On weakly symmetric Kenmotsu manifolds

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Abstract. We consider weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds. We show that there exist no weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds unless $\alpha+\sigma+\gamma$ and $\rho+\mu+\upsilon$ are everywhere zero, respectively.

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1 Introduction

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T. Q. Binh in [8] and [9].

A non-flat n-dimensional differentiable manifold (M,g), $n>3$, is called *pseudosymmetric* if there exists a 1-form α on M such that

$$\begin{gathered}(\nabla_X R)(Y,Z,V)=2\alpha(X)R(Y,Z)V+\alpha(Y)R(X,Z)V\\ +\alpha(Z)R(Y,X)V+\alpha(V)R(Y,Z)X+g(R(Y,Z)V,X)A,\end{gathered}$$

where $X,Y,Z,V\in\chi(M)$ are arbitrary vector fields and α is a 1-form on M. $A\in\chi(M)$ is the vector field corresponding through g to the 1-form α which is given by $g(X,A)=\alpha(X)$ (see [2]).

A non-flat n-dimensional differentiable manifold (M,g), $n>3$, is called *weakly symmetric* (see [8] and [9]) if there exist 1-forms α,β,γ and σ such that the condition

$$\begin{gathered}(\nabla_X R)(Y,Z,V)=\alpha(X)R(Y,Z)V+\beta(Y)R(X,Z)V\\ +\gamma(Z)R(Y,X)V+\sigma(V)R(Y,Z)X+g(R(Y,Z)V,X)P\end{gathered}\tag{1.1}$$

holds for all vector fields $X,Y,Z,V\in\chi(M)$. A weakly symmetric manifold (M,g) is pseudosymmetric if $\beta=\gamma=\sigma=\frac{1}{2}\alpha$ and $P=A$, locally symmetric if $\alpha=\beta=\gamma=\sigma=0$ and $P=0$. A weakly symmetric manifold is said to be *proper* if at least one of the 1-forms α,β,γ and σ is not zero or $P\neq 0$.

A non-flat n-dimensional differentiable manifold (M,g), $n>3$, is called *weakly Ricci-symmetric (*see [8] and [9]) if there exist 1-forms ρ,μ,υ such that the condition

$$(\nabla_X S)(Y,Z)=\rho(X)S(Y,Z)+\mu(Y)S(X,Z)+\upsilon(Z)S(X,Y),\tag{1.2}$$

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holds for all vector fields $X, Y, Z \in \chi(M)$. If $\rho = \mu = \upsilon$ then M is called pseudo Ricci-symmetric (see [3]).

If M is weakly symmetric, from (1.1), we have

$$(\nabla_X S)(Z,V) = \alpha(X)S(Z,V) + \beta(R(X,Z)V) + \gamma(Z)S(X,V) + \sigma(V)S(X,Z) + p(R(X,V)Z), \tag{1.3}$$

where p is defined by $p(X) = g(X,P)$ for all $X \in \chi(M)$ (see [9]). In [9], it was considered weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds. In [5] and [7], the authors studied weakly symmetric and weakly Ricci-symmetric K-contact manifolds and Lorentzian para-Sasakian manifolds, respectively.

In this study, we consider weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds.

2 Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be an n-dimensional almost contact metric manifold, where φ is a (1,1) tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well-known that (φ, ξ, η, g) satisfy

$$\eta(\xi) = 1 \quad , \quad \varphi\xi = 0, \tag{2.1}$$

$$\eta(X) = g(X,\xi),\ \varphi^2 X = -X + \eta(X)\xi, \tag{2.2}$$

$$\eta(\varphi X) = 0,\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M (see [1]).

If moreover

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X, \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g, then $(M, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold.

In a Kenmotsu manifold (see [6]), the following relations hold

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

$$S(X,\xi) = (1-n)\eta(X), \tag{2.8}$$

for any vector fields X, Y where R is the Riemannian curvature tensor and S is the Ricci tensor.

3 Weakly symmetric Kenmotsu Manifolds

In this chapter, we investigate weakly symmetric and weakly Ricci-symmetric Kenmotsu manifolds. Firstly we have;

Theorem 3.1 *There is no weakly symmetric Kenmotsu manifold* M*,* $n > 3$*, unless* $\alpha + \sigma + \gamma$ *is everywhere zero.*

Proof. Assume that M is a weakly symmetric Kenmotsu manifold. By the covariant differentiation of the Ricci tensor S with respect to X we have

$$(\nabla_X S)(Z,V) = \nabla_X S(Z,V) - S(\nabla_X Z, V) - S(Z, \nabla_X V). \tag{3.1}$$

So replacing V with ξ in (3.1) and using (2.8), (2.5) and (2.2) we obtain

$$(\nabla_X S)(Z,\xi) = (1-n)g(X,Z) - S(X,Z). \tag{3.2}$$

On the other hand replacing V with ξ in (1.3) and by the use of (2.7), (2.8), (2.1) and (2.2) we get

$$\begin{aligned}(\nabla_X S)(Z,\xi) \;=\;& (1-n)\alpha(X)\eta(Z) + \eta(X)\beta(Z) - \eta(Z)\beta(X)\\ &+(1-n)\gamma(Z)\eta(X) + \sigma(\xi)S(X,Z) + g(X,Z)p(\xi)\\ &-\eta(Z)p(X).\end{aligned} \tag{3.3}$$

Hence, comparing the right hand sides of the equations (3.2) and (3.3) we have

$$\begin{aligned}&(1-n)g(X,Z) - S(X,Z)\\ =\;& (1-n)\alpha(X)\eta(Z) + \eta(X)\beta(Z) - \eta(Z)\beta(X)\\ &+(1-n)\gamma(Z)\eta(X) + \sigma(\xi)S(X,Z) + g(X,Z)p(\xi) - \eta(Z)p(X).\end{aligned} \tag{3.4}$$

Therefore putting $X = Z = \xi$ in (3.4) and using (2.8) and (2.1) we get

$$(1-n)\left(\alpha(\xi) + \sigma(\xi) + \gamma(\xi)\right) = 0.$$

Since $n > 3$, so we obtain

$$\alpha(\xi) + \sigma(\xi) + \gamma(\xi) = 0 \tag{3.5}$$

holds on M.

