Integrability conditions for the homogeneous almost product structure

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Abstract. In this paper we study the integrability conditions for the homogeneous almost product structure on the tangent bundle, equipped with nonlinear connection and adapted frame. The space (M,∇) with linear connection and Riemann space with Levi-Civita connection are studied.

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1 Introduction

Let (TM,π,M) be the tangent bundle, where M is a C^∞ -differentiable, real n-dimensional manifold. If (U,φ) is a local chart on M then the coordinates of a point $u=(x,y)\in\pi^{-1}(U)\in TM$ will be denotes (x^i,y^i), $(i,j,k=\overline{1,n})$. The natural basis of the module $\mathcal{X}(TM)$ is given by $(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i})$.

Given a nonlinear connection N on TM, there exist a single system of functions $N_i^j(x,y)$ which are 1-homogeneous with respect to y^i such that $\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^j\frac{\partial}{\partial y^j}$ and $(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i})$ is a local basis of $\mathcal{X}(TM)$, which is called the adapted basis to N. The vector fields $\frac{\delta}{\partial x^i},\frac{\partial}{\partial y^i}$ are 1 and 0-homogeneous, respectively. The tensors of curvature and torsion of nonlinear connection N are given by $R_{jk}^{(i)}=\frac{\delta N_k^i}{\delta x^j}-\frac{\delta N_j^i}{\delta x^k}$ and $t_{kr}^s=\frac{\partial N_k^s}{\partial y^r}-\frac{\partial N_r^s}{\partial y^k}$, respectively. Let us prove that the almost complex structure F, defined by $F(\frac{\delta}{\delta x^i})=-\frac{\partial}{\partial y^i}$, $F(\frac{\partial}{\partial y^i})=\frac{\delta}{\delta x^i}$ does not preserve the property of homogeneity of the vector fields. R. Miron in [3], [4] eliminates this inconvenient by defining a new kind of almost complex structure, setting $\overset{0}{F}(\frac{\delta}{\delta x^i})=-\frac{\|y\|}{a}\frac{\partial}{\partial y^i}$, $\overset{0}{F}(\frac{\partial}{\partial y^i})=\frac{a}{\|y\|}\frac{\delta}{\delta x^i}$ where $\|y\|=\mathbf{F}=\sqrt{g_{ij}(x,y)y^iy^j}$ and g_{ij} is metric tensor of the Finsler space $(M,\mathbf{F})$. In this case, the integrability conditions for the homogeneous almost complex structure has been studied. In the case of Riemann space $(M,g_{ij}(x))$, $\overset{0}{F}$ is integrable if and only if the space has the positive constant sectional curvature [2]. Using the cotangent bundle, P. Stavre and L. Popescu have studied the properties of homogeneous structures in [5], [6], [7], [8]. Other point of view on the homogeneity of the geometric structures on tangent bundle is given by D. Bao, S. Chen and Z. Zhen in [1].

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2 Homogeneous almost product structure

We can define the almost product structure $P{:}\mathcal{X}(\widetilde{TM}) \to \mathcal{X}(\widetilde{TM})$ considering $P(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial y^i}$, $P(\frac{\partial}{\partial y^i}) = \frac{\delta}{\delta x^i}$, but does not preserve the property of homogeneity of vector fields. Let $L : \widetilde{TM} = TM\backslash\{0\} \to \mathbb{R}$ a differentiable function which is 1-homogeneous with respect to y^i. If we consider linear mapping $\mathbf{P}{:}\mathcal{X}(\widetilde{TM}) \to \mathcal{X}(\widetilde{TM})$ given by:

$$\mathbf{P}(\frac{\delta}{\delta x^i}) = \frac{L}{r}\frac{\partial}{\partial y^i}, \quad \mathbf{P}(\frac{\partial}{\partial y^i}) = \frac{r}{L}\frac{\delta}{\delta x^i}, \tag{2.1}$$

($r > 0$ is a constant) then the following results can be proved without difficulties:

Proposition 1 *1° $\mathbf{P}$ is an almost product structure $\mathbf{P}^2 = I$.*
2° $\mathbf{P}$ preserves the property of homogeneity of vector fields from $\mathcal{X}(TM)$.

