# Classification of cubics up to affine transformations

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**Abstract.** Classification of cubics (that is, third order planar curves in the $\mathbb{R}^2$) up to certain transformations is interested since Newton, and treated by several authors. We classify cubics up to affine transformations, in seven class, and give a complete set of representatives of the these classes. This result is complete and briefer than the similar results.

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## 1 Introduction

An algebraic curve over a field $K$ is the solution set of an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial in $x$ and $y$ with coefficients in $K$, and the degree of $f$ is the maximum degree of each of its terms (monomials). A **cubic** curve is an algebraic curve of order 3 with $K = \mathbb{R}$ real numbers.

One of Isaac Newton's many accomplishments was the classification of the cubic curves. He showed that all cubics can be generated by the projection of the five divergent cubic parabolas. Newton's classification of cubic curves appeared in the chapter "Curves" in Lexicon Technicum [4], by John Harris published in London in 1710. Newton also classified all cubics into 72 types, missing six of them. In addition, he showed that any cubic can be obtained by a **suitable projection** of the elliptic curve

$$y^2 = ax^3 + bx^2 + cx + d \tag{1.1}$$

where the projection is a birational transformation, and the general cubic can also be written as

$$y^2 = x^3 + ax + b \tag{1.2}$$

This classification was criticized by Euler, because of it's lack of generality. There are also other classifications, ranging from 57 to 219 types. The one with the 219 classes was given by Plucker.

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We classify cubics up to affine transformations, in seven class, and give a complete set of representatives of the these classes. This result is complete and briefer than the similar results. We done this by studying the structure of symmetries of the of an special differential equation which $\mathcal{M}$ is it's solution.

# 2 Forming the problem

Let $\mathbb{R}^2$ have the standard structure with the identity chart $(x, y)$ and $\partial_x := \partial/\partial x$ and $\partial_y := \partial/\partial y$ are the standard vector fields on $\mathbb{R}^2$. Let

$$\begin{aligned}\mathcal{C} \,:\, c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3 + c_{20}x^2 \\ +c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y + c_{00} = 0\end{aligned} \tag{2.1}$$

have the induced differentiable structure from $\mathbb{R}^2$. Let $\mathcal{M}$ be the set of all sub-manifolds in the form (2.1) with ${c_{30}}^2 + {c_{21}}^2 + {c_{12}}^2 + {c_{03}}^2 \neq 0$. $\mathcal{M}$ can be regarded as an open sub-manifold $\mathbb{R}^{10} - \{0\}$ of $\mathbb{R}^{10}$, with the trivial chart:

$$\varphi(\mathcal{C}) = (c_{30}, c_{21}, c_{12}, c_{03}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}, c_{00}) \tag{2.2}$$

Let $Aff(2)$ be the Lie group of real affine transformations in the plan:

$$\left\{ g := \begin{pmatrix} a_{11} & a_{12} & \alpha \\ a_{21} & a_{22} & \beta \\ 0 & 0 & 1 \end{pmatrix} \Big| \, a_{ij}, \alpha, \beta \in \mathbb{R}^2 \,,\, a_{11}a_{22} \neq a_{12}a_{21} \right\} \tag{2.3}$$

Which, act on $\mathbb{R}^2$ as

$$g \cdot (x, y) := \Big(a_{11}x + a_{12}y + \alpha, a_{21}x + a_{22}y + \beta\Big) \tag{2.4}$$

The Lie algebra $\mathfrak{aff}(2) := \mathcal{L}(Aff(2))$ of this Lie group spanned by

$$\partial_x \,,\, \partial_y, x\partial_x \,,\, y\partial_x \,,\, x\partial_y \,,\, y\partial_y \tag{2.5}$$

over $\mathbb{R}$. $Aff(2)$ as a Lie group act on $\mathbb{R}^2$ and so on the set $\mathcal{M}$ of all sub-manifolds in the form (2.1) of $\mathbb{R}^2$. Two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in the form (2.1) are said to be equivalence, if there exists an element $T$ of $Aff(2)$ as (2.10) such that $T(\mathcal{C}_1) = \mathcal{C}_2$. This relation partition $\mathcal{M}$ into disjoint cosets. The main idea of this paper is to realize these cosets by giving a complete set of representatives.

**Theorem 1.** *Every cubic curve (2.1) can be transformed into a cubic in the form*

$$x^3 + x^2y = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \tag{2.6}$$

*with $C \geq 0$, by an affine transformation.*

*Proof:* Let $g \in Aff(2)$ act on $\mathcal{C} := \varphi^{-1}(a_{30}, \cdots, a_{00})$ and result be $\tilde{\mathcal{C}} := g \cdot \mathcal{C} = \varphi^{-1}(\tilde{a}_{30}, \cdots, \tilde{a}_{00})$. There are two cases,

