Parallel displacements in the bunch of connections of first type on distributions of planes

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Abstract. In multi-dimensional projective space, the distribution of planes is considered. The composite equipment of the distribution, representing an analog to Cartan's equipment and Norden's normalization of 2-nd type, is made. The concepts of the bunch of connections of first type and its pre-bunches is introduced. The parallel displacements of equipping planes in the bunch of connections of first type are described.

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The given paper is continuation of research of distribution of planes NS_n, which represents an n-dimensional family of centered m-dimensional planes $P_m^* = \{A, P_m\}$, where $A \in P_m$. The equations of distributions NS_n in n-dimensional projective space P_n look like ([2]):

$$\omega_i^a = \Lambda_{iJ}^a \omega^J \qquad (I, J, K = \overline{1, n}; i, j, k = \overline{1, m}; a, b, c = \overline{m+1, n}),$$

where the components of the fundamental object Λ satisfy the differential equations

$$\Delta\Lambda_{iJ}^a - \delta_J^a \omega_i = \Lambda_{iJK}^a \omega^K \qquad (\Lambda_{i[JK]}^a = 0),$$

and the differential operator Δ acts as follows:

$$\Delta\Lambda_{iJ}^a = d\Lambda_{iJ}^a + \Lambda_{iJ}^b \omega_b^a - \Lambda_{jJ}^a \omega_i^j - \Lambda_{iK}^a \omega_J^K .$$

The principal bundle $G(NS_n)$ is associated to the distribution NS_n, whose base is the distribution, and the fiber — the subgroup of a stationarity $G \subset GP(n)$ of the centered plane P_m^*, and $\dim G = n^2 + m^2 - n(m-1)$. The fundamental-group connection in the bundle $G(NS_n)$ is set by Laptev's mode [1] by means of a field of connection object

$$\Gamma = \{\Gamma_{jk}^i, \Gamma_{ja}^i, \Gamma_{ij}, \Gamma_{ia}, \Gamma_{bi}^a, \Gamma_{bc}^a, \Gamma_{aj}^i, \Gamma_{ab}^i, \Gamma_{ai}, \Gamma_{ab}\}.$$

The composite equipment of the distribution NS_n, consisting in an assignment of fields of two planes in it, is produced via

$$C_{n-m-1} : P_m \oplus C_{n-m-1} = P_n, \qquad N_{m-1} : A \notin N_{m-1} \subset P_m,$$

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and the equipping planes are defined by the aggregate of points

$$B_a = A_a + \lambda_a^i A_i + \lambda_a A, \qquad B_i = A_i + \lambda_i A. \tag{1}$$

The object $\lambda = \{\lambda_a^i, \lambda_a, \lambda_i\}$ refers to ([3]) the equipping quasitensor of distribution NS_n.

We further transform the differentials of points B_a and B_i, substituting instead of differentials of components of equipping quasitensor their expressions through covariant differentials

$$d\lambda_a^i = \nabla\lambda_a^i + \lambda_b^i \tilde{\omega}_a^b - \lambda_a^j \tilde{\omega}_j^i - \tilde{\omega}_a^i,$$

$$d\lambda_a = \nabla\lambda_a + \lambda_b \tilde{\omega}_a^b - \lambda_a^i \tilde{\omega}_i - \tilde{\omega}_a, \qquad d\lambda_i = \nabla\lambda_i + \lambda_j \tilde{\omega}_i^j - \tilde{\omega}_i, \tag{2}$$

we get

$$dB_a = (\dots)_a^b B_b + (\nabla\lambda_a^i + l_{aj}^i \omega^j + l_{ab}^i \omega^b) B_i + (\Omega_a + (l_{ai} - \lambda_j l_{ai}^j)\omega^i +$$

$$+ (l_{ab} - \lambda_i l_{ab}^i)\omega^b) A, \tag{3}$$

$$dB_i = (\dots)_i^j B_j + (\dots)_i^a B_a + (\nabla\lambda_i + l_{ij}\omega^j + l_{ia}\omega^a) A, \tag{4}$$