It is important to know when $\mathbf{P}$ is integrable.

Theorem 1 *The homogeneous almost product structure $\mathbf{P}$ is integrable if and only if the tensors of torsion and curvature of nonlinear connection N satisfy the relations:*

$$R^{(k)}_{rs} = \frac{L}{r^2}(\frac{\partial L}{\partial y^r}\delta^k_s - \frac{\partial L}{\partial y^s}\delta^k_r), \tag{2.2}$$

$$t^s_{kr} = \frac{1}{L}(\frac{\delta L}{\delta x^r}\delta^s_k - \frac{\delta L}{\delta x^k}\delta^s_r). \tag{2.3}$$

Proof. Let $\mathbf{N}$ be the Nijenhuis tensor of the homogeneous almost product structure $\mathbf{P}$

$$\mathbf{N}(X,Y) = [\mathbf{P}X, \mathbf{P}Y] - \mathbf{P}[\mathbf{P}X, Y] - \mathbf{P}\ [X, \mathbf{P}Y] + \mathbf{P}^2[X,Y]. \tag{2.4}$$

In the adapted basis we have the unique decomposition

$$\begin{cases} \mathbf{N}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = \mathbf{N}^k_{ji}\frac{\delta}{\delta x^k} + \mathbf{N}^{(k)}_{ji}\frac{\partial}{\partial y^k}, \\ \mathbf{N}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = \mathbf{N}^{\ k}_{(j)i}\frac{\delta}{\delta x^k} + \mathbf{N}^{(k)}_{(j)i}\frac{\partial}{\partial y^k}, \\ \mathbf{N}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = \mathbf{N}^k_{(j)(i)}\frac{\delta}{\delta x^k} + \mathbf{N}^{(k)}_{(j)(i)}\frac{\partial}{\partial y^k}, \end{cases}$$

where

$$\begin{cases} \mathbf{N}^r_{jk} = \frac{\partial N^r_j}{\partial y^k} - \frac{\partial N^r_k}{\partial y^j} + \frac{1}{L}(\frac{\delta L}{\delta x^j}\delta^r_k - \frac{\delta L}{\delta x^k}\delta^r_j), \\ \mathbf{N}^{(r)}_{jk} = \frac{L}{r^2}(\frac{\partial L}{\partial y^k}\delta^r_j - \frac{\partial L}{\partial y^j}\delta^r_k) + R^{(r)}_{jk}. \end{cases} \tag{2.5}$$

$$\begin{cases} \mathbf{N}^s_{(r)k} = \frac{1}{L}(\frac{\partial L}{\partial y^r}\delta^s_k - \frac{\partial L}{\partial y^k}\delta^s_r) - \frac{r^2}{L^2}R^{(s)}_{rk}, \\ \mathbf{N}^{(s)}_{(r)k} = \frac{\partial N^s_k}{\partial y^r} - \frac{\partial N^s_r}{\partial y^k} - \frac{1}{L}(\frac{\delta L}{\delta x^r}\delta^s_k - \frac{\delta L}{\delta x^k}\delta^s_r). \end{cases} \tag{2.6}$$

$$\begin{cases} \mathbf{N}^i_{(r)(s)} = \frac{r^2}{L^2}(\frac{\partial N^i_r}{\partial y^s} - \frac{\partial N^i_s}{\partial y^r}) + \frac{1}{L}(\frac{\delta L}{\delta x^r}\delta^i_s - \frac{\delta L}{\delta x^s}\delta^i_r), \\ \mathbf{N}^{(i)}_{(r)(s)} = \frac{r^2}{L^2}(R^{(i)}_{rs} + \frac{L}{r^2}(\frac{\partial L}{\partial y^s}\delta^i_r - \frac{\partial L}{\partial y^r}\delta^i_s)). \end{cases} \tag{2.7}$$