(a) Let $c_{30} \neq 0$ or $c_{03} \neq 0$. If one of these numbers is zero, we can transform $\mathcal{C}$ into a similar curve with non-zero $c_{30}$ and $c_{03}$, by using the affine transformation $(x, y) \mapsto (x + y, x)$. Therefore, we can assume that $c_{30}$ and $c_{03}$ are non-zero. Now, we have

$$
\begin{array}{lll}
\tilde{c}_{30} & = & c_{30}{a_{11}}^3 + c_{21}{a_{11}}^2 a_{21} + c_{12}a_{11}{a_{21}}^2 + c_{03}{a_{21}}^3 \\
\tilde{c}_{21} & = & 3c_{30}a_{12}{a_{11}}^2 + c_{21}{a_{11}}^2 a_{22} + 2c_{21}a_{12}a_{21}a_{11} \\
 & & +2c_{12}a_{11}a_{22}a_{21} + c_{12}a_{12}{a_{21}}^2 + 3c_{03}a_{22}{a_{21}}^2 \\
\tilde{c}_{12} & = & 3c_{30}{a_{12}}^2 a_{11} + 2c_{21}a_{12}a_{22}a_{11} + c_{21}{a_{12}}^2 a_{21} \\
 & & +c_{12}a_{11}{a_{22}}^2 + 2c_{12}a_{12}a_{22}a_{21} + 3c_{03}{a_{22}}^2 a_{21} \\
\tilde{c}_{03} & = & 3c_{30}{a_{12}}^3 + 3c_{21}{a_{12}}^2 a_{22} + 3c_{12}a_{12}{a_{22}}^2 + 3c_{03}{a_{22}}^3
\end{array}
\tag{2.7}
$$

Let $\tilde{c}_{12} = 0$ and solving the result equation for $a_{11}$, we find that

$$
a_{11} = -\frac{a_{21}(2c_{12}a_{12}a_{22} + 3c_{03}{a_{22}}^2 + c_{21}{a_{12}}^2)}{3c_{30}{a_{12}}^2 + 2c_{21}a_{12}a_{22} + c_{12}{a_{22}}^2}
\tag{2.8}
$$

Thus, by putting $a_{11}$ in $\tilde{c}_{30}$ and $\tilde{c}_{03}$, we have

$$
\begin{array}{lll}
\tilde{c}_{30} & = & \tilde{a}_{21} - {a_{21}}^2 \dfrac{c_{30}{a_{12}}^3 + c_{21}{a_{12}}^2 a_{22} + c_{12}a_{12}{a_{22}}^2 + c_{03}{a_{22}}^3}{(3c_{30}{a_{12}}^2 + 2c_{21}a_{12}a_{22} + c_{12}{a_{22}}^2)^3} \\
 & & \times\Big(27{a_{12}}^4 c_{12}{c_{30}}^2 - 9{a_{12}}^4{c_{21}}^2 c_{30} + 81{a_{12}}^3 a_{22}c_{03}{c_{30}}^2 \\
 & & -27{a_{12}}^3{c_{30}}^2 c_{03}a_{21} + 9{a_{12}}^3 c_{30}c_{21}c_{12}a_{21} \\
 & & -6{a_{12}}^3{c_{21}}^3 a_{22} - 2{a_{12}}^3 a_{21}{c_{21}}^3 + 81{a_{12}}^2{a_{22}}^2 c_{30}c_{03}c_{21} \\
 & & +18{a_{12}}^2 c_{30}{c_{12}}^2 a_{21}a_22 - 27{a_{12}}^2 c_{30}c_{03}a_{21}c_{21}a_22 \\
 & & -9{a_{12}}^2{a_{22}}^2 c_{12}{c_{21}}^2 - 3{a_{12}}^2 a_{21}c_{12}a_{22}{c_{21}}^2 \\
 & & +27a_{12}c_{30}c_{12}{a_{22}}^2 c_{03}a_{21} + 27a_{12}{a_{22}}^3 c_{30}c_{12}c_{03} \\
 & & -18a_{12}a_{21}{a_{22}}^2 c_{03}{c_{21}}^2 - 9a_{12}{a_{22}}^3{c_{12}}^2 c_{21} \\
 & & +3a_{12}a_{21}{c_{12}}^2{a_{22}}^2 c_{21} + 18a_{12}{a_{22}}^3 c_{03}{c_{21}}^2 \\
 & & +27{c_{03}}^2{a_{22}}^3 a_{21}c_{30} + 2a_{21}{c_{12}}^3{a_{22}}^3 + 9{a_{22}}^4 c_{12}c_{03}c_{21} \\
 & & -3{a_{22}}^4{c_{12}}^3 - 9c_{03}{a_{22}}^3 c_{21}c_{12}a_{21} + 9{a_{12}}^3 a_{22}c_{30}c_{12}c_{21}\Big) \\
\tilde{c}_{03} & = & 3c_{30}{a_{12}}^3 + 3c_{21}{a_{12}}^2 a_{22} + 3c_{12}a_{12}{a_{22}}^2 + 3c_{03}{a_{22}}^3
\end{array}
\tag{2.9}
$$

Now, if we can assume that

$$
c_{30}{a_{12}}^3 + c_{21}{a_{12}}^2 a_{22} + c_{12}a_{12}{a_{22}}^2 + c_{03}{a_{22}}^3 = 0 \tag{2.10}
$$

$$
3c_{30}{a_{12}}^2 + 2c_{21}a_{12}a_{22} + c_{12}{a_{22}}^2 \neq 0 \tag{2.11}
$$

then, $\tilde{c}_{30} = \tilde{c}_{21}$ and $\tilde{c}_{12} = \tilde{c}_{03} = 0$. The equation (2.10) is an third order in $a_{22}$ and the coefficient of ${a_{22}}^3$ is $c_{03} \neq 0$. Therefore, has a real solution. On the other hand, if the left hand side of the (2.11) be zero, for all $a_{12} \neq 0$, then it must be $c_{30} = 0$, which is impossible. Therefore, the relations (2.10) and (2.11) can be valid. Because $a_{12} \neq 0$, we can choose $a_{21}$ such that $\det(g) = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

(b) Let $c_{30} = c_{03} = 0$. Therefore, $c_{21} \neq 0$ or $c_{12} \neq 0$. Assume, $g \cdot (x, y) = (x+\alpha y, y)$. Then, we have

$$\tilde{c}_{30} = 0, \qquad \tilde{c}_{03} = 3\alpha(\alpha c_{21} + c_{12}). \tag{2.12}$$

Because $c_{21} \neq 0$ or $c_{12} \neq 0$, we can choose $\alpha$ such that $\tilde{c}_{03} \neq 0$, and this is the case (a).

