 & 0 \\ \sinh y & 0 & \cosh y \end{pmatrix}. \tag{3.2}$$

3. *The matrix corresponds to the rotation about ξ_0 is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos y & -\sin y \\ 0 & \sin y & \cos y \end{pmatrix} \tag{3.3}$$

4. *The matrix corresponds to the rotation about $\xi_0 + \xi_1$ is*

$$\begin{pmatrix} 1+\frac{y^2}{2} & -\frac{y^2}{2} & y \\ \frac{y^2}{2} & 1-\frac{y^2}{2} & y \\ y & -y & 1 \end{pmatrix}. \tag{3.4}$$

5. *The matrix corresponds to the rotation about $\xi_0 - \xi_1$ is*

$$\begin{pmatrix} 1+\frac{y^2}{2} & \frac{y^2}{2} & -y \\ -\frac{y^2}{2} & 1-\frac{y^2}{2} & y \\ -y & -y & 1 \end{pmatrix}. \tag{3.5}$$

6. *The matrix corresponds to the rotation about $\xi_0 + \xi_2$ is*

$$\begin{pmatrix} 1+\frac{y^2}{2} & y & -\frac{y^2}{2} \\ y & 1 & -y \\ \frac{y^2}{2} & y & 1-\frac{y^2}{2} \end{pmatrix}. \tag{3.6}$$

7. *The matrix corresponds to the rotation about $\xi_0 - \xi_2$ is*

$$\begin{pmatrix} 1+\frac{y^2}{2} & -y & \frac{y^2}{2} \\ -y & 1 & -y \\ -\frac{y^2}{2} & y & 1-\frac{y^2}{2} \end{pmatrix}. \tag{3.7}$$

Proof. For instance, let A be a 3×3 rotational matrix corresponds to the rotation about the spacelike axis ξ_1. Then this rotation fixes the spacelike vector $e_1 = (0, 1, 0)$. The matrix A is a Lorentz transformation, i.e., it preserves the Lorentzian metric (*Lorentz isometry*). Hence, the matrix A can be found by solving the following equations simultaneously:

1. $A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
2. $A\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
3. $\det A = 1$.

The other rotational matrices are found in the same manner. $\square$

Remark 3 *The rotational matrices* (3.1) *and* (3.2) *are called the* Lorentz boosts. *These are regarded as rotations take place between space and time.*

By the formula (2.10), we can easily prove the following theorem which characterizes spacelike constant mean curvature surfaces of revolution in $\mathbb{E}^3_1$.

Theorem 5 *Let* $\varphi : M \longrightarrow \mathbb{E}^3_1$ *be an immersed spacelike surface of revolution with constant mean curvature* H *in* $\mathbb{E}^3_1$. *Then* φ *is parameterized in one of the following ways.*

1. φ *is a spacelike surface of revolution about the spacelike axis* ξ_1*:*

   $$\varphi(x,y) = (h(x)\cosh y, x, h(x)\sinh y), \tag{3.8}$$

   where $(h(x), x, 0)$ *is a profile curve in* $\xi_0\xi_1$*-plane with* $h(x) > 0$.

   $h(x)$ *satisfies the differential equation*

   $$H = \frac{1}{2}\frac{h^{''}h - (h^{'})^2 + 1}{h(1-(h^{'})^2)^{\frac{3}{2}}}. \tag{3.9}$$

2. φ *is a spacelike surface of revolution about the spacelike axis* ξ_2*:*

   $$\varphi(x,y) = (h(x)\cosh y, h(x)\sinh y, x), \tag{3.10}$$

   where $(h(x), 0, x)$ *is a profile curve in* $\xi_0\xi_2$*-plane with* $h(x) > 0$.

   $h(x)$ *satisfies the differential equation*

   $$H = -\frac{1}{2}\frac{h^{''}h - (h^{'})^2 + 1}{h(1-(h^{'})^2)^{\frac{3}{2}}}. \tag{3.11}$$

3. φ *is a spacelike surface of revolution about the timelike axis* ξ_0*:*

   $$\varphi(x,y) = (x, h(x)\cos y, h(x)\sin y), \tag{3.12}$$

   where $(x, 0, h(x))$ *is a profile curve in* $\xi_0\xi_2$*-plane with* $h(x) > 0$.

   $h(x)$ *satisfies the differential equation*

   $$H = \frac{1}{2}\frac{h^{''}h - (h^{'})^2 + 1}{h((h^{'})^2 - 1)^{\frac{3}{2}}}. \tag{3.13}$$

   For the rotations about null axes, we consider only $\xi_0 + \xi_1$ *as our conventional axis.*

4. φ *is a spacelike surface of revolution about the null axis* $\xi_0 + \xi_1$*:*

   $$\varphi(x,y) = (\varphi_0(x,y), \varphi_1(x,y), \varphi_2(x,y)), \tag{3.14}$$

where

$$\varphi_0(x,y) = x\left(1+\frac{y^2}{2}\right) - \frac{1}{2}h(x)y^2,$$
$$\varphi_1(x,y) = \frac{xy^2}{2} + h(x)\left(1-\frac{y^2}{2}\right),$$
$$\varphi_2(x,y) = xy - h(x)y.$$

The profile curve is $(x, h(x), 0)$ *with* $h(x) > 0$.

