

# On $N$ -linear connections compatible with conformal structures on the 2-osculator bundle

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**Abstract.** The present paper deals with some problems on the conformal structure on  $Osc^2M$ . The transformation of the coefficients of an  $N$ -linear connection on  $Osc^2M$ , by a transformation of nonlinear connections is determined. The concepts of conformal structure and  $N$ -linear connection compatible with the conformal structure corresponding to a 1-form are introduced in the bundle of accelerations. Some properties of these notions are under consideration. The problem of determining the set of all  $N$ -linear connections compatible with the conformal structure, corresponding to a 1-form is solved for an arbitrary nonlinear connection. Some important particular cases are considered.

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**Key words:** 2-osculator bundle, conformal structure,  $N$ -linear connection compatible with the conformal structure corresponding to a 1-form.

## 1 Introduction

The conformal geometry of Finsler or Lagrange spaces was studied by many geometers (see eg. H.Rund in [12], M.Matsumoto in [5], K.Yano and S.Ishihara in [13], M. Hashiguchi in [3], R.Miron and M.Hashiguchi in [10], Y.Ichijyo in [4], P.L.Antonelli, R.S.Ingarden and M.Matsumoto in [1], R.Miron and M.Anastasiu in [7], R.Miron in [6], D.Bao, S.Chern and Z.Shen in [2] and others).

The literature on the higher order Lagrange spaces geometry highlights the theoretical and practical importance of these spaces, (see eg.R.Miron, M.Anastasiu and Gh.Atanasiu in [6]-[9]).

Due to its content, the present paper continues a trend of interest with a long tradition in the modern differential geometry, i.e. the study of remarkable geometrical structures.

All these problems are extensions of those studied in the paper [11]. The terminology and notations, are usually retained, which are essentially based on R.Miron's book: [6].

## 2 The set of the transformations of $N$ -linear connections in the 2-osculator bundle

Let  $M$  be a real  $n$ -dimensional  $C^\infty$ -manifold and let  $(Osc^2M, \pi, M)$  be its 2-osculator bundle, with  $E = Osc^2M$  the total space.

The local coordinates on  $E$  are denoted by:  $(x^i, y^{(1)i}, y^{(2)i})$ , briefly:  $(x, y^{(1)}, y^{(2)})$ .

If  $N$  is a nonlinear connection on  $E$ , with the coefficients  $(N_{(1)j}{}^i, N_{(2)j}{}^i)$ , then let  $D$  be an  $N$ -linear connection on  $E$ , with the coefficients  $DF(N) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ .

If  $\bar{N}$  is another nonlinear connection on  $E$ , with the coefficients  $(\bar{N}_{(1)j}{}^i, \bar{N}_{(2)j}{}^i)$ , then there exists a uniquely determined tensor fields  $A_{(\alpha)j}{}^i \in \tau_1^1(E)$ ,  $(\alpha = \bar{1}, \bar{2})$  such that:

$$(2.1) \quad \bar{N}_{(\alpha)j}{}^i = N_{(\alpha)j}{}^i - A_{(\alpha)j}{}^i, \quad (\alpha = \bar{1}, \bar{2}).$$

Conversely, if  $N_{(\alpha)j}{}^i$  and  $A_{(\alpha)j}{}^i$ ,  $(\alpha = \bar{1}, \bar{2})$  are given, then  $\bar{N}_{(\alpha)j}{}^i$ ,  $(\alpha = \bar{1}, \bar{2})$ , given by (2.1) is a nonlinear connection.

Let us suppose that the mapping  $N \rightarrow \bar{N}$  is given by (2.1).