If $C < 0$, we can apply the transformation $(x, y) \mapsto (-x, -y)$, and take a curve with $C \geq 0$. □

**Definition 1.** The set of all cubics of the form (2.6) is denoted by $M$, and define $\varphi : M \to \mathbb{R}^6$ by

$$\begin{aligned} &\varphi\big(\{x^3 + x^2y = Ax^2 + Bxy + Cy^2 + Dx \\ &\qquad +Ey + F\}\big) := (A, B, C, D, E, F) \in \mathbb{R}^6 \end{aligned} \tag{2.13}$$

as a chart of $M$. Then, $M$ has a six-dimensional manifold structure isomorphic to $\mathbb{R}^5 \times \mathbb{R}^+ \cong \mathbb{H}^6$:

$$\begin{aligned} M := \Big\{ &\{x^3 + x^2y = Ax^2 + Bxy + Cy^2 + Dx \\ &\quad +Ey + F\} \,\big|\, A, B, C, D, E, F \in \mathbb{R}, C \geq 0 \Big\} \end{aligned} \tag{2.14}$$

**Conclusion 1.** *$M$ is a regular six-dimensional sub-manifold with boundry of $\mathcal{M}$, and a section of action $Aff(2)$ on $\mathcal{M}$.* □

**Conclusion 2.** *We can classify $M$ up to affine transformations, instead of $\mathcal{M}$.* □

## 3 Reducing the problem

Let $\mathcal{C} = \varphi(A, B, C, D, E, F) \in M$ and $(y, x) \in \mathcal{M}$. The equation (2.6) can be writen as $f := x^3 + x^2y - Ax^2 - Bxy - Cy^2 - Dx - Ey - F = 0$. Since $\partial f/\partial y = x^2 - Bx - 2Cy - E$ and by the initial assumption $\mathcal{C}$ is of order 3 in $x$, then $\partial f/\partial y \neq 0$ for all $(x, y) \in \mathcal{C}$, without a finite set of points. Therefore, we can assume $y$ is a function of $x$ in almost every points of $\mathcal{C}$. Now, we can prolong $y$ up to sixth order $j^6y$, and forming $J^6\mathcal{C}$ the sixth order jet space of $\mathcal{M}$. For this, it is enough to compute the sixth order total derivative of equation (2.6). That is, we apply

$$\frac{d}{dx} := \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots \tag{3.1}$$

in six times. In this manner, we find six equation. By solving these equations for $A$, $B$, $C$, $D$, $E$ and $F$, and substitute these values into (2.6), we find that

**Theorem 2.** *If* $(x, y, y', y'', y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)})$ *be the standard chart of* $J^6(\mathbb{R}^2)$*, then the curve* $\varphi^{-1}(A, B, C, D, E, F)$ *in* $\mathbb{R}^2$ *prolonged into the hyper-surface*

$$
\begin{aligned}
&600y''y^{(3)3}y^{(4)} - 225y^{(4)3}y'' + 120y^{(3)3}y^{(5)} - 300y^{(3)2}y^{(4)2} \\
&\quad -54y^{(5)2}y''^2 + 460y^{(5)}y^{(4)}y'y''y^{(3)} + 360y^{(5)}y^{(3)}y^{(4)}y'' \\
&\quad -120y^{(6)}y'y^{(3)2}y'' + 45y^{(6)}y^{(4)}y'y''^2 - 400y^{(3)5} \\
&\quad +90y^{(6)}y^{(3)}y''^3 - 120y^{(6)}y''y^{(3)2} - 225y^{(4)3}y''y' \\
&\quad +225y^{(4)2}y''^2y^{(3)} - 135y^{(4)}y''^3y^{(5)} - 150y^{(3)2}y^{(4)2}y' \\
&\quad +120y^{(5)}y'y^{(3)3} - 54y''^2y^{(5)2}y' - 360y^{(5)}y^{(3)2}y''^2 \\
&\quad +45y^{(6)}y^{(4)}y''^2 = 0
\end{aligned}
\tag{3.2}
$$

*in* $J^6(\mathbb{R}^2)$*. Which is a five order algebraic curve in* $J^6(\mathbb{R}^2)$*, or a six order and five degree ordinary differential equation in* $\mathbb{R}^2$*.* □

**Conclusion 3.** *We can classify* $\mathcal{E}$ *the solution set of (3.2) up to affine transformations, instead of* $\mathcal{M}$*. That is, we find the* $Aff(2)-$ *invariant solutions of the ODE (3.2).* □