$h(x)$ *satisfies the differential equation*

$$H = -\frac{1}{2}\frac{(x-h)h^{''} - (h^{'}-1)((h^{'})^2-1)}{(x-h)((h^{'})^2-1)^{\frac{3}{2}}}. \tag{3.15}$$

4 Spacelike maximal surfaces of revolution in $\mathbb{E}_1^3$

A conformal spacelike immersion $\varphi : M \longrightarrow \mathbb{E}_1^3$ is said to be *maximal* if $H = 0$. Such a surface locally maximizes its surface area. See [2] for more details. Spacelike maximal surfaces of revolution in $\mathbb{E}_1^3$ can be found by solving those differential equations in Theorem 5 with $H = 0$.

It should be remarked that spacelike maximal surfaces of revolution have been completely classified by L. McNertney in her Ph.D. thesis [6].

In this section, we assume that $\varphi : M \longrightarrow \mathbb{E}_1^3$ is an oriented conformal spacelike immersion from a Riemann surface M into $\mathbb{E}_1^3$.

4.1 Spacelike maximal surfaces of revolution about spacelike axes in $\mathbb{E}_1^3$

In this subsection, we consider spacelike maximal surfaces of revolution about ξ_2 axis.

Set $H = 0$ in the equation (3.11). Then

$$\frac{h^{''}h - (h^{'})^2 + 1}{h(1-(h^{'})^2)^{\frac{3}{2}}} = 0. \tag{4.1}$$

Note that, in this case, the quantities E, F, G are computed to be

$$E = -(h^{'})^2 + 1,\ F = 0,\ G = h^2.$$

The conformality condition (2.2) implies then

$$-(h^{'})^2 + 1 = h^2.$$

This reduces the equation (4.1) to the equation of a simple harmonic oscillator

$$h^{''} + h = 0,$$

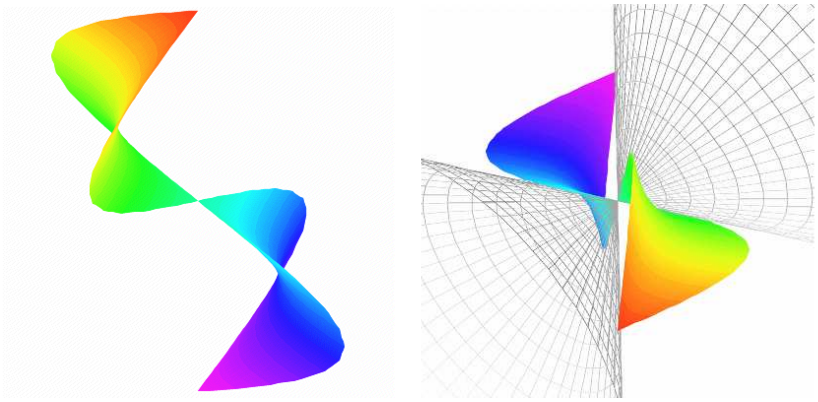
whose solution is

$$h(x) = C_1 \cos x + C_2 \sin x.$$

Hence, by (3.10) spacelike maximal surfaces of revolution about ξ_2 axis is given by

$$\varphi(x,y) = ((C_1 \cos x + C_2 \sin x)\cosh y, (C_1 \cos x + C_2 \sin x)\sinh y, x).$$

Figure 1 shows an example of such surfaces with $C_1 = 0$ and $C_2 = 1$. In particular, the picture (b) in figure 1 shows the surface with the *light cone* so it helps to see how the surface is positioned in $\mathbb{E}^3_1$ and where the rotation takes place.



(a) (b)

Figure 1: $\varphi(x,y) = (\sin x \cosh y, \sin x \sinh y, x)$

4.2 Spacelike maximal surfaces of revolution about timelike axis

Recall that the mean curvature formula for spacelike surfaces of revolution about ξ_0 axis is (3.13). Set $H = 0$. Then

$$\frac{h^{''}h - (h^{'})^2 + 1}{h((h^{'})^2 - 1)^{\frac{3}{2}}} = 0. \tag{4.2}$$

In this case, the conformality condition (2.2) is

$$(h^{'})^2 - 1 = h^2.$$

This reduces (4.2) to the equation of a simple harmonic oscillator

$$h^{''} - h = 0,$$

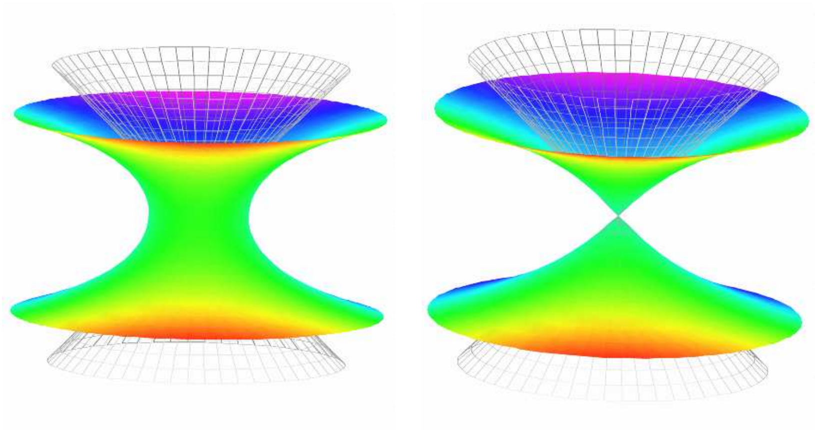
whose solution is

$$h(x) = C_1 e^x + C_2 e^{-x}.$$

Hence, by (3.12) spacelike maximal surfaces of revolution about ξ_0 axis is given by

$$\varphi(x,y) = (x, (C_1 e^x + C_2 e^{-x})\cos y, (C_1 e^x + C_2 e^{-x})\sin y).$$