where the following labels are defined

$$l_{aj}^i = \Gamma_{aj}^i + \Gamma_{kj}^i \lambda_a^k - \Gamma_{aj}^b \lambda_b^i - \Lambda_{kj}^b \lambda_a^k \lambda_b^i + \delta_j^i \lambda_a,$$

$$l_{ab}^i = \Gamma_{ab}^i + \Gamma_{jb}^i \lambda_a^j - \Gamma_{ab}^c \lambda_c^i - \Lambda_{jb}^c \lambda_a^j \lambda_c^i - \lambda_a \lambda_b^i,$$

$$l_{ai} = \Gamma_{ai} - \Gamma_{ai}^b \lambda_b + \Gamma_{ji} \lambda_a^j - \Lambda_{ji}^b \lambda_a^j \lambda_b, \tag{5}$$

$$l_{ab} = \Gamma_{ab} - \Gamma_{ab}^c \lambda_c + \Gamma_{ib} \lambda_a^i - \Lambda_{ib}^c \lambda_a^i \lambda_c - \lambda_a \lambda_b,$$

$$l_{ij} = \Gamma_{ij} - \Gamma_{ij}^k \lambda_k - \Lambda_{ij}^a (\lambda_a - \lambda_a^k \lambda_k) - \lambda_i \lambda_j,$$

$$l_{ia} = \Gamma_{ia} - \Gamma_{ia}^j \lambda_j - \Lambda_{ia}^b (\lambda_b - \lambda_b^j \lambda_j) - \lambda_i \lambda_a + \lambda_i \lambda_j \lambda_a^j.$$

$$\Omega_a = \nabla\lambda_a - \lambda_i \nabla\lambda_a^i \tag{5'}$$

Differentiating the expressions (5) and taking into consideration the differential relations ([2]) on components of next objects (connection object Γ, fundamental object Λ and equipping quasitensor λ [3]), we infer the following relations modulo the basic forms ω^I:

$$\Delta l_{aj}^i \equiv 0, \qquad \Delta l_{ab}^i - l_{aj}^i \omega_b^j \equiv 0, \qquad \Delta l_{ai} + l_{ai}^j \omega_j \equiv 0, \tag{6_1}$$

$$\Delta l_{ab} - l_{ai}\omega_b^i + l_{ab}^i \omega_i \equiv 0, \qquad \Delta l_{ij} \equiv 0, \qquad \Delta l_{ia} - l_{ij}\omega a^j \equiv 0. \tag{6_2}$$

The relations $(6_1, 6_2)$ allow us to formulate the following

Proposition. *The object* $l = \{l_{aj}^i, l_{ab}^i, l_{ai}, l_{ab}, l_{ij}, l_{ia}\}$, *whose components are determined by the formulas (5), is a tensor containing two elementary subtensors* $\{l_{aj}^i\}, \{l_{ij}\}$ *and four simple subtensors* $\{l_{aj}^i, l_{ab}^i\}, \{l_{aj}^i, l_{ai}\}, \{l_{aj}^i, l_{ab}^i, l_{ai}, l_{ab}\}, \{l_{ij}, l_{ia}\}$.

By analogy to ([4]), we provide the following

Definition. *The group connection* Γ *belongs to the bunch of connections of 1-st type, if the tensor* l *vanishes, i.e. the following equalities are fulfilled:*

$$a_1) l^i_{aj} = 0, \qquad a_2) l^i_{ab} = 0, \qquad b_1) l_{ai} = 0, \tag{7_1}$$

$$b_2) l_{ab} = 0, \qquad c_1) l_{ij} = 0, \qquad c_2) l_{ia} = 0. \tag{7_2}$$

The connections from the bunch of 1-st type will be called $a_{12}b_{12}c_{12}$-connections. If only a part of the conditions $(7_1, 7_2)$ are fulfilled, we shall speak of a pre-bunch of appropriate connections.