# 4 Solving the problem

In order to find the symmetries of the differential equation (3.2), we use the method which is described in the page 104 of [5]. Let $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be an arbitrary element of $\mathfrak{aff}(2)$, and prolong it to the $\mathfrak{aff}^{(6)}(2)$, which actis on (3.2). Because the variables $x$, $y$, $y'$, $y''$, $y^{(3)}$, $y^{(4)}$, $y^{(5)}$ and $y^{(6)}$ are independent in $J^6(\mathbb{R}^2)$, we obtain a system of 422 partial differential equations for $\xi$ and $\eta$. Reducing this system by the method of Gauss-Jordan, and find the following

$$
\begin{aligned}
&\xi_x + \eta_x = \eta_y, \\
&\eta_{xy} = 0,\ \eta_{y^2} = 0,\ \eta_{x^2} = 0, \\
&\xi_y = 0,\ \xi_{x^2} = 0,\ \xi_{xy} = 0,\ \xi_{y^2} = 0, \\
&\eta_{x^3} = 0,\ \eta_{x^2y} = 0,\ \eta_{xy^2} = 0,\ \eta_{y^3} = 0, \\
&\xi_{x^3} = 0,\ \xi_{x^2y} = 0,\ \xi_{xy^2} = 0,\ \xi_{y^3} = 0, \\
&\eta_{x^4} = 0,\ \eta_{x^3y} = 0,\ \eta_{x^2y^2} = 0,\ \eta_{xy^3} = 0,\ \eta_{y^4} = 0, \\
&\xi_{x^4} = 0,\ \xi_{x^3y} = 0,\ \xi_{x^2y^2} = 0,\ \xi_{xy^3} = 0,\ \xi_{y^4} = 0, \\
&\eta_{x^5} = 0,\ \eta_{x^4y} = 0,\ \eta_{x^3y^2} = 0,\ \eta_{x^2y^3} = 0,\ \eta_{xy^4} = 0,\ \eta_{y^5} = 0, \\
&\xi_{xy^4} = 0,\ \xi_{y^5} = 0,\ \xi_{x^5} = 0,\ \xi_{x^4y} = 0,\ \xi_{x^3y^2} = 0,\ \xi_{x^2y^3} = 0, \\
&\eta_{x^6} = 0,\ 6\eta_{x^5y} = \xi_{x^6},\ 3\xi_{x^4y^2} = 4\eta_{x^3y^3},\ 4\xi_{x^3y^3} = 3\eta_{x^2y^4}, \\
&5\xi_{x^2y^4} = 2\eta_{xy^5},\ 6\xi_{xy^5} = \eta_{y^6},\ \xi_{y^6} = 0,\ 2\xi_{x^5y} = 5\eta_{x^4y^2}.
\end{aligned}
\tag{4.1}
$$

This system too, in turn equivalented to

$$
\begin{aligned}
&\xi_x + \eta_x = \eta_y, \quad \xi_y = 0, \quad \xi_{xx} = 0, \\
&\eta_{xx} = 0, \quad \eta_{xy} = 0, \quad \eta_{yy} = 0.
\end{aligned}
\tag{4.2}
$$

The general solution of this system is

$$\xi(x,y) = C_3 x + C_1, \qquad (4.3)$$
$$\eta(x,y) = C_4 x + (C_3 + C_4)y + C_2.$$

Which $C_1$, $C_2$, $C_3$ and $C_4$ are arbitrary numbers. Therefore,

**Theorem 3.** *There is only four linearly independent infinitesimal generators for the action of $Aff(2)$ on the solution of (3.2):*

$$X_1 := \partial_x, \quad X_2 := \partial_y, \quad X_3 := x\partial_x + y\partial_y, \quad X_4 := (x+y)\partial_y. \qquad (4.4)$$

*The commutator table of $\mathfrak{g} := \mathrm{span}\{X_1, X_2, X_3, X_4\}$ is:*

(4.5)

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | $0$ | $0$ | $X_1$ | $X_2$ |
| $X_2$ | $0$ | $0$ | $X_2$ | $X_2$ |
| $X_3$ | $-X_1$ | $-X_2$ | $0$ | $0$ |
| $X_4$ | $-X_2$ | $-X_2$ | $0$ | $0$ |

□

**Definition 2.** Let $G$ be the closed connected Lie sub-group of $Aff(2)$, which it's Lie algebra is $\mathfrak{g}$.

**Conclusion 4.** *The necessary and sufficient condition for a one-parameter Lie transformation $T$ leaves $\mathcal{E}$ invariant, is that the corresponding infinitesimal transformation belongs to $\mathfrak{g}$. That is, be a linear combination of $X_1$, $X_2$, $X_3$ and $X_4$ of (4.4)* □

Now, we study the action of $G$ on $M$. To this end, we find the one-parameter transformation group corresponding to any generators of $\mathfrak{g}$, and then, applying those on $\varphi(A,B,C,D,E,F) \in M$. For example, if $\exp(tX_4)\cdot(x,y) = (\bar{x},\bar{y})$, then must have

$$\begin{cases} \tilde{x}'(t) = 0 & , \quad \tilde{y}(0) = y \\ \tilde{y}'(t) = \tilde{x}(t) + \tilde{y}(t) & , \quad \tilde{x}(0) = x \end{cases} \qquad (4.6)$$

Therefore, $\tilde{x}(t) = x$ and $\tilde{y}(t) = \pm e^t(y-x) - x$. In a similar fashion, we can prove that

**Theorem 4.** *A set of generating infinitesimal one-parameter sub-groups of action $G$ are*

$$\begin{aligned} g_1(t) &: \exp(tX_1)\cdot\big(x,y\big) \mapsto \big(x+t, y\big), \\ g_2(t) &: \exp(tX_2)\cdot\big(x,y\big) \mapsto \big(x, y+t\big), \\ g_3(t) &: \exp(tX_3)\cdot\big(x,y\big) \mapsto \big(e^t x, e^t y\big), \\ g_4(t) &: \exp(tX_4)\cdot\big(x,y\big) \mapsto \big(x, e^t(x+y) - x\big). \end{aligned} \qquad (4.7)$$