Figure 2 shows examples of such surfaces. $C_1 = C_2 = \frac{1}{2}$ are used in picture (a) and $C_1 = \frac{1}{2}, C_2 = -\frac{1}{2}$ are used in picture (b). Those surfaces in pictures (a) and (b) are usually called *spacelike catenoid*. In particular, notice the similarity of the surface in picture (a) to the Euclidean catenoid.



(a) (b)

Figure 2: Spacelike maximal surfaces of revolution about ξ_0 axis

4.3 Spacelike maximal surfaces of revolution about null axes

In this subsection, we consider spacelike maximal surfaces of revolution about the null axis $\xi_0 + \xi_1$.

In the equation (3.15), set $H = 0$. Then

$$\frac{(x-h)h^{''} - (h^{'}-1)((h^{'})^2-1)}{(x-h)((h^{'})^2-1)^{\frac{3}{2}}} = 0. \tag{4.3}$$

In this case, the conformality condition (2.2) is

$$(h^{'})^2 - 1 = (x-h)^2.$$

This reduces (4.3) to the equation

$$h^{''} - (x-h)(h^{'}-1) = 0,$$

whose solution is

$$h(x) = x - C_1 \tan\frac{C_1}{2}(x - C_2).$$

Hence, by (3.14) the coordinate functions $\varphi_0, \varphi_1, \varphi_2$ of spacelike maximal surfaces of revolution about $\xi_0 + \xi_1$ axis are given by

$$\begin{aligned}
\varphi_0(x,y) &= x\left(1+\frac{y^2}{2}\right) - \frac{y^2}{2}\left(x - C_1 \tan\frac{C_1}{2}(x - C_2)\right),\\
\varphi_1(x,y) &= \frac{xy^2}{2} + \left(1-\frac{y^2}{2}\right)\left(x - C_1 \tan\frac{C_1}{2}(x - C_2)\right),\\
\varphi_2(x,y) &= xy - y\left(x - C_1 \tan\frac{C_1}{2}(x - C_2)\right).
\end{aligned}$$

Figure 3 shows an example of such surfaces with $C_1 = C_2 = 1$.

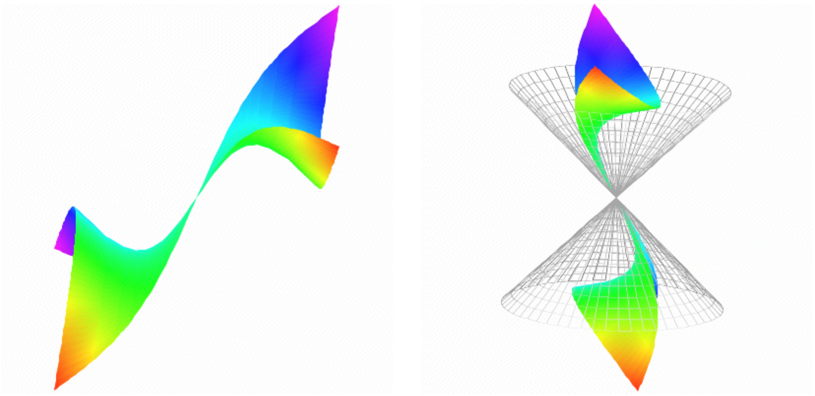

(a) (b)

Figure 3: A spacelike maximal surface of revolution about $\xi_0 + \xi_1$ axis

5 Spacelike surfaces of revolution in $\mathbb{E}^3_1$ with non-zero constant mean curvature

If H is a non-zero constant, it is difficult to solve those mean curvature equations in Theorem 5. In this section, using *calculus of variation* we obtain simpler equations (first-order ordinary differential equations) which give rise to profile curves of spacelike constant mean curvature (abbreviated as CMC) H surfaces of revolution. These equations are still nonlinear and cannot be solved analytically. However, they can be solved numerically using well-known numerical methods such as the Euler's method or Runge-Kutta's method.

5.1 Spacelike CMC $H \neq 0$ surfaces of revolution about spacelike axes

In this subsection, we discuss spacelike CMC $H \neq 0$ surfaces of revolution about ξ_1 axis.

Let $x := \xi_0$ and $y := \xi_1$. Let $x = h(y)$ be a profile curve. Then the area functional with volume constraint is given by

$$\begin{aligned} J(x, x_y, y) &= S - \lambda V \\ &= \int_{y_1}^{y_2} (2\pi h\sqrt{1-(h')^2} - \lambda h^2)dy. \end{aligned}$$

Here, x_y denotes $\frac{dx}{dy} = h'(y)$, S is area functional, V is volume constraint, and λ is the Lagrangian multiplier.