According to [6], we have:

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^{(\alpha)j}} = L_{jk}{}^i \frac{\delta}{\delta y^{(\alpha)i}}, \quad D_{\frac{\delta}{\delta y^{(\beta)k}}} \frac{\delta}{\delta y^{(\alpha)j}} = C_{(\beta)jk}{}^i \frac{\delta}{\delta y^{(\alpha)i}},$$

$(\beta = \bar{1}, \bar{2}; \alpha = \bar{0}, \bar{2}; y^{(0)i} = x^i)$  and

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}{}^j \frac{\partial}{\partial y^{(1)j}} - N_{(2)i}{}^j \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - N_{(1)i}{}^j \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.$$

It follows first of all that the transformation (2.1) preserves the coefficients  $C_{(2)jk}{}^i$ . Taking in account the fact that:

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + A_{(1)i}{}^j \frac{\partial}{\partial y^{(1)j}} + A_{(2)i}{}^j \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + A_{(1)i}{}^j \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\bar{\delta}}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.$$

it follows:

$$\begin{aligned} D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\bar{\delta}}{\delta y^{(2)j}} &= D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} = \bar{L}_{jk}{}^i \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta x^k} + A_{(1)k}{}^l \frac{\partial}{\partial y^{(1)l}} + A_{(2)k}{}^l \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} = \\ &= D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} + A_{(1)k}{}^l D_{(\frac{\delta}{\delta y^{(1)l}} + N_{(1)l}{}^m \frac{\partial}{\partial y^{(2)m}})} \frac{\partial}{\partial y^{(2)j}} + A_{(2)k}{}^l D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} = \\ &= L_{jk}{}^i \frac{\partial}{\partial y^{(2)i}} + A_{(1)k}{}^l C_{(1)jl}{}^i \frac{\partial}{\partial y^{(2)i}} + A_{(1)k}{}^l N_{(1)l}{}^m C_{(2)jm}{}^i \frac{\partial}{\partial y^{(2)i}} + \\ &\quad A_{(2)k}{}^l C_{(2)jl}{}^i \frac{\partial}{\partial y^{(2)i}} = \\ &= (L_{jk}{}^i + A_{(1)k}{}^l C_{(1)jl}{}^i + A_{(1)k}{}^l N_{(1)l}{}^m C_{(2)jm}{}^i + A_{(2)k}{}^l C_{(2)jl}{}^i) \frac{\partial}{\partial y^{(2)i}}. \\ D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\bar{\delta}}{\delta y^{(2)j}} &= D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} = \bar{C}_{(1)jk}{}^i \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta y^{(1)k}} + A_{(1)k}{}^l \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} = \\ &= D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} + A_{(1)k}{}^l D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} = C_{(1)jk}{}^i \frac{\partial}{\partial y^{(2)i}} + A_{(1)k}{}^l C_{(2)jl}{}^i \frac{\partial}{\partial y^{(2)i}} = \\ &= (C_{(1)jk}{}^i + A_{(1)k}{}^l C_{(2)jl}{}^i) \frac{\partial}{\partial y^{(2)i}}. \end{aligned}$$

Therefore the change we are looking for is:

$$(2.2) \quad \begin{cases} \bar{L}_{jk}{}^i = L_{jk}{}^i + A_{(1)k}{}^l C_{(1)jl}{}^i + A_{(1)k}{}^l N_{(1)l}{}^m C_{(2)jm}{}^i + A_{(2)k}{}^l C_{(2)jl}{}^i \\ \bar{C}_{(1)jk}{}^i = C_{(1)jk}{}^i + A_{(1)k}{}^l C_{(2)jl}{}^i, \\ \bar{C}_{(2)jk}{}^i = C_{(2)jk}{}^i. \end{cases}$$

So, we have proved:

**Proposition 2.1** *The transformation (2.1) of nonlinear connections implies the transformations (2.2) for the local coefficients  $D\Gamma(N) = (L_{jk}^i, C_{(1)jk}^i, C_{(2)jk}^i)$  of the  $N$ -linear connection  $D$ .*

Now, using the result proved by M.Matsumoto in [5] we can easily prove:

**Theorem 2.1** *Let  $N$  and  $\bar{N}$  be two nonlinear connections on  $E$ , with local coefficients  $(N_{(1)j}^i, N_{(2)j}^i), (\bar{N}_{(1)j}^i, \bar{N}_{(2)j}^i)$ -respectively. If  $D\Gamma(N) = (L_{jk}^i, C_{(1)jk}^i, C_{(2)jk}^i)$  and  $D\bar{\Gamma}(\bar{N}) = (\bar{L}_{jk}^i, \bar{C}_{(1)jk}^i, \bar{C}_{(2)jk}^i)$  are the local coefficients of two  $N$ -, respectively  $\bar{N}$ -linear connections:  $D$ , respectively  $\bar{D}$  on the differentiable manifold  $E$ , then there exists only one quintet of tensor fields  $(A_{(1)j}^i, A_{(2)j}^i, B_{jk}^i, D_{(1)jk}^i, D_{(2)jk}^i)$  such that:*

