Warped products of constant curvature

M. T. Mustafa

Abstract. Constraints on the existence of constant curvature warped products are investigated. In particular, it is shown that only constant curvature warped products with compact base are flat Riemannian products.

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1 Introduction

In [2], Bishop and O'Neill introduced the concept of warped products, which were used to construct a large variety of complete Riemannian manifolds with negative sectional curvature. Since then, warped products have been used to construct new examples with interesting curvature properties cf. [1, 3, 4, 5].

Definition 1.1 *Let (B, g_B) and (F, g_F) be Riemannian manifolds with $f : B \to (0, \infty)$ a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric*

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : B \times F \to B$, $\sigma : B \times F \to F$ are usual projections and $*$ denotes pullback. (B, g_B) is called the base, (F, g_F) is called the fiber and f the warping function of the warped product.

If the warping function 'f' is constant then the warped product $B \times_f F$ (up to a change of scale) is a (global) Riemannian product, which we call as trivial warped product. Though the class of warped products with non-constant warping functions provides a rich class of examples in Riemannian and semi-Riemannian geometry, there are no known examples of constant curvature (non-trivial) warped products with a compact base. The purpose of this paper is to investigate restrictions on (non-trivial) warped products $B \times_f F$ of constant curvature and prove their non-existence if the base B is a compact Riemannian manifold.

For the fundamental results and properties of warped products, the reader is referred to [1, 5].

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2 Restrictions on warped products of constant curvature

In this section we obtain restrictions on constant curvature warped products, using fundamental curvature relations of warped products [5].

Theorem 2.1 *Let $M^m = B \times_f F$ be a warped product of an $(m$-$n)$-dimensional Riemannian manifold B and an n-dimensional Riemannian manifold F. Suppose that M has constant sectional curvature K.*

(i) If f achieves its maximum on B then $K \geq 0$.

(ii) If f achieves its minimum on B then $K \leq 0$.

Proof. From the curvature identity ([5], p. 211), relating the Ricci curvature of M and B, we have

$$\sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) = \mathrm{Scal}^B - \frac{n}{f}\Delta^B f \tag{2.1}$$

where Scal^B is the scalar curvature of B and Δ^B is the Laplacian on B.

Since, for any $q \in F$, the leaves $B \times q$ are totally geodesic and M is of constant curvature K, we can write

$$\sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) = (m-n)K + \mathrm{Scal}^B. \tag{2.2}$$

From Equations 2.1 and 2.2, we have

$$-\frac{1}{f}\Delta^B f = (m-n)K \tag{2.3}$$

(i) If 'p' is a maximum point of f on B, then

$$f(p) > 0 \ \text{ and } \ \Delta^B f(p) \leq 0.$$

Hence, from Equation 2.3 it follows that $K \geq 0$.

(ii) If 'p' is a minimum point of f on B, then

$$f(p) > 0 \ \text{ and } \ \Delta^B f(p) \geq 0.$$

Hence, from Equation 2.3 it follows that $K \leq 0$.

□

Corollary 2.2 *Let $M = B \times_f F$ be a warped product of constant curvature K. If B is compact then M is a Riemannian product with zero curvature.*

Proof. Since B is compact, Theorem 2.1 implies that $K = 0$. Then from Equation 2.3, f is a harmonic function and hence must be constant (due to compactness of B). Therefore, up to a change of scale, M is a Riemannian product. □

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Author's address:

M. T. Mustafa
Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia.
email: tmustafa@kfupm.edu.sa