Let $f(x, x_y, y) := 2\pi h\sqrt{1-(h')^2} - \lambda h^2$. Then it is well-known that J has an extremum if and only if $f(x, x_y, y)$ satisfies the *Euler-Lagrange equation*

$$\frac{\partial f}{\partial y} - \frac{d}{dy}\left(f - x_y\frac{\partial f}{\partial x_y}\right) = 0.$$

For more details, see [1] or [3] for instance. The Euler-Lagrange equation is equivalent to

$$h^2 \pm \frac{2ah}{\sqrt{-(h')^2+1}} = \pm b^2, \tag{5.1}$$

where a and b are constants. This is a nonlinear first-order ordinary differential equation (abbreviated as ODE) and it cannot be solved explicitly. We use the *Euler's method* to solve (5.1) numerically.

Figure 4 shows a numerical solution (profile curve) to the equation

$$h^2 - \frac{2ah}{\sqrt{-(h')^2+1}} = b^2$$

with $a = 3$ and $b = 1$. In the numerical solution, the initial condition $h(-0.45) = 0.16$ and the step size $\Delta y = 0.0045$ are used.

Now, we obtain the spacelike CMC[1] $H \neq 0$ surface of revolution in figure 5 by rotating the profile curve $x = h(y)$ in figure 4 about ξ_1 axis.

5.2 Spacelike CMC $H \neq 0$ surfaces of revolution about the timelike axis ξ_0

In this subsection, we construct an example of spacelike CMC $H \neq 0$ surfaces of revolution about ξ_0 axis by the same method as in the previous section.

Let $x := \xi_0$ and $y := \xi_1$. Let $y = h(x)$ be a profile curve of a spacelike CMC $H \neq 0$ surface of revolution about ξ_0. Then the area functional with volume constraint is given by

$$\begin{aligned} J(y, y_x, x) &= S - \lambda V \\ &= \int_{x_1}^{x_2} (2\pi h\sqrt{(h')^2-1} - \lambda h^2)dx. \end{aligned}$$

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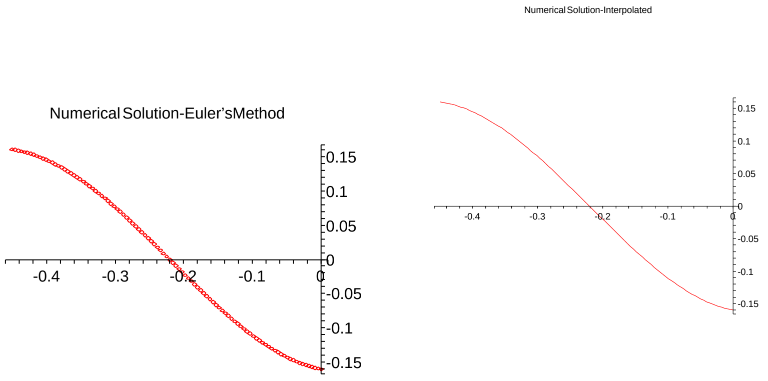
[1] At the moment, we are not able to determine the value of the non-zero constant mean curvature. We will discuss how to determine the mean curvature in the next section (section 6).



(a) (b)

Figure 4: A numerical solution to (5.1)

Let $f(y, y_x, x) = 2\pi h\sqrt{(h')^2 - 1} - \lambda h^2$. Then J has an extremum if and only if $f(y, y_x, x)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dy}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0.$$

This equation is equivalent to

$$h^2 \pm \frac{2ah}{\sqrt{(h')^2 - 1}} = \pm b^2. \tag{5.2}$$

Again, we use the Euler's method to obtain a numerical solution to the equation (5.2). Figure 6 shows a numerical solution to the equation

$$h^2 - \frac{2ah}{\sqrt{(h')^2 - 1}} = b^2$$

with $a = 3$ and $b = 1$. In the numerical solution, the initial condition $h(-0.67) = 0.8$ and the step size $\Delta x = 0.0067$ are used.

By rotating the profile curve in figure 6 about ξ_0 axis, we obtain the spacelike CMC $H \neq 0$ surface of revolution in figure 7.

5.3 Spacelike CMC $H \neq 0$ surfaces of revolution about null axes

In this subsection, we construct an example of spacelike CMC $H \neq 0$ surfaces of revolution about the null axis $\xi_0 + \xi_1$.

This case is tricky because it is difficult to set up area functional in terms of the rectangular coordinates x, y.

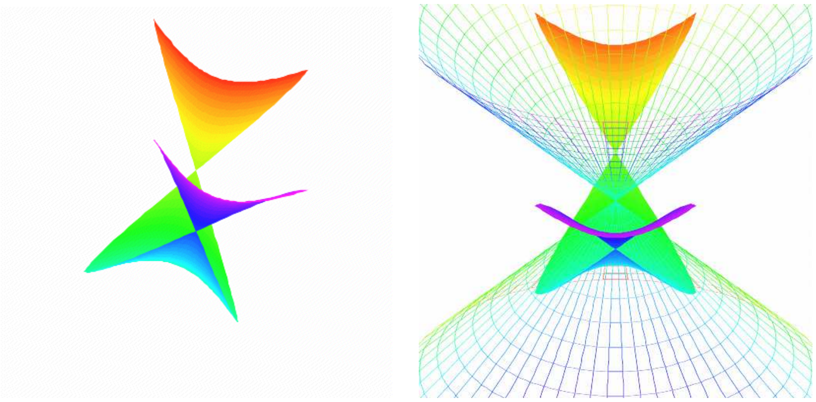

(a) (b)

Figure 5: A spacelike CMC $H \neq 0$ surface of revolution about ξ_1 axis in $\mathbb{E}^3_1$

Let $x := \xi_0$ and $y := \xi_1$. Let $u = x + y$ and $v = -x + y$. Then (u, v) is the *null coordinate system*. In terms of null coordinates, we are able to set up area functional with volume constraint.