$$(2.3) \quad \begin{cases} \bar{N}_{(\alpha)j}^i = N_{(\alpha)j}^i - A_{(\alpha)j}^i, & (\alpha = \overline{1,2}), \\ \bar{L}_{jk}^i = L_{jk}^i + A_{(1)k}^l C_{(1)jl}^i + A_{(1)k}^l N_{(1)l}^m C_{(2)jm}^i + A_{(2)k}^l C_{(2)jl}^i - B_{jk}^i, \\ \bar{C}_{(1)jk}^i = C_{(1)jk}^i + A_{(1)k}^l C_{(2)jl}^i - D_{(1)jk}^i, \\ \bar{C}_{(2)jk}^i = C_{(2)jk}^i - D_{(2)jk}^i. \end{cases}$$

**Proof.** *The first equality (2.3) determines uniquely the tensor fields  $A_{(\alpha)j}^i$ ,  $(\alpha = \overline{1,2})$ . Since  $C_{(\alpha)jk}^i$ ,  $(\alpha = \overline{1,2})$  are tensor fields, the second equation (2.3) determines uniquely the tensor field  $B_{jk}^i$ . Similarly the third and the fourth equation (2.3) determine the tensor fields  $D_{(1)jk}^i$  and  $D_{(2)jk}^i$  respectively.*

We have immediately:

**Theorem 2.2** *If  $D\Gamma(N) = (L_{jk}^i, C_{(1)jk}^i, C_{(2)jk}^i)$  are the local coefficients of an  $N$ -linear connection  $D$  on  $E$  and  $(A_{(1)j}^i, A_{(2)j}^i, B_{jk}^i, D_{(1)jk}^i, D_{(2)jk}^i)$  is a quintet of tensor fields on  $E$ , then:  $D\bar{\Gamma}(\bar{N}) = (\bar{L}_{jk}^i, \bar{C}_{(1)jk}^i, \bar{C}_{(2)jk}^i)$  given by (2.3) are the local coefficients of an  $\bar{N}$ -linear connection,  $\bar{D}$ , on  $E$ .*

The tensor fields  $(A_{(1)j}^i, A_{(2)j}^i, B_{jk}^i, D_{(1)jk}^i, D_{(2)jk}^i)$  are called the difference tensor fields of  $D\Gamma(N)$  to  $D\bar{\Gamma}(\bar{N})$  and the mapping  $D\Gamma(N) \rightarrow D\bar{\Gamma}(\bar{N})$  given by (2.3) is called a transformation of  $N$ -linear connection to  $\bar{N}$ -linear connection, on  $Osc^2M$ .

### 3 The notion of $N$ -linear connection compatible with a conformal structure

We consider on  $E$  a  $d$ -structure, defined by a  $d$ -tensor field of the type  $(0, 2)$ , let us say  $g_{ij}(x^i, y^{(1)i}, y^{(2)i})$ , (on denotes the  $d$ -tensor field by its components), nondegenerate and symmetric (alternate), in the last case it is necessary that  $n=2n'$ .

We associate to this  $d$ -structure the Obata's operators:

$$(3.1) \quad \Omega_{sj}^{ir} = \frac{1}{2}(\delta_s^i \delta_j^r - g_{sj} g^{ir}), \Omega_{sj}^{*ir} = \frac{1}{2}(\delta_s^i \delta_j^r + g_{sj} g^{ir}),$$

where  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ .

Obata’s operators have the same properties as the ones associated with the metrical d-structure on E, ([10] for Finsler connections).

Let  $\mathcal{S}_2(E)$  be the set of all symmetric d-tensor fields of the type (0, 2) on E ( $\mathcal{A}_2(E)$  the set of all alternate d-tensor fields of type (0,2) on E). As is easily shown, the relation for  $b_{ij}, c_{ij} \in \mathcal{S}_2(E), (b_{ij}, c_{ij} \in \mathcal{A}_2(E))$  defined by:

$$(3.2) \quad b_{ij} \sim c_{ij} \iff \{\exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) | b_{ij} = e^{2\rho} c_{ij}\}$$

is an equivalent relation on  $\mathcal{S}_2(E) (\mathcal{A}_2(E))$ .