Now, In order to a complete list of $G-$invariants on $M$, applying the each of infinitesimals of (4.7) on $\varphi^{-1}(A,B,C,D,E,F)$: For example, for $\tilde{X}_4$ we have

$$\begin{aligned} \tilde{X}_4\left(\varphi^{-1}(A,B,C,D,E,F)\right) &= \frac{d}{dt}\Big|_{t=0} \exp(tX_4)\cdot(x,y) \\ &= (B-A)\partial_A + 2C\partial_B + C\partial_C + (E-D)\partial_D - F\partial_F \end{aligned} \tag{4.8}$$

In a similar fashion, we can prove that

**Theorem 5.** *If $\tilde{X}_i$ be the infinitesimal generator corresponding to one-parameter Lie group $g_i$, then*

$$\begin{aligned} \tilde{X}_1 &= -3\partial_A - 2\partial_B + 2A\partial_D + B\partial_E + D\partial_F \\ \tilde{X}_2 &= -\partial_A + B\partial_D + 2C\partial_E + E\partial_F \\ \tilde{X}_3 &= -A\partial_A - B\partial_B - C\partial_C - 2D\partial_D - 2E\partial_E - 3F\partial_F \\ \tilde{X}_4 &= (B-A)\partial_A + 2C\partial_B + C\partial_C + (E-D)\partial_D - F\partial_F \end{aligned} \tag{4.9}$$

If $I$ be an $G-$invariant, then it is necessary and sufficient that $I$ be a solution of the PDE

$$\left\{\tilde{X}_1(I)=0, \tilde{X}_2(I)=0, \tilde{X}_3(I)=0, \tilde{X}_4(I)=0\right\}. \tag{4.10}$$

Therefore,

**Theorem 6.** *Every sixth order $G-$invariant of action $Aff(2)$ on $M$ is a function of following invariants*

$$\begin{aligned} I_1 &= \frac{C^2(D+AB-B^2-E-2AC+3CB-2C^2)^2}{(4E+8AC+B^2-12CB+12C^2)^3} \\ I_2 &= \frac{C}{(4E+8AC+B^2-12CB+12C^2)^2}\Big(4CE+8AC^2 \\ &\quad +7CB^2-12C^2B+2F+2CA^2+2AE-B^3 \\ &\quad -3BE-8BAC+BD+AB^2-2CD+6C^3\Big) \end{aligned} \tag{4.11}$$

**Conclusion 5.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves in the form (2.6). The necessary condition for $\mathcal{C}_1$ and $\mathcal{C}_2$ being equivalent, is that $I_1(\mathcal{C}_1)=I_1(\mathcal{C}_1)$ and $I_2(\mathcal{C}_1)=I_2(\mathcal{C}_1)$.*

Now, we choose one representative from every equivalent coset, by applying the generating one-parameter groups of $Aff(2)$ on $\varphi^{-1}(A,B,C,D,E,F)\in M$. This, can reduce the section $M$ to a minimal section of the group action $Aff(2)$. on $\mathcal{M}$. That is, a section with one element from any coset. this complete the Conclusion 5, and prove the sufficient version of this fact.

Let $g_i(t_i)$, with $i=1,2,3,4$ of (4.10) operate respectively on $\varphi^{-1}(A,B,C,D,E,F)$, where $t_i\in\mathbb{R}$. Consider following seven cases:

**Case 1.** If $C > 0$ and

$$\Delta_1 := D + AB - B^2 - E - 2AC + 3CB - 2C^2 \neq 0 \tag{4.12}$$

then, we can assume

$$\begin{aligned} t_1 &= (e^{t_4} - 1)C + B/2, \\ t_2 &= \frac{1}{2e^{t_4}}\Big(2C(1 + e^{t_4} - 2e^{2t_4}) - B(e^{t_4} + 2) + 2A\Big), \\ t_3 &= t_4 + \ln(C), \qquad t_4 = (1/2)\ln\Big(|\Delta_1|/12C^2\Big), \end{aligned} \tag{4.13}$$

and obtain the curve $\varphi^{-1}(0, 0, 1, D_1, 0, F_1)$, with $\varepsilon = \operatorname{sgn}(\Delta_1) \in \{-1, 1\}$,

$$D_1 = 2 + 24\varepsilon\sqrt{3|I_1|}, \qquad F_1 = -1 + 72I_2 + 72\varepsilon\sqrt{|I_1|}. \tag{4.14}$$

We define

$$\mathcal{C}^1_{a,b} \,:\, x^3 + x^2y = y^2 + ax + b\,, \quad a, b \in \mathbb{R} \tag{4.15}$$

**Case 2.** If $C > 0$, $\Delta_1 = 0$ and

$$\Delta_2 := 4D + 4C^2 - 3B^2 + 4AB \neq 0 \tag{4.16}$$

then, $E = 4C(B - 1) - (B^2 + 8AC)/3$ and we can assume

$$\begin{aligned} t_1 &= B/2 + C(e^{t_4} - 1), \\ t_2 &= A - B + C - t_1 + e^{t_4}(B - 2t_1 - 2C) + Ce^{2t_4}, \\ t_3 &= (1/3)\ln\left(C|\Delta_2|/4\right), \qquad t_4 = t_3 - \ln C. \end{aligned} \tag{4.17}$$

and obtain the curve $\varphi^{-1}(0, 0, 1, \varepsilon - 1, -3, F_1)$, with a constant $F_1$.