Let $v = h(u)$ be a profile curve. If one regards the profile curve as $y = g(x)$, then the arc-length element $d\ell$ is given by

$$\begin{aligned} d\ell &= \sqrt{-dx^2 + dy^2} \\ &= \sqrt{dudv} \\ &= \sqrt{h'}du. \end{aligned}$$

Here, $h^{'}$ denotes $\frac{dh}{du}$. So, the area functional with volume constraint is

$$\begin{aligned} J(v, v_u, u) &= S - \lambda V \\ &= \int_{u_1}^{u_2} (2\pi h\sqrt{h'} - \lambda h^2)du. \end{aligned}$$

Let $f(v, v_u, u) = 2\pi h\sqrt{h'} - \lambda h^2$. Then J has an extremum if and only if $f(v, v_u, u)$ satisfies the Euler-Lagrange equation

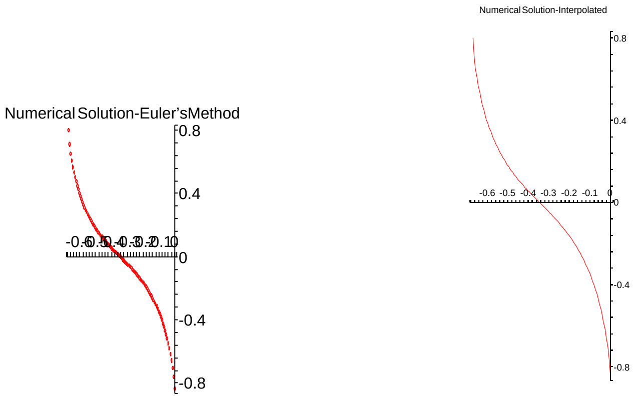
$$\frac{\partial f}{\partial u} - \frac{d}{du}\left(f - v_u \frac{\partial f}{\partial v_u}\right) = 0.$$

The Euler-Lagrange equation is equivalent to

$$h^2 \pm 2ah\sqrt{h'} = \pm b^2. \tag{5.3}$$

Now, we construct an example of spacelike CMC $H \neq 0$ surface of revolution about $\xi_0 + \xi_1$ axis by solving the equation

$$h^2 - 2ah\sqrt{h'} = b^2$$



(a) (b)

Figure 6: A numerical solution to the equation (5.2)

with $a = 1$ and $b = 2$.

We convert this equation back in terms of the rectangular coordinate system (x, y):

$$\frac{-1 + \frac{dy}{dx}}{1 + \frac{dy}{dx}} = \frac{[(-x + y)^2 - 4]^2}{(-x + y)^2}.$$

Note that the LHS of this equation is the same as $h^{'}(u) = \frac{dh}{du}$.

Again, we apply the euler's method to obtain a solution to this equation. Figure 8 shows a numerical solution to this equation with the initial condition $y(-0.66) = 0.864138$ and the step size $\Delta x = 0.0132$.

Figure 9 shows the spacelike CMC $H \neq 0$ surface obtained by rotating the profile curve in figure 8 about $\xi_0 + \xi_1$ axis.

6 Spacelike Undulods and Nodoids in $\mathbb{E}^3_1$

In section 5, we constructed some examples of spacelike surfaces of revolution with non-zero constant mean curvature. Although we know that those resulting surfaces have non-zero constant mean curvatures, we do not know how to determine their mean curvatures, because they are constructed numerically.

In this section, we study how to determine the mean curvatures of those spacelike surfaces of revolution with non-zero constant mean curvature. Furthermore, we study whether we can consider both *unduloid* and *nodoid* type spacelike surfaces in $\mathbb{E}^3_1$.

Proposition 6 *If* $h(x)$ *satisfies the equation* (5.1) *or* (5.2)*, then the mean curvature* H *of the resulting surface of revolution is* $H = \pm\frac{1}{2a}$.

Proof. Assume that $h(x)$ satisfies the equation (5.1). Differentiating the equation

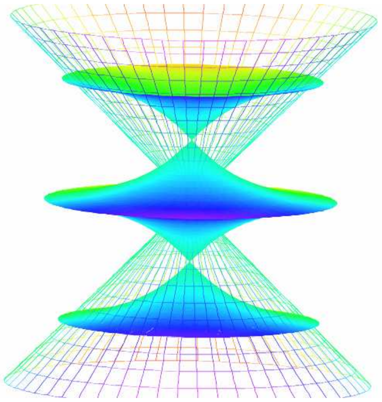


Figure 7: A spacelike CMC $H \neq 0$ surface of revolution about ξ_0 axis in $\mathbb{E}^3_1$

(5.1), we get

$$\frac{1}{2a} = \pm\frac{1}{2}\frac{h^{''}h - (h^{'})^2 + 1}{h(1-(h^{'})^2)^{\frac{3}{2}}}.$$

Comparing this equation with (3.9) or (3.11), we see that $H = \pm\frac{1}{2a}$. □

By proposition 6, we are now able to determine the mean curvatures of surfaces in figures 5 and 7. For both surfaces in figures 5 and 7 $a = 3$ was used, so both surfaces have the mean curvature[2] $H = \pm\frac{1}{6}$.