We shall denote this bunch of connections by $\overset{1}{\Gamma}$. The formulas for its components have the following form:

$$\begin{aligned}
\overset{1}{\Gamma}_{ij} &= \Gamma^k_{ij}\lambda_k + \Lambda^a_{ij}(\lambda_a - \lambda^k_a\lambda_k) + \lambda_i\lambda_j,\\
\overset{1}{\Gamma}_{ia} &= \Gamma^j_{ia}\lambda_j + \Lambda^b_{ia}(\lambda_b - \lambda^j_b\lambda_j) + \lambda_i\lambda_a - \lambda_i\lambda_j\lambda^j_a,\\
\overset{1}{\Gamma}{}^i_{aj} &= \Gamma^b_{aj}\lambda^i_b - \Gamma^i_{kj}\lambda^k_a + \Lambda^b_{kj}\lambda^k_a\lambda^i_b - \delta^i_j\lambda_a,\\
\overset{1}{\Gamma}{}^i_{ab} &= \Gamma^c_{ab}\lambda^i_c - \Gamma^i_{jb}\lambda^j_a + \Lambda^c_{jb}\lambda^j_a\lambda^i_c + \lambda_a\lambda^i_b,\\
\overset{1}{\Gamma}_{ai} &= \Gamma^b_{ai}\lambda_b - \overset{1}{\Gamma}_{ji}\lambda^j_a + \Lambda^b_{ji}\lambda^j_a\lambda_b,\\
\overset{1}{\Gamma}_{ab} &= \Gamma^c_{ab}\lambda_c - \overset{1}{\Gamma}_{ib}\lambda^i_a + \Lambda^c_{ib}\lambda^i_a\lambda_c + \lambda_a\lambda_b.
\end{aligned} \tag{8}$$

From the formulas (8) it follows that the components of the subobject $\Gamma_0 = \{\Gamma^i_{jk}, \Gamma^i_{ja}, \Gamma^a_{bi}, \Gamma^a_{bc}\} \subset \Gamma$ are parameters of the bunch.

Theorem 1. *The composite equipment of distribution of planes* NS_n *induces the bunch of group connections of 1-st type.*

In the bunch of group connections of first type it is possible to pick out an unique connection of 1-st type, as the scopes of parameters of the bunch are found in ([3]), and the remaining components of connection object are determined by the relations (8), in which the scopes of parameters of the bunch are substituted. Thus, the connection of 1-st type is set by the object

$$\overset{01}{\Gamma} = \{\overset{0}{\Gamma}{}^i_{jk}, \overset{0}{\Gamma}{}^i_{ja}, \overset{01}{\Gamma}_{ij}, \overset{01}{\Gamma}_{ia}, \overset{0}{\Gamma}{}^a_{bi}, \overset{0}{\Gamma}{}^a_{bc}, \overset{01}{\Gamma}{}^i_{aj}, \overset{01}{\Gamma}{}^i_{ab}, \overset{01}{\Gamma}_{ai}, \overset{01}{\Gamma}_{ab}\}.$$

From the expressions (3, 4) and the definition of the bunch, follow the statements below:

Theorem 2. *The Cartan's plane* C_{n-m-1} *is transferred in a parallel way if and only if, when it is displaced*

a) in Norden's normal of 1-st type $P_{n-m} = [A, B_a]$, *the displacement takes place relative to the subconnection* $\Gamma_1 = \{\Gamma^i_{jk}, \Gamma^i_{ja}, \Gamma^a_{bi}, \Gamma^a_{bc}, \Gamma^i_{aj}, \Gamma^i_{ab}\}$, *if* Γ_1 *belongs to* a_{12}-*pre-bunch of group subconnections;*

b) in Bortolotti's hyperplane $P_{n-1} = [B_a, B_i]$, *the displacement happens in linear combination of group connection* Γ, *defined by forms* $(5')$, *if* Γ *belongs to* $a_{12}b_{12}$-*pre-bunch of group subconnections;*

c) arbitrarily, the displacement is carried out in group connection Γ, if Γ does not belong to $a_{12}b_{12}$-pre-bunch of group subconnections.

Theorem 3. *Norden's normal of 2-nd type N_{m-1} is transferred in a parallel way in subconnection $\Gamma_2 = \{\Gamma^i_{jk}, \Gamma^i_{ja}, \Gamma_{ij}, \Gamma_{ia}\}$ if and only if, it is displaced*

a) in Bortolotti's hyperplane P_{n-1}, if Γ_2 belongs to c_{12}-pre-bunch of group subconnections;

b) arbitrarily, if Γ_2 does not belong to a c_{12}-pre-bunch of group subconnections.

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