We define

$$\mathcal{C}^2_{c,d} \,:\, x^3 + x^2y = y^2 + cx - 3y + d\,, \quad c \in \{-2, 0\}, \;\; d \in \mathbb{R} \tag{4.18}$$

**Case 3.** If $C > 0$ and $\Delta_1 = \Delta_2 = 0$, then $E = 4C(B - 1) - (B^2 + 8AC)/3$ and we can assume

$$\begin{aligned} t_1 &= C(e^{t_4} - 1) + B/2, \\ t_2 &= (2A - 3B + 4C)/2 - Ce^{t_4} - Ce^{2t_4}, \\ t_3 &= t_4 + \ln(C), \end{aligned} \tag{4.19}$$

and obtain the curve $\varphi^{-1}(0, 0, 1, -1, -3, 2 + \Delta_3/(4C^3e^{4t_4}))$, where

$$\begin{aligned} \Delta_3 \;:=\; & C(4E + 7B^2 + 2A^2 - 8BA - 2D) + 4C^2(2A - 3B) \\ & + 6C^3 + B(D - 3E) + AB^2 - B^3 + 2F + 2AE. \end{aligned} \tag{4.20}$$

If $\Delta_3 \neq 0$, we can assume $4t_4 = \ln(|\Delta_3|/4C^3)$, and obtain

$$2 + \Delta_3/(4C^3e^{4t_4}) = 2 + \operatorname{sgn}(\Delta_3) \tag{4.21}$$

Therefore, we obtain following curves:

$$\mathcal{C}^3_e \,:\, x^3 + x^2y = y^2 - x - 3y + e\,, \quad e \in \{1, 2, 3\} \tag{4.22}$$

**Case 4.** If $C = \Delta_1 = 0$ and $\Delta_2$ and $\Delta_3$ are not zero, then we can assume

$$\begin{aligned} t_1 &= B/2, \quad t_2 = (2A - 3B)/2, \\ t_3 &= \ln(2|\Delta_2 \Delta_3|), \\ t_4 &= 3 \ln|\Delta_2| - 2 \ln|\Delta_3|, \end{aligned} \tag{4.23}$$

and obtain the curve $\varphi^{-1}(0, 0, 0, \varepsilon, 0, \gamma)$, where $\varepsilon = \operatorname{sgn}(\Delta_2)$ and $\gamma = \operatorname{sgn}(\Delta_3)$. If $\gamma = -1$, we can use the transformation $(x, y) \mapsto (-x, -y)$, and obtain a curve with $\gamma = 1$. Therefore, we obtain following curves:

$$\mathcal{C}^4_f \, : \, x^3 + x^2 y = fx + 1\,, \quad f \in \{-1, 1\} \tag{4.24}$$

**Case 5.** If $C = \Delta_1 = \Delta_2 = 0$ and $\Delta_3$ be not zero, then we can assume

$$\begin{aligned} t_1 &= B/2, \quad t_2 = (2A - 3B)/2, \\ t_4 &= -3t_3 + \ln|\Delta_3|, \end{aligned} \tag{4.25}$$

and obtain the curve $\varphi^{-1}(0, 0, 0, 0, 0, \varepsilon)$, with $\varepsilon = \operatorname{sgn}(\Delta_3)$. If $\varepsilon = -1$, we can use the transformation $(x, y) \mapsto (-x, -y)$, and obtain a curve with $\varepsilon = 1$. Therefore, we obtain following curve:

$$\mathcal{C}^5 \, : \, x^3 + x^2 y = 1. \tag{4.26}$$

**Case 6.** If $C = \Delta_1 = \Delta_3 = 0$ and $\Delta_2$ be not zero, then we can assume

$$\begin{aligned} t_1 &= B/2, \quad t_2 = (2A - 3B)/2, \\ 2t_3 &= -t_4 + \ln|\Delta_2/4|, \end{aligned} \tag{4.27}$$

and obtain the curve $\varphi^{-1}(0, 0, 0, \varepsilon, 0, 0)$, with $\varepsilon = \operatorname{sgn}(\Delta_2)$. Therefore, we obtain following curves:

$$\mathcal{C}^6_h \, : \, x^3 + x^2 y = hx\,, \quad h \in \{-1, 1\}. \tag{4.28}$$

**Case 7.** If $C$, $\Delta_1$, $\Delta_2$ and $\Delta_3$ are zero, then we can assume

$$t_1 = B/2, \quad t_2 = (2A - 3B)/2, \tag{4.29}$$

and obtain the curve $\varphi^{-1}(0, 0, 0, 0, 0, 0)$, or

$$\mathcal{C}^7 \, : \, x^3 + x^2 y = 0. \tag{4.30}$$

Therefore, we prove that

**Theorem 7.** *Let $\mathcal{C}$ be a curve in the form (2.6), then only one of the following cases are possible:*

*1)* $\mathcal{C}^1_{a,b} \, : \, x^3 + x^2 y = y^2 + ax + b$*, with* $a, b \in \mathbb{R}$*;*

*2)* $\mathcal{C}^2_{c,d} \, : \, x^3 + x^2 y = y^2 + cx - 3y + d$*, with* $c \in \{-2, 0\}$ *and* $d \in \mathbb{R}$*;*

*3)* $\mathcal{C}^3_e : x^3 + x^2y = y^2 - x - 3y + e$, *with* $e \in \{1, 2, 3\}$;

*4)* $\mathcal{C}^4_f : x^3 + x^2y = fx + 1$, *with* $f \in \{-1, 1\}$;

*5)* $\mathcal{C}^5 : x^3 + x^2y = 1$;

*6)* $\mathcal{C}^6_g : x^3 + x^2y = gy$, *with* $g \in \{-1, 1\}$;

*7)* $\mathcal{C}^7 : x^3 + x^2y = 0$.