Remark 4 *The authors are unable to determine the mean curvature of spacelike surfaces of revolution about null axes such as the one in figure 9. The reason is that the authors are unable to convert the equation* (5.3) *in terms of the rectangular coordinate system* (x, y).

The following lemma can be easily proved by a direct computation.

Lemma 7 *1. The mean curvature equation* (3.9) *is equivalent to*

$$\frac{d}{dx}\left(Hh^2 - \frac{h}{\sqrt{1-(h^{'})^2}}\right) = 0. \tag{6.1}$$

2. The mean curvature equation (3.11) *is equivalent to*

$$\frac{d}{dx}\left(Hh^2 + \frac{h}{\sqrt{1-(h^{'})^2}}\right) = 0. \tag{6.2}$$

3. The mean curvature equation (3.13) *is equivalent to*

$$\frac{d}{dx}\left(Hh^2 + \frac{h}{\sqrt{(h^{'})^2-1}}\right) = 0. \tag{6.3}$$

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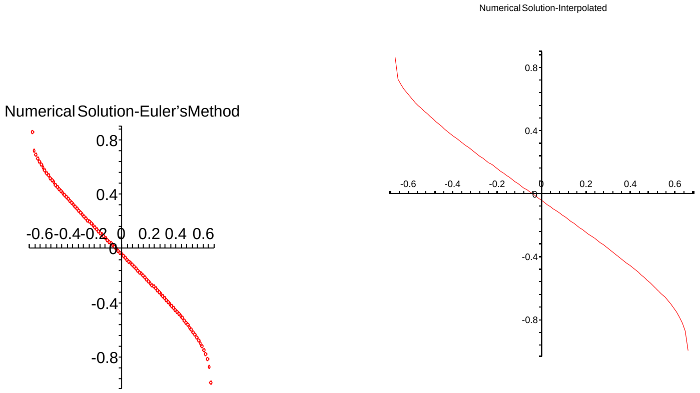
[2] The sign of mean curvature depends on the orientation of a surface, or equivalently the orientation of normal vector field of a surface.



(a) (b)

Figure 8: A numerical solution to the equation (5.3)

The following theorem distinguishes spacelike surfaces of revolution with nonzero constant mean curvature in Minkowski 3-space and surfaces of revolution in Euclidean 3-space with nonzero constant mean curvature.

Theorem 8

1. *There exist only* nodoid *type spacelike surfaces of revolution about spacelike axes, which are globally defined.*
2. *There exist only* unduloid *type spacelike surfaces of revolution about timelike axis, which are globally defined.*

Proof. Let $\varphi(x,y) = (h(x)\cosh y, x, h(x)\sinh y)$ be a spacelike surface of revolution about ξ_1 axis with the constant mean curvature $H \neq 0$. Then $h(x)$ satisfies the equation (6.1). That is,

$$Hh^2 - \frac{h}{\sqrt{1-(h^{'})^2}} = c$$

for some non-zero constant[3] c. Solving this equation for $h^{'}$, we get

$$h^{'} = \pm\frac{\sqrt{(Hh^2-c)^2-h^2}}{Hh^2-c}.$$

In order to consider globally defined unduloid or nodoid type surfaces, we require $(Hh^2-c)^2 - h^2 > 0$ which is equivalent to the inequality $cH < -\frac{1}{4}$. This inequality tells us that the signs of c and H must be different. So, there are only nodoid type surfaces of revolution about ξ_1 axis, which are globally defined.

Now, let $\varphi(x,y) = (x, h(x)\cos y, h(x)\sin y)$ be a spacelike surface of revolution about ξ_0 axis with the constant mean curvature $H \neq 0$. Then $h(x)$ satisfies the

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[3] If $c = 0$, then the spacelike CMC surface is a *totally umbilic* surface which is part of a hyperbolic 2-space in $\mathbb{E}^3_1$. This will be discussed in Theorem 9.

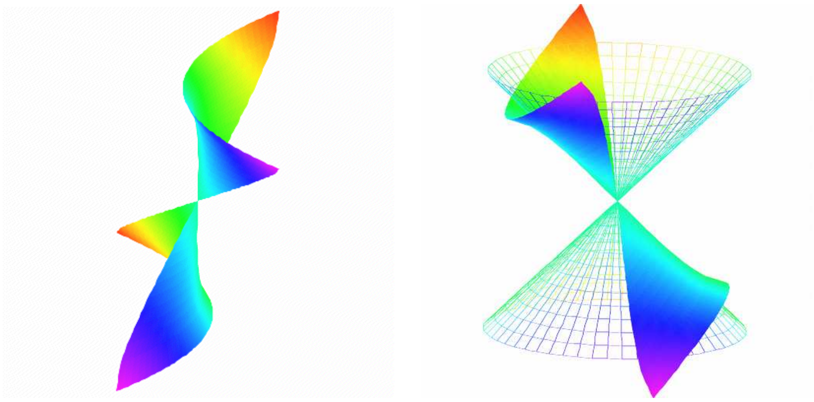

(a) (b)

