Nonlinear connections in Lagrange spaces with (α, β)-metrics

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Abstract. Within the framework of Geometry of Lagrange spaces endowed with (α, β)-metrics, is continued the study developed in [5], by examining the canonic nonlinear connection and investigating the properties of torsion and curvature.

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1 Introduction

In the Ph.D. Thesis [5] has been studied the theory of Lagrange spaces with (α, β)-metrics $L^n = (M, \check{L}(\alpha, \beta))$. This class of spaces includes the well known categories of Randers spaces $(RF^n = (M, (\alpha, \beta)^2)$ and the Lagrange spaces L^n of electrodynamics with have the fundamental function

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{mc}A_i(x)y^i + U(x).$$

In this paper, we introduce the concept of canonical nonlinear connection N with coefficients $N^i_j(x, y)$ in (3.1), which derives from the canonic semispray S described in (2.4) and (2.5). We study then the torsion t^i_{jk} and the curvature R^i_{jk} of N, and show that $t^i_{jk} = 0$. The condition $R^i_{jk} = 0$ gives the case when the nonlinear connection N is integrable. Throughout the paper, we use the results regarding the space $L^n = (M, \check{L})$ obtained in [5].

2 Lagrange spaces with (α, β)-metrics

This class of Lagrange spaces was studied by the author in his Ph.D. thesis [5] in 2003, presented at University of Craiova. Now we study problems related to the notion of nonlinear connection of a Lagrange space with (α, β)-metrics.

Let $\alpha = \sqrt{\gamma_{ij}(x)y^i y^j}$ and $\beta = A_i(x)y^i$ be a Riemannian norm produced by an underlying Riemannian metric $\gamma_{ij}(x)$ on M and respectively a transvected linear term

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produced by an electromagnetic field $A_i(x)$. Then the following Lagrangians:

1) $L(x,y) = \alpha^2(x,y) + \beta(x,y)$;
2) $L(x,y) = a\alpha(x,y) + b\beta(x,y) + c\beta^2(x,y), \quad a,b,c \in \mathbb{R}, a \neq 0$;
3) $L(x,y) = (\alpha(x,y) + \beta(x,y))^2$,

are (α,β)-type Lagrangians, being expressed in terms of α and β. We further focus on a Lagrangian of the type

$$L(x,y) = \check{L}(\alpha(x,y) + \beta(x,y)). \tag{2.1}$$

Then the fundamental metric tensor field of the space is given by

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 \check{L}}{\partial y^i \partial y^j} = \rho\gamma_{ij} + \rho_0 A_i A_j + \rho_{-1}(y_i A_j + y_j A_i) + \rho_{-2} y_i y_j, \tag{2.2}$$

where $y_i = \gamma_{ij} y^j$ and $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ are the corresponding coefficients in the detailed expression of g_{ij},

$$\begin{aligned} g_{ij} = \; & \tfrac{1}{2}\{\alpha^{-2}\check{L}_{\alpha\alpha} y_i y_j + \alpha^{-1}\check{L}_{\alpha\beta}(y_i A_j + y_j A_i) + \\ & + \check{L}_{\beta\beta} A_i A_j + \check{L}_\alpha \alpha^{-1}(\gamma_{ij}(x) - \alpha^{-2} y_i y_j)\}. \end{aligned} \tag{2.3}$$

The canonic semispray S of the space $L^n = (M, \check{L}(\alpha,\beta))$ obtained in [5] is

$$S = y^i \frac{pp}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{2.4}$$

where the coefficients G^i are given by

$$2G^i(x,y) = \left\{ {i \atop jk} \right\} y^j y^k - \lambda(x,y) F^i_j y^j, \tag{2.5}$$

with

$$\lambda(x,y) = \frac{\check{L}_\beta}{\check{L}_\alpha}, \quad F^i_j = \gamma^{ih} F_{hj}, \quad F_{hj} = \frac{1}{2}\frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j}. \tag{2.6}$$

Obviously, here $F_{ij}(x)$ is *the electromagnetic tensor* of the space, and $\left\{ {i \atop jk} \right\}(x)$ are the Christoffel symbols of $\gamma_{ij}(x)$. The semispray S is called *canonic*, since it depends only on the fundamental function $\check{L}(\alpha,\beta)$.