From (2.5), (2.6), (2.7) we have

$$\mathbf{N}^r_{jk} = -\mathbf{N}^{(r)}_{(j)k} = \frac{L^2}{r^2}\mathbf{N}^r_{(j)(k)},$$

$$\mathbf{N}^{(r)}_{jk} = -\frac{L^2}{r^2}\mathbf{N}^r_{(j)k} = \frac{L^2}{r^2}\mathbf{N}^{(i)}_{(r)(s)},$$

so the Nijenhuis tensor vanishes if and only if $\mathbf{N}^r_{jk} = 0$ and $\mathbf{N}^{(r)}_{jk} = 0$ which imply (2.2) and (2.3). □

2.1 Special cases

Let (M, ∇) be the space with linear connection ∇ having the coefficients $\Gamma^i_{jk}(x)$. We consider the nonlinear connections N with the coefficients given by $\overset{c}{N^r_k}(x, y) = y^s\Gamma^r_{sk}(x)$. In this case we have:

Proposition 2 *The homogeneous almost product structure is integrable if and only if ∇ is semi-symmetric connection and*

$$y^k\mathbf{R}^i_{ksr} = \frac{L}{r^2}\left(\frac{\partial L}{\partial y^r}\delta^i_s - \frac{\partial L}{\partial y^s}\delta^i_r\right) \tag{2.8}$$

where $\mathbf{R}^i_{skj}$ is the tensor of curvature of linear connection ∇.

Proof. We have $t^s_{kr} = \frac{\partial N^s_k}{\partial y^r} - \frac{\partial N^s_r}{\partial y^k} = \Gamma^s_{kr} - \Gamma^s_{rk} = \mathbf{T}^s_{kr}$ and $R^{(i)}_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k} = y^s\mathbf{R}^i_{skj}$, where $\mathbf{T}^s_{kr}$ is the tensor of torsion of linear connection ∇. It follows

$$\mathbf{T}^s_{kr} = \frac{1}{L}\left(\frac{\delta L}{\delta x^r}\delta^s_k - \frac{\delta L}{\delta x^k}\delta^s_r\right) \tag{2.9}$$

From (2.9) we have $\mathbf{T}_k = \frac{1-n}{L}\frac{\delta L}{\delta x^k}$ and it results

$$\mathbf{T}^r_{jk} = \frac{1}{n-1}(\mathbf{T}_j\delta^r_k - \mathbf{T}_k\delta^r_j)$$

If we denote $\frac{1}{n-1}\mathbf{T}_j = \omega_j$ then we have

$$\mathbf{T}^r_{jk} = \omega_j\delta^r_k - \omega_k\delta^r_j \tag{2.10}$$

so ∇ is semi-symmetric connection. Conversely, if ∇ is semi-symmetric connection we have $\mathbf{T}(X, Y) = \omega(Y)X - \omega(X)Y$, so $\mathbf{T}^r_{jk} = \omega_j\delta^r_k - \omega_k\delta^r_j$. But $\mathbf{N}^r_{jk} = \mathbf{T}^r_{jk} + \frac{1}{L}(\frac{\delta L}{\delta x^j}\delta^r_k - \frac{\delta L}{\delta x^k}\delta^r_j) = (\omega_j + \frac{1}{L}\frac{\delta L}{\delta x^j})\delta^r_k - (\omega_k + \frac{1}{L}\frac{\delta L}{\delta x^k})\delta^r_j$. Considering $\omega_j = -\frac{1}{L}\frac{\delta L}{\delta x^j}$ we have $\mathbf{N}^r_{jk} = 0$. From $y^k\mathbf{R}^i_{ksr} = R^{(i)}_{rs} = \frac{L}{r^2}(\frac{\partial L}{\partial y^r}\delta^i_s - \frac{\partial L}{\partial y^s}\delta^i_r)$ we get $\mathbf{N}^{(r)}_{jk} = 0$ which ends the proof. □