Figure 9: A spacelike CMC $H \neq 0$ surface of revolution about $\xi_0 + \xi_1$ axis in $\mathbb{E}^3_1$

equation (6.3) which is equivalent to the differential equation

$$h^{'} = \pm \frac{\sqrt{(c - Hh^2)^2 + h^2}}{c - Hh^2},$$

where c is a nonzero constant. Again, we require $(c - Hh^2)^2 + h^2 > 0$ which is equivalent to the inequality $cH > \frac{1}{4}$. This implies that the signs of c and H must be the same, i.e., there are only unduloid type surfaces of revolution about ξ_0 axis, which are globally defined. □

Remark 5 *Spacelike surfaces of revolution with nonzero constant mean curvature such that H and c have the same (opposite) sign are called the* unduloid type surfaces *(*nodoid type surfaces*). Note that, in Euclidean case, surfaces of revolution with nonzero constant mean curvature such that H and c have the same (opposite) sign is an unduloid (nodoid). Please see John Oprea's book ([8], pp. 114–116) for more details.*

Remark 6 *Note that there exist unduloid type spacelike surfaces of revolution about spacelike axes, which are locally defined. Also, there exist nodoid type spacelike surfaces of revolution about timelike axis, which are locally defined.*

Theorem 9 *1. If a spacelike surface of revolution about spacelike axis ξ_1 or ξ_2 with $H \neq 0$ satisfies*

$$Hh^2 \mp \frac{h}{\sqrt{1 - (h^{'})^2}} = 0,$$

then it is part of the hyperbolic 2-space $\mathbb{H}^2(-\frac{1}{H^2})$ of radius $\frac{1}{H}$, and hence it is totally umbilic.

2. *If a spacelike surface of revolution about timelike axis ξ_0 with $H \neq 0$ satisfies*

$$Hh^2 + \frac{h}{\sqrt{(h')^2 - 1}} = 0,$$

then it is part of the hyperbolic 2-space $\mathbb{H}^2(-\frac{1}{H^2})$ of radius $\frac{1}{H}$, and hence it is totally umbilic.

Proof. Let $\varphi(x, y) = (h(x)\cosh y, x, h(x)\sinh y)$ be a spacelike surface of revolution about ξ_1 axis with the constant mean curvature $H \neq 0$. Assume that the profile curve $h(x)$ satisfies the equation

$$Hh^2 - \frac{h}{\sqrt{1 - (h')^2}} = 0,$$

i.e.,

$$\frac{dh}{dx} = \pm\frac{\sqrt{H^2h^2 - 1}}{Hh}.$$

This is a simple separable equation and its solution is

$$h(x) = \pm\frac{\sqrt{(xH + d)^2 + 1}}{H}.$$

Without loss of generality, we may assume that $d = 0$, so that

$$h(x) = \pm\frac{\sqrt{x^2H^2 + 1}}{H}.$$

Now,

$$\begin{aligned}
||\varphi||^2 &= -h(x)^2\cosh^2 y + x^2 + h(x)^2\sinh^2 y \\
&= -h(x)^2 + x^2 \\
&= -\frac{x^2H^2 + 1}{H^2} + x^2 \\
&= -\frac{1}{H^2}.
\end{aligned}$$

Hence, φ is part of the hyperbolic 2-space in $\mathbb{H}^2(-\frac{1}{H^2})$ with radius $\frac{1}{H}$. Since $\mathbb{H}^2(-\frac{1}{H^2})$ is totally umbilic, so is φ.

The other cases are proved in the same manner. □

Remark 7 *It is well-known that* a totally umbilic spacelike surface of constant mean curvature H in $\mathbb{E}^3_1$ is part of spacelike plane $(H = 0)$ or part of the hyperbolic 2-space $\mathbb{H}^2(-\frac{1}{H^2})$ of radius $\frac{1}{H}$.

Example 1 $\varphi(x, y) = (\sqrt{x^2 + 1}\cosh y, x, \sqrt{x^2 + 1}\sinh y)$ *is a totally umbilic spacelike surface of revolution about ξ_1 axis with $H = 1$. (Figure 10.)*

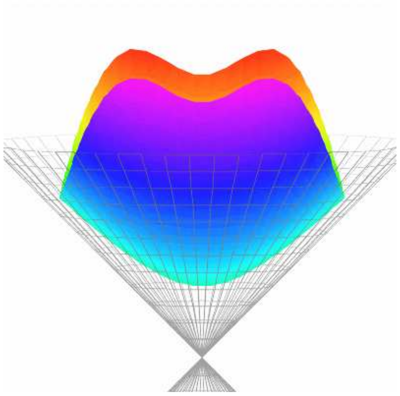


Figure 10: $\varphi(x,y) = (\sqrt{x^2+1}\cosh y, x, \sqrt{x^2+1}\sinh y)$

The equations (6.1), (6.2), and (6.3) can be also used to construct examples of spacelike maximal surfaces of revolution and spacelike surfaces of revolution with nonzero constant mean curvature.

Proposition 10 *1. The solution to the equation* (6.1) *or* (6.2) *with* $H = 0$ *is*

$$h(x) = \pm\frac{1}{a}\sin(ax \pm b),$$

where $a \neq 0$ *and* b *are constants. The spacelike surface obtained by rotating* $h(x)$ *about* ξ_1 *axis or* ξ_2 *axis is conformal if and only if* $a = \pm 1$.