3 Nonlinear connections of the space $L^n = (M, \check{L})$

The canonic semispray S allows us to determine the nonlinear connection N of the Lagrange space with (α,β)-metrics$L^n = (M, \check{L}(\alpha,\beta))$, which is canonical, since it depends on $\check{L}$ only ([1]). Following the general theory in [1], it follows that the coefficients of N are

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{3.1}$$

From (2.5) and (2.6), we obtain

$$N^i_j = \left\{{i \atop jk}\right\} y^k - \frac{1}{2}\left\{\lambda F^i_j + \frac{\partial \lambda}{\partial y^i} F^i_r y^r\right\}. \tag{3.2}$$

where

$$\frac{\partial \lambda}{\partial y^i} = \frac{1}{\check{L}_\alpha}\left\{\check{L}_\alpha \frac{\partial \check{L}_\beta}{\partial y^i} - \check{L}_\beta \frac{\partial \check{L}_\alpha}{\partial y^j}\right\}. \tag{3.3}$$

A convenient form for N^i_j is

$$N^i_j = \left\{{i \atop jk}\right\}(x) y^k - \frac{1}{2}\lambda^k_j F^i_k, \text{ where } \lambda^k_j = \lambda \delta^k_j + \frac{\partial \lambda}{\partial y^j}^k. \tag{3.4}$$

The horizontal distribution determined by N has the property

$$T_p(TM) = N_p \oplus V_p, \quad \forall p \in TM. \tag{3.5}$$

The associated adapted dual basis is $(dx^i, \delta y^i)_p$, $\delta y^i = dy^i + N^i_j dx^j$. The weak torsion of N is

$$t^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j},$$

and the curvature of N is

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}.$$

Obviously, N depends on the fundamental function $\check{L}(\alpha, \beta)$ only. Hence, we have the following

Theorem 1. *The non-linear connection N of the space L^n produced by the canonical spray S has the local coefficients (3.4). Moreover, N depends only on the fundamental function $\check{L}(\alpha, \beta)$ and its weak curvature t^i_{jk} identically vanishes.*

Indeed, the coefficients N^i_j defined by (3.1) have the form (3.4), and depend only on $\check{L}(\alpha, \beta)$; as well, the weak torsion is $t^i_{jk} = -\frac{1}{2}\left\{\frac{\partial \lambda^i_j}{\partial y^k} - \frac{\partial \lambda^i_k}{\partial y^j}\right\}$, q.e.d..

The connection N is called *the canonical nonlinear connection of the space L^n*. It is straightforward to prove

Theorem 2. The autoparallel curves of the canonical nonlinear connection N are locally given by

$$\frac{dx^i}{dt} = y^i, \quad \frac{\delta y^i}{\delta t} \equiv \frac{dy^i}{dt} + \left\{{i \atop jk}\right\} y^j y^k - \frac{1}{2}\lambda^k_j F^i_k y^j = 0.$$

Example. For the Randers spaces ($RF^n = (M, \alpha + \beta)$, we have $L = (\alpha + \beta)^2$, hence $\lambda = 1$. The canonical nonlinear connection has the coefficients

$$N^i_j = \left\{{i \atop jk}\right\}(x) y^k - \frac{1}{2} F^i_j.$$

The autoparallel curves of N are in fact the Lorentz equations corresponding to the potentials $A_i(x)$,

$$\frac{dx^i}{dt} = y^i, \quad \frac{d^2 x^i}{dt^2} + \left\{{i \atop jk}\right\} \frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2} F^i_j \frac{dx^j}{dt}.$$

Returning to the general case, we shall consider further the curvature R^i_{jk} of the canonical nonlinear connection N. We obtain from (3.4) that

$$R^i_{jk} = y^s \rho_s{}^i{}_{jk} + \frac{1}{2}(\lambda^s_j F^i_{s|k} - \lambda^s_k F^i_{s|j}) - \frac{1}{2}\left(\frac{\delta\lambda^s_j}{\delta x^k} - \frac{\delta\lambda^s_k}{\delta x^j}\right) F^i_s,$$

where $\rho_s{}^i{}_{jk}$ is the tensor of curvature of the Riemannian metric γ_{ij} and $_{|k}$ is the covariant derivative w.r.t. $\frac{\partial}{\partial x^k}$ corresponding to the Levi-Civita connection associated to γ_{ij}.

In the case of Randers spaces $\check{L} = (\alpha + \beta)^2$, the tensor R^i_{jk} has the expression

$$R^i_{jk} = y^s \rho_s{}^i{}_{jk} + \frac{1}{2}(F^i_{j|k} - F^i_{k|j}).$$

In this case, the condition $R^i_{jk} = 0$ is equivalent to $\rho_s{}^i{}_{jk} = 0, F^i_{j|k} = 0$. In general, the space $L^n = (M, \check{L})$ has the canonical nonlinear connection integrable iff the curvature tensor R^i_{jk} of N identically vanishes.

Analogously, one can study the equation $R^i_{jk} = 0$ in the notable cases considered in [5], this being subject of further concern.

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