**Definition 3.1** *The equivalent class:  $\hat{g}$  of  $\mathcal{S}_2(E) / \sim (\mathcal{A}_2(E) / \sim)$  to which the d-structure  $g_{ij}$  belongs, is called a conformal structure on E.*

*Every  $g'_{ij} \in \hat{g}$  is a nondegenerate, symmetric (alternate) d-tensor field, expressed by:*

$$(3.3) \quad g'_{ij} = e^{2\rho} g_{ij}.$$

*Obata’s operators are defined for  $g'_{ij} \in \hat{g}$  by putting  $(g'^{ij}) = (g'_{ij})^{-1}$ . Since equation (3.3) is equivalent to:*

$$(3.4) \quad (g'^{ij}) = e^{-2\rho} g^{ij},$$

we have:

**Proposition 3.1** *Obata’s operators depend on the conformal structure  $\hat{g}$ , and do not depend on its representative  $g'_{ij} \in \hat{g}$ .*

Let N be a nonlinear connection on E with the coefficients  $(N_{(1)j}{}^i, N_{(2)j}{}^i)$  and let D be an N-linear connection on E with the local coefficients in the adapted basis  $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\} : D\Gamma(N) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ .

**Definition 3.2** *An N-linear connection D on E, for which there exists a 1-form  $\omega$  in  $\mathcal{X}^*(Osc^2M)$ , ( $\omega = \tilde{\omega}_i dx^i + \dot{\omega}_{(1)i} \delta y^{(1)i} + \dot{\omega}_{(2)i} \delta y^{(2)i}$ ) such that:*

$$(3.5) \quad g_{ij|k} = 2\tilde{\omega}_k g_{ij}, \quad g_{ij} \Big|_k^{(\alpha)} = 2 \dot{\omega}_{(\alpha)k} g_{ij}, \quad (\alpha = \overline{1, 2}),$$

where  $\Big|, \Big|^{(\alpha)}$  denote the h- and  $v_\alpha$ -covariant derivatives,  $(\alpha = \overline{1, 2})$ , with respect to D, is said to be compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , and is denoted by:  $D(N, \omega)$ .

For any representative  $g'_{ij} \in \hat{g}$  we have:

**Theorem 3.1** *For  $g'_{ij} = e^{2\rho} g_{ij}$ , an N-linear connection compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ ,  $D(N, \omega)$ , satisfies:*

$$(3.6) \quad g'_{ij|k} = 2\tilde{\omega}'_k g'_{ij}, \quad g'_{ij} \Big|_k^{(\alpha)} = 2 \dot{\omega}'_{(\alpha)k} g'_{ij}, \quad (\alpha = \overline{1, 2}),$$

where  $\omega' = \omega + d\rho$ .

Since in Theorem 3.1.  $\omega' = 0$  is equivalent to  $\omega = d(-\rho)$ , we have:

**Theorem 3.2** *An  $N$ -linear connection compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ ,  $D(N, \omega)$ , is a metrical (almost symplectic)  $N$ -linear connection with respect to some  $g'_{ij} \in \hat{g}$ , (i.e.  $g'_{ij|k} = 0, g'^{(\alpha)}_{ij|k} = 0, (\alpha = \overline{1, 2})$ ), if and only if  $\omega$  is exact.*

### 4 The set of all $N$ -linear connections compatible with the conformal structure $\hat{g}$

Let  $\overset{0}{N}$  and  $N$  be two nonlinear connections on  $E$ , with the coefficients  $(N_{(1)j}^0)^i, N_{(2)j}^0)^i$  and  $(N_{(1)j}^i, N_{(2)j}^i)$  respectively.