Notice that this coincides with the solution we found in subsection 4.1.

2. The solution to the equation (6.3) *with* $H = 0$ *is*

$$h(x) = \pm\frac{1}{a}\sinh(ax \pm b),$$

where $a \neq 0$ *and* b *are constants. The spacelike surface obtained by rotating* $h(x)$ *about* ξ_0 *axis is conformal if and only if* $a = \pm 1$.

Notice that this coincides with the solution we found in subsection 4.2.

Proof. The equations (6.1), (6.2), and (3.13) with $H = 0$ are separable first-order ODEs which can be easily solved by direct integration. □

The equation

$$Hh^2 - \frac{h}{\sqrt{1-(h')^2}} = c$$

with $c \neq 0$, or equivalently

$$\frac{dh}{dx} = \pm\frac{\sqrt{(Hh^2-c)^2-h^2}}{Hh^2-c}$$

is a separable equation. However, the integration involves elliptic functions with $c \neq 0$. Thus, we still need to use numerical methods such as the Euler's method to construct examples of spacelike CMC $H \neq 0$ surfaces of revolution about ξ_1 axis. Figure 11 is a numerical solution to the equation with $H = c = 1$ and the initial condition $h(0) = -0.61$.

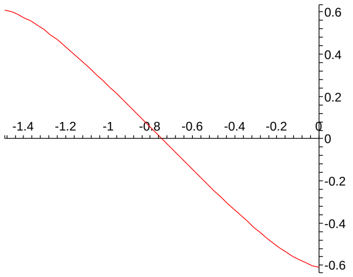
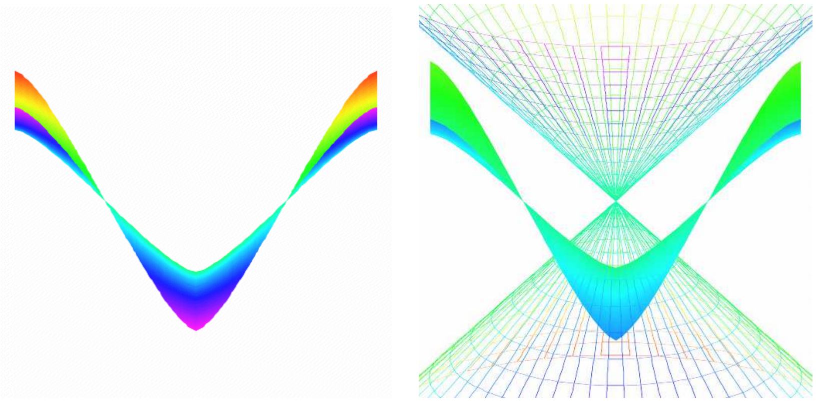


Figure 11: A numerical solution to $\frac{dh}{dx} = \frac{\sqrt{h^4-3h^2+1}}{h^2-1}$.

Figure 12 shows the locally defined spacelike unduloid which is obtained by rotating $h(x)$ in figure 11 about ξ_1 axis.



(a) (b)

Figure 12: A locally defined spacelike unduloid in $\mathbb{E}^3_1$.

Figure 13 is a numerical solution to the equation

$$Hh^2 + \frac{h}{\sqrt{(h')^2 - 1}} = c$$

or

$$\frac{dh}{dx} = \pm \frac{\sqrt{h^2 + (c - Hh^2)^2}}{c - Hh^2}$$

with $H = c = 1$ and the initial condition $h(0) = 0$.

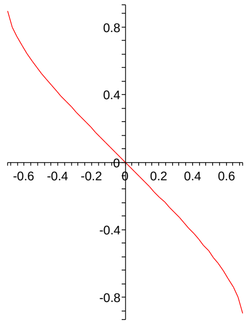
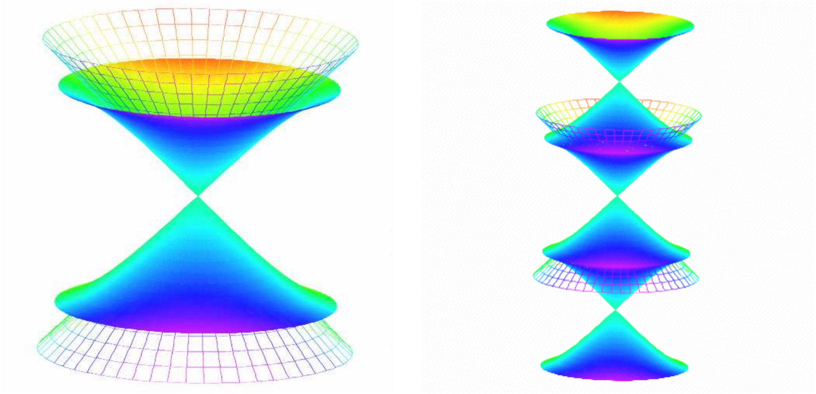


Figure 13: A numerical solution to $\frac{dh}{dx} = \frac{\sqrt{h^4-h^2+1}}{h^2-1}$.

Figure 14 shows the globally defined spacelike unduloid which is obtained by rotating $h(x)$ in figure 13 about ξ_0 axis.



(a) (b)

Figure 14: A globally defined spacelike unduloid in $\mathbb{E}^3_1$.

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