Finally, we prove the main Theorem:

**Theorem 8.** *Any cubic can be transformed by an affine transformation to one and only one of the following cubics:*

*1)* $\mathcal{C}^1_{a,b} : x^3 + x^2y = y^2 + ax + b$, *with* $a, b \in \mathbb{R}$ *and* $a \geq 2$;

*2)* $\mathcal{C}^2_{c,d} : x^3 + x^2y = y^2 + cx - 3y + d$, *with* $c \in \{-2, 0\}$ *and* $d \in \mathbb{R}$;

*3)* $\mathcal{C}^3_e : x^3 + x^2y = y^2 - x - 3y + e$, *with* $e \in \{1, 2, 3\}$;

*4)* $\mathcal{C}^4_f : x^3 + x^2y = fx + 1$, *with* $f \in \{-1, 1\}$;

*5)* $\mathcal{C}^5 : x^3 + x^2y = 1$;

*6)* $\mathcal{C}^6_h : x^3 + x^2y = hx$, *with* $h \in \{-1, 1\}$; *and*

*7)* $\mathcal{C}^7 : x^3 + x^2y = 0$.

*Furthermore, the isotropic sub-group of this curves are*

*1)* $\mathrm{Iso}(\mathcal{C}^1_{a,b}) = \{\mathrm{Id}\}$, *if* $a > 2$;

*2)* $\mathrm{Iso}(\mathcal{C}^1_{2,b}) = \{\mathrm{Id}, T\} \cong \mathbb{Z}_2$, *with* $T(x, y) = (-x - 2, 2x + y + 2)$;

*3)* $\mathrm{Iso}(\mathcal{C}^2_{c,d}) = \mathrm{Iso}(\mathcal{C}^3_e) = \mathrm{Iso}(\mathcal{C}^4_f) = \{\mathrm{Id}\}$;

*4)* $\mathrm{Iso}(\mathcal{C}^5) = \{T_a \,|\, a \in \mathbb{R}^+\}$, *where* $T_a(x, y) = (ax, (a^{-2} - a)x + a^{-2}y)$;

*5)* $\mathrm{Iso}(\mathcal{C}^6_h) = \{T_a \,|\, a \in \mathbb{R}^+\}$, *where* $T_a(x, y) = (ax, (a^{-1} - a)x + a^{-1}y)$; *and*

*6)* $\mathrm{Iso}(\mathcal{C}^7) = \{T_{a,b} \,|\, a, b \in \mathbb{R}^+\}$, *where* $T_{a,b}(x, y) = (ax, a(b - 1)x + aby)$.

*Proof:* By the Theorem 7, every cubic belongs to one of these seven families. It is therefore enough to show that any cubic of a family is not equivalent to one of the other families.

(1-1) Let $\mathcal{C}^1_{a,b}$ and $\mathcal{C}^1_{A,B} = g \cdot \mathcal{C}^1_{a,b}$ are equivalent, and

$$g := g_1(t_1) \circ g_2(t_2) \circ g_3(t_3) \circ g_4(t_4). \tag{4.31}$$

By solving the corresponding system of equations, and find that $\{t_1 = t_2 = 2, t_3 = t_4 = -1, A = 4 - a, B = b + 4 - 2a\}$, or $\{t_1 = t_2 = 0, t_3 = t_4 =$

$1, A = a, B = b\}$. Therefore $\mathcal{C}^1_{a,b}$ is equivalent to $\mathcal{C}^1_{4-a,b+4-2a}$, by the action $T : (x, y) \longmapsto (-x-2, 2x+y+2)$, where $T^2 = \mathrm{Id}$. If $a < 2$, then $4-a > 2$, and we can restrict the curves $\mathcal{C}^1_{a,b}$ in $a \geq 2$. Therefore, every two cubics of type $\mathcal{C}^1_{a,b}$ with $a \geq 4$, are not equivalent. If $a > 2$, then the isotropic sub-group $\mathrm{Iso}(\mathcal{C}^1_{a,b})$ of $\mathcal{C}^1_{a,b}$ is $\{\mathrm{Id}\}$; and otherwise, $\mathrm{Iso}(\mathcal{C}^1_{2,b}) = \{\mathrm{Id}, T\}$, with $T^2 = \mathrm{Id}$.

(1-2) If a curve be in the form $\mathcal{C}^2_{c,d}$, then $\Delta_1 = 0$, and $I_1$ and $I_2$ are not "undefined". Thus, every cubic of the type $\mathcal{C}^1_{a,b}$ is not equivalent with any cubic of type $\mathcal{C}^2_{c,d}$.

(1-3) This is similar to (1-2).

(1-C $\geq 4$) Let $\varphi^{-1}(0,0,0,\alpha,\beta,\gamma) = g \cdot \mathcal{C}^1_{a,b}$ are equivalent. By solving the corresponding system of equations, we find that $e^{t_4 - t_3} = 0$, which is imposible. Therefore, any curve of type $\mathcal{C}^1_{a,b}$ is not equivalent to any cubic of type $\mathcal{C}^a$ with $a \geq 4$.