Now we will show that $\alpha + \sigma + \gamma = 0$ holds for all vector fields on M. In (1.3) taking $Z = \xi$ similar to the previous calculations it follows that

$$\begin{aligned}&(1-n)g(X,V) - S(X,V) = (1-n)\alpha(X)\eta(V)\\ &+g(X,V)\beta(\xi) - \eta(V)\beta(X) + \gamma(\xi)S(X,V)\\ &+(1-n)\sigma(X)\eta(X) + \eta(X)p(V) - \eta(V)p(X).\end{aligned} \tag{3.6}$$

Putting $V = \xi$ in (3.6) by virtue of (2.1) and (2.8) we get

$$\begin{aligned}0 = (1-n)\alpha(X) + \eta(X)\beta(\xi) - \beta(X) + (1-n)\gamma(\xi)\eta(X)\\ +(1-n)\sigma(\xi)\eta(X) + \eta(X)p(\xi) - p(X).\end{aligned} \tag{3.7}$$

Now taking $X = \xi$ in (3.6) we have

$$\begin{aligned} 0 &= (1-n)\alpha(\xi)\eta(V) \\ &\quad +(1-n)\gamma(\xi)\eta(V) + (1-n)\sigma(V) + p(V) - \eta(V)p(\xi). \end{aligned} \tag{3.8}$$

Replacing V with X in (3.8) and summing with (3.7), in view of (3.5), we find

$$(1-n)\alpha(X) + \eta(X)\beta(\xi) - \beta(X) + (1-n)\gamma(\xi)\eta(X) + (1-n)\sigma(X) = 0. \tag{3.9}$$

Now putting $X = \xi$ in (3.4) we have

$$(1-n)\alpha(\xi)\eta(Z) + \beta(Z) - \eta(Z)\beta(\xi) + (1-n)\gamma(Z) + (1-n)\sigma(\xi)\eta(Z) = 0. \tag{3.10}$$

Replacing Z with X in (3.10) and taking the summation with (3.9), we have

$$\begin{aligned} 0 &= (1-n)\eta(X)\left[\alpha(\xi) + \sigma(\xi) + \gamma(\xi)\right] \\ &\quad +(1-n)\left[\alpha(X) + \sigma(X) + \gamma(X)\right]. \end{aligned}$$

So in view of (3.5) we obtain $\alpha(X) + \sigma(X) + \gamma(X)$ for all X on M. This completes the proof of the theorem. □

Theorem 3.2 *There is no weakly Ricci-symmetric Kenmotsu manifold M, $n > 3$, unless $\rho + \mu + \upsilon$ is everywhere zero.*

Proof. Suppose that M is a weakly Ricci-symmetric Kenmotsu manifold. Replacing Z with ξ in (1.2) and using (2.8) we have

$$(\nabla_X S)(Y, \xi) = (1-n)\rho(X)\eta(Y) + (1-n)\mu(Y)\eta(X) + \upsilon(\xi)S(X,Y). \tag{3.11}$$

So in view of (3.11) and (3.2) we obtain

$$\begin{aligned} &(1-n)g(X,Y) - S(X,Y) \\ &= (1-n)\rho(X)\eta(Y) + (1-n)\mu(Y)\eta(X) + \upsilon(\xi)S(X,Y). \end{aligned} \tag{3.12}$$

Taking $X = Y = \xi$ in (3.12) and by the use of (2.8), (2.1) and (2.2) we get

$$(1-n)\left(\rho(\xi) + \mu(\xi) + \upsilon(\xi)\right) = 0,$$

which gives (since $n > 3$)

$$\rho(\xi) + \mu(\xi) + \upsilon(\xi) = 0. \tag{3.13}$$

Now putting $X = \xi$ in (3.12) we have

$$(1-n)\eta(Y)\left[\rho(\xi) + \upsilon(\xi)\right] + (1-n)\mu(Y) = 0.$$

So by virtue of (3.13) this yields

$$(1-n)\left[-\mu(\xi)\eta(Y) + \mu(Y)\right] = 0,$$

which gives us (since $n > 3$)

$$\mu(Y) = \mu(\xi)\eta(Y). \tag{3.14}$$

Similarly taking $Y = \xi$ in (3.12) we also have

$$\rho(X) + \eta(X)\,[\mu(\xi) + \upsilon(\xi)] = 0,$$

hence applying (3.13) into the last equation we find

$$\rho(X) = \rho(\xi)\eta(X). \tag{3.15}$$

Since $(\nabla_{\xi} S)(\xi, X) = 0$, then from (1.2) we obtain

$$\eta(X)\,[\rho(\xi) + \mu(\xi)] + \upsilon(X) = 0.$$

So by making use of (3.13) the last equation reduces to

$$\upsilon(X) = \upsilon(\xi)\eta(X). \tag{3.16}$$

Therefore changing Y with X in (3.14) and by the summation of the equations (3.14), (3.15) and (3.16) we obtain

$$\rho(X) + \mu(X) + \upsilon(X) = (\rho(\xi) + \mu(\xi) + \upsilon(\xi))\,\eta(X)$$

and so in view of (3.13) it follows that

$$\rho(X) + \mu(X) + \upsilon(X) = 0,$$

for all X, which implies $\rho + \mu + \upsilon = 0$ on M. Our theorem is thus proved. □

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