Let  $D \overset{0}{\Gamma}(\overset{0}{N}) = (L_{jk}^0)^i, C_{(1)jk}^0)^i, C_{(2)jk}^0)^i$ , be the local coefficients of an arbitrary fixed  $\overset{0}{N}$ -linear connection on  $E, \overset{0}{D}$ . Then any  $N$ -linear connection,  $D$ , on  $E$ , with the local coefficients  $D\Gamma(N) = (L_{jk}^i, C_{(1)jk}^i)^i, C_{(2)jk}^i)^i$ , can be expressed in the form (2.3), taking  $D$  for  $\bar{D}$  and  $\overset{0}{D}$  for  $D$ , where  $(A_{(1)j}^i, A_{(2)j}^i, B_{jk}^i, D_{(1)jk}^i, D_{(2)jk}^i)$  is the difference tensor fields of  $D \overset{0}{\Gamma}(\overset{0}{N})$  to  $D\Gamma(N)$ .

In order that  $D$  is an  $N$ -linear connection compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , (i.e.  $D(N, \omega)$ ), that is (3.5) holds for  $D(N, \omega)$ , it is necessary and sufficient that  $B_{jk}^i, D_{(1)jk}^i, D_{(2)jk}^i$  satisfy:

$$(4.1) \quad \left\{ \begin{array}{l} \Omega_{sj}^{*ir} B_{rk}^s = -\frac{1}{2} g^{im} [g_{mj|k}^0 + A_{(1)k}^l g_{mj|l}^{(1)} + \\ \quad + (A_{(2)k}^l + A_{(1)k}^r N_{(1)r}^l) g_{mj|l}^{(2)} - 2\tilde{\omega}_k g_{mj}], \\ \Omega_{sj}^{*ir} D_{(1)rk}^s = -\frac{1}{2} g^{im} (g_{mj|k}^{(1)} + A_{(1)k}^l g_{mj|l}^{(2)} - 2\dot{\omega}_{(1)k} g_{mj}), \\ \Omega_{sj}^{*ir} D_{(2)rk}^s = -\frac{1}{2} g^{im} (g_{mj|k}^{(2)} - 2\dot{\omega}_{(2)k} g_{mj}), \end{array} \right.$$

where  $\omega = \tilde{\omega}_i dx^i + \dot{\omega}_{(1)i} \delta y^{(1)i} + \dot{\omega}_{(2)i} \delta y^{(2)i}$  is an arbitrary fixed 1-form in  $\mathcal{X}^*(Osc^2 M)$

and  $\overset{0}{|}$  and  $\overset{(\alpha)}{|}$ ,  $(\alpha = \overline{1, 2})$ , denote the  $h$ - and  $v_\alpha$ -covariant derivatives,  $(\alpha = \overline{1, 2})$ , with respect to  $\overset{0}{D}$ .

Thus, we have:

**Proposition 4.1** *Let  $\overset{0}{D}$  be a fixed  $\overset{0}{N}$ -linear connection on  $E$ . Then the set of all  $N$ -linear connections compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ ,  $D(N, \omega)$  is given by (2.3), where  $B_{jk}^i, D_{(\alpha)jk}^i, (\alpha = \overline{1, 2})$ , are arbitrary tensor fields satisfying (4.1). Especially, if  $\overset{0}{D}$  is an  $N$ -linear connection compatible*

with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , (i.e.  $\overset{0}{D}(\overset{0}{N}, \omega)$ ), then (4.1) becomes:

$$(4.2) \quad \begin{cases} \Omega_{sj}{}^{*ir} B_{rk}{}^s = -\dot{\omega}_{(1)l} \delta_j^i A_{(1)k}{}^l - (A_{(2)k}{}^l + A_{(1)k}{}^r N_{(1)r}{}^l) \dot{\omega}_{(2)l} \delta_j^i, \\ \Omega_{sj}{}^{*ir} D_{(1)rk}{}^s = -\dot{\omega}_{(2)l} \delta_j^i A_{(1)k}{}^l \\ \Omega_{sj}{}^{*ir} D_{(2)rk}{}^s = 0, \end{cases}$$

where  $\overset{0}{\mathbb{D}}$  and  $\overset{(\alpha)}{\mathbb{D}}$ , ( $\alpha = \overline{1, 2}$ ), denote the  $h$ - and  $v_\alpha$ -covariant derivatives, ( $\alpha = \overline{1, 2}$ ), with respect to  $\overset{0}{D}(\overset{0}{N}, \omega)$ .