(2-2) Let $\mathcal{C}^2_{0,d}$ and $\mathcal{C}^2_{\bar{c},\bar{d}} = g \cdot \mathcal{C}^2_{0,d}$ are equivalent. By solving the corresponding system of equations, we find that $\{t_1 = e^{t_4} - 1, t_2 = 1 - t_1 - 2e^{t_4}(t_1+1) + e^{2t_4}, t_3 = t_4, \bar{c} = e^{-3t_4} - 1, \bar{d} = 2 + e^{-4t_4}(d-3) + e^{-3t_4}\}$. If $\bar{c} = 0$, then $t_4 = 0$ and $\bar{d} = -2$. If $\bar{c} = -2$, then $e^{-3t_4} = -3$, which is impossible. Thus cubics $\mathcal{C}^2_{0,d}$ and $\mathcal{C}^2_{-2,\bar{d}}$, with $d \neq \bar{d}$, are not equivalent; and $\mathrm{Iso}(\mathcal{C}^2_{c,d}) = \{\mathrm{Id}\}$, for any $c$ and $d$.

(2-3) Let $\mathcal{C}^3_e = g \cdot \mathcal{C}^2_{c,d}$ are equivalent. By solving the corresponding system of equations, and find that $e^{-3t_4}(c+1) = 0$. But $c \in \{-2, 0\}$, which is imposible. Therefore, any curve of type $\mathcal{C}^3_e$ is not equivalent to any cubic of type $\mathcal{C}^2_{c,d}$.

(2-C $\geq 4$) This is similar to (1-C).

(3-3) Let $\mathcal{C}^3_e$ and $\mathcal{C}^3_{\bar{e}} = g \cdot \mathcal{C}^3_e$ are equivalent. By solving the corresponding system of equations, we find that $\{t_1 = e^{t_4} - 1, t_2 = 1 - t_1 - 2e^{t_4}(t_1+1) + e^{2t_4}, t_3 = t_4, \bar{e} - 2 = e^{-4t_4}(e-2)\}$. But $e, \bar{e} \in \{1,2,3\}$, therefore $e = \bar{e}$ and $t_4 = 0$. Thus, the curves $\mathcal{C}^3_e$ and $\mathcal{C}^3_{\bar{e}}$ are equivalent, if and only if $e = \bar{e}$; and $\mathrm{Iso}(\mathcal{C}^3_e) = \{\mathrm{Id}\}$, for any $e$.

(3-C $\geq 4$) This is similar to (1-C).

(4-4) Let $\mathcal{C}^4_f$ and $\mathcal{C}^4_{\bar{f}} = g \cdot \mathcal{C}^4_f$ are equivalent. By solving the corresponding system of equations, we find that $\{t_1 = t_2 = 0, t_4 = -3t_3, \bar{f} = e^{t_3} f\}$. But $f, \bar{f} \in \{-1, 1\}$, therefore $f = \bar{f}$ and $t_3 = 0$. Thus, the curves $\mathcal{C}^4_f$ and $\mathcal{C}^4_{\bar{f}}$ are equivalent, if and only if $f = \bar{f}$; and $\mathrm{Iso}(\mathcal{C}^4_f) = \{\mathrm{Id}\}$, for any $f$.

(4-C $\geq 5$) Let $\varphi^{-1}(0,0,0,0,\alpha,\beta) = g \cdot \mathcal{C}^4_f$ are equivalent. By solving the corresponding system of equations, we find that $e^{-t_4 - 2t_3} f = 0$, which is imposible. Therefore, any curve of type $\mathcal{C}^4_f$ is not equivalent to any cubic of type $\mathcal{C}^a$ with $a \geq 5$.

(5-5) Let $\mathcal{C}^5 = g \cdot \mathcal{C}^5$. By solving the corresponding system of equations, we find that $\{t_1 = t_2 = 0, t_4 = 3t_3\}$. Therefore, $\mathrm{Iso}(\mathcal{C}^5) = \{T_a \,|\, a \in \mathbb{R}^+\}$, where $T_a(x, y) = (ax, (a^{-2} - a)x + a^{-2}y)$.

(5-C $\geq 6$) This is similar to (4-C).

(6-6) Let $\mathcal{C}^6_h = g \cdot \mathcal{C}^6_{\bar{h}}$. By solving the corresponding system of equations, we find that $\{t_1 = t_2 = 0, \bar{h} = e^{t_4+2t_3}h\}$. But $h, \bar{h} \in \{-1, 1\}$, therefore $h = \bar{h}$ and $t_4 = -2t_3$. Thus, the curves $\mathcal{C}^6_h$ and $\mathcal{C}^6_{\bar{h}}$ are equivalent, if and only if $h = \bar{h}$; and $\mathrm{Iso}(\mathcal{C}^6_h) = \{T_a \mid a \in \mathbb{R}^+\}$, where $T_a(x, y) = (ax, (a^{-1} - a)x + a^{-1}y)$.

(6-7) This is similar to (4-C).

(7-7) Let $\mathcal{C}^7 = g \cdot \mathcal{C}^7$. By solving the corresponding system of equations, we find that $\{t_1 = t_2 = 0\}$. Therefore $\mathrm{Iso}(\mathcal{C}^7) = \{T_{a,b} \mid a, b \in \mathbb{R}^+\}$, where $T_{a,b}(x, y) = (ax, a(b - 1)x + aby)$. □

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