Proposition 3 *If ∇ is symmetric connection then the homogeneous almost product structure is integrable if and only if*

$$\begin{gathered}\tfrac{\partial L}{\partial x^k} - y^r\Gamma^j_{rk}(x)\tfrac{\partial L}{\partial y^j} = 0,\\ y^k\mathbf{R}^i_{ksr} = \tfrac{L}{r^2}\left(\tfrac{\partial L}{\partial y^r}\delta^i_s - \tfrac{\partial L}{\partial y^s}\delta^i_r\right).\end{gathered} \tag{2.11}$$

Proof. We have $\frac{\delta L}{\delta x^r}\delta^s_k - \frac{\delta L}{\delta x^k}\delta^s_r = 0 \Rightarrow (n-1)\frac{\delta L}{\delta x^r} = 0 \Rightarrow \frac{\delta L}{\delta x^r} = 0$. □

Proposition 4 *If L is differentiable of the null section $y = 0$ then the homogeneous almost product structure is integrable if and only if*

$$\mathbf{R}^r_{skj} = \frac{c_s(x)}{r^2}\left(c_j(x)\delta^r_k - c_k(x)\delta^r_j\right), \tag{2.12}$$

where the functions $c_s(x)$ satisfies the equations

$$\frac{\partial c_s(x)}{\partial x^k} = \Gamma^r_{sk}(x)c_r(x). \tag{2.13}$$

Proof. If L is differentiable at $y = 0$ there exist the functions $c_i(x)$ such as $L(x, y) = c_i(x)y^i$. In this case we have $\frac{\delta L}{\delta x^k} = y^s\left(\frac{\partial c_s(x)}{\partial x^k} - \Gamma^r_{sk}c_r(x)\right) = 0$, so $\frac{\partial c_s(x)}{\partial x^k} = \Gamma^r_{sk}(x)c_r(x)$. From $y^s\mathbf{R}^r_{skj} = \frac{1}{r^2}y^s c_s(x)\left(c_j(x)\delta^r_k - c_k(x)\delta^r_j\right)$ we get (2.12). □
From (2.12) we have:

Remark 1 *The Ricci tensor satisfies the equation*

$$\mathbf{r}_{sk}(x) = \frac{c_s(x)c_k(x)}{r^2}(1-n). \tag{2.14}$$

2.2 The case of Riemann space

Let $(M, g_{ij}(x))$ be a Riemann space with Levi-Civita connection ∇. We consider $L = \sqrt{g_{ij}(x)y^iy^j}$ which is 1-homogeneous. In this case we get:

Theorem 2 *The homogeneous almost product structure is integrable if and only if (M, g) is a Riemann space with negative constant sectional curvature $K = -\frac{1}{r^2}$.*

Proof. From (2.2) we have

$$R^{(r)}_{jk} = \frac{1}{r^2}(g_{js}y^s\delta^r_j - g_{ks}y^s\delta^r_j) \tag{2.15}$$

and it results

$$\mathbf{R}^r_{skj} = -\frac{1}{r^2}(g_{sk}\delta^r_j - g_{sj}\delta^r_k), \tag{2.16}$$

so $K = -\frac{1}{r^2}$. For $L = \sqrt{g_{ij}(x)y^iy^j}$ we have $\frac{\delta L}{\delta x^k} = 0$. □

Remark 2 *The homogeneous almost product structure and homogeneous almost complex structure cannot be simultaneous integrable on the same manifold.*

If $\mathbf{P}$ is integrable, we get:

Remark 3 *For $n = 2$ the Riemann space (M, g) is locally isometric with a pseudosphere of radium r.*

From (2.16) we have

$$\mathbf{r}_{ij} = -\frac{n-1}{r^2} g_{ij} = (n-1)K g_{ij}, \quad (n > 1) \tag{2.17}$$

where $\mathbf{r}_{ij}$ is the Ricci tensor and

$$\overline{r} = -\frac{n(n-1)}{r^2} < 0, \quad \overline{r} = n(n-1)K. \tag{2.18}$$

($\overline{r}$ is the scalar curvature and $K = -\frac{1}{r^2} < 0$ is the curvature of $(M, g_{ij}(x))$.

Remark 4 *If* $\mathbf{P}$ *is integrable then* (M, g) *is an Einstein space with negative scalar curvature.*

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