From [10], however, the system (4.1) has solutions in  $B_{jk}{}^i$ ,  $D_{(\alpha)jk}{}^i$ , ( $\alpha = \overline{1, 2}$ ). Substituting in (2.3) from the general solution we have:

**Theorem 4.1** Let  $\overset{0}{D}$  be a fixed  $\overset{0}{N}$ -linear connection on  $E$  with the local coefficients  $D \overset{0}{\Gamma}(\overset{0}{N}) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ . The set of all  $N$ -linear connections compatible with the conformal structure  $\hat{g}$  corresponding to the 1-form  $\omega$ , with the local coefficients  $D\Gamma(N, \omega) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ , is given by:

$$(4.3) \quad \begin{cases} L_{jk}{}^i = L_{jk}{}^i + X_{(1)k}{}^l C_{(1)jl}{}^i + X_{(1)k}{}^l N_{(1)l}{}^m C_{(2)jm}{}^i + X_{(2)k}{}^l C_{(2)jl}{}^i + \\ \quad + \frac{1}{2} g^{im} [g_{mj|k} + X_{(1)k}{}^l g_{mj|l} + (X_{(2)k}{}^l + X_{(1)k}{}^r N_{(1)r}{}^l) g_{mj|l} - \\ \quad - 2\tilde{\omega}_k g_{mj}] + \Omega_{sj}{}^{ir} X_{rk}{}^s, \\ C_{(1)jk}{}^i = C_{(1)jk}{}^i + C_{(2)jl}{}^i X_{(1)k}{}^l + \frac{1}{2} g^{im} (g_{mj|k} + X_{(1)k}{}^l g_{mj|l} - \\ \quad - 2\dot{\omega}_{(1)k} g_{mj}) + \Omega_{sj}{}^{ir} Y_{(1)rk}{}^s, \\ C_{(2)jk}{}^i = C_{(2)jk}{}^i + \frac{1}{2} g^{im} (g_{mj|k} - 2\dot{\omega}_{(2)k} g_{mj}) + \Omega_{sj}{}^{ir} Y_{(2)rk}{}^s, \end{cases}$$

where  $N_{(\alpha)j}{}^i = N_{(\alpha)j}{}^i - X_{(\alpha)j}{}^i$ ,  $X_{(\alpha)j}{}^i$ ,  $X_{jk}{}^i$ ,  $Y_{(\alpha)jk}{}^i$ , ( $\alpha = \overline{1, 2}$ ) are arbitrary tensor fields,  $\overset{0}{\mathbb{D}}$ ,  $\overset{(\alpha)}{\mathbb{D}}$ , denote the  $h$ - and  $v_\alpha$ -covariant derivatives, ( $\alpha = \overline{1, 2}$ ), with respect to  $\overset{0}{D}$  and  $\omega = \tilde{\omega}_i dx^i + \dot{\omega}_{(1)i} \delta y^{(1)i} + \dot{\omega}_{(2)i} \delta y^{(2)i}$ , is an arbitrary fixed 1-form in  $\mathcal{X}^*(Osc^2 M)$ .

#### Particular cases

1. If  $X_{(\alpha)j}{}^i = X_{jk}{}^i = Y_{(\alpha)jk}{}^i = 0$ , ( $\alpha = \overline{1, 2}$ ) in Theorem 4.1 we have:

**Theorem 4.2** Let  $\overset{0}{D}$  be a given  $\overset{0}{N}$ -linear connection on  $E$  with the local coefficients  $D \overset{0}{\Gamma}(\overset{0}{N}) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ . Then the following  $\overset{0}{N}$ -linear connection,

$\tilde{D}(\overset{0}{N}, \omega)$ , is compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , with the local coefficients  $D\overset{0}{\Gamma}(\overset{0}{N}, \omega) = (\tilde{L}_{jk}{}^i, \tilde{C}_{(1)jk}{}^i, \tilde{C}_{(2)jk}{}^i)$  given by:

$$(4.4) \quad \begin{cases} \tilde{L}_{jk}{}^i = L_{jk}{}^i + \frac{1}{2}g^{im}(g_{mj|k}{}^0 - 2\tilde{\omega}_k g_{mj}), \\ \tilde{C}_{(\alpha)jk}{}^i = C_{(\alpha)jk}{}^i + \frac{1}{2}g^{im}(g_{mj|k}{}^{(\alpha)} - 2\dot{\omega}_{(\alpha)k} g_{mj}), \quad (\alpha = \overline{1,2}), \end{cases}$$

where  $\overset{0}{|}$  and  $\overset{(\alpha)}{|}$  denote the  $h$ - and  $v_\alpha$ -covariant derivatives,  $(\alpha = \overline{1,2})$ , with respect to  $\overset{0}{D}$  and  $\omega = \tilde{\omega}_i dx^i + \dot{\omega}_{(1)i} \delta y^{(1)i} + \dot{\omega}_{(2)i} \delta y^{(2)i}$  is an arbitrary fixed 1-form in  $\mathcal{X}^*(Osc^2 M)$ .

2.If we take a metrical (almost symplectic)  $\overset{0}{N}$ -linear connection as  $\overset{0}{D}$  in Theorem 4.2., then (4.4) becomes:

$$(4.5) \quad \begin{cases} \tilde{L}_{jk}{}^i = L_{jk}{}^i - \delta_j^i \tilde{\omega}_k, \\ \tilde{C}_{(\alpha)jk}{}^i = C_{(\alpha)jk}{}^i - \delta_j^i \dot{\omega}_{(\alpha)k}, \quad (\alpha = \overline{1,2}). \end{cases}$$

3.If we consider an  $\overset{0}{N}$ -linear connection compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ ,  $\overset{0}{D}(\overset{0}{N}, \omega)$ , as  $\overset{0}{D}$  in Theorem 4.1., we have:

**Theorem 4.3** Let  $\overset{0}{D}(\overset{0}{N}, \omega)$  be a given  $\overset{0}{N}$ -linear connection compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , with the local coefficients  $D\overset{0}{\Gamma}(\overset{0}{N}, \omega) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$ . The set of all  $N$ -linear connections compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , with the local coefficients  $D\Gamma(N, \omega) = (L_{jk}{}^i, C_{(1)jk}{}^i, C_{(2)jk}{}^i)$  is given by:

$$(4.6) \quad \begin{cases} L_{jk}{}^i = L_{jk}{}^i + (C_{(1)jl}{}^i + N_{(1)l}{}^m C_{(2)jm}{}^i + \delta_j^i \dot{\omega}_{(1)l}) X_{(1)k}{}^l + \\ \quad + (C_{(2)jl}{}^i + \delta_j^i \dot{\omega}_{(2)l}) X_{(2)k}{}^l + \delta_j^i \dot{\omega}_{(2)l} X_{(1)k}{}^r N_{(1)r}{}^l + \Omega_{sj}{}^{ir} X_{rk}{}^s, \\ C_{(1)jk}{}^i = C_{(1)jk}{}^i + (C_{(2)jl}{}^i + \delta_j^i \dot{\omega}_{(2)l}) X_{(1)k}{}^l + \Omega_{sj}{}^{ir} Y_{(1)rk}{}^s, \\ C_{(2)jk}{}^i = C_{(2)jk}{}^i + \Omega_{sj}{}^{ir} Y_{(2)rk}{}^s, \end{cases}$$

where  $N_{(\alpha)j}{}^i = N_{(\alpha)j}{}^i - X_{(\alpha)j}{}^i$ ,  $X_{(\alpha)j}{}^i, X_{jk}{}^i, Y_{(\alpha)jk}{}^i$ ,  $(\alpha = \overline{1,2})$  are arbitrary tensor fields,  $\overset{0}{|}$ ,  $\overset{(\alpha)}{|}$ , denote the  $h$ - and  $v_\alpha$ -covariant derivatives,  $(\alpha = \overline{1,2})$ , with respect to  $\overset{0}{D}(\overset{0}{N}, \omega)$  and  $\omega = \tilde{\omega}_i dx^i + \dot{\omega}_{(1)i} \delta y^{(1)i} + \dot{\omega}_{(2)i} \delta y^{(2)i}$  is an arbitrary fixed 1-form in  $\mathcal{X}^*(Osc^2 M)$ .

4.If we take  $X_{(\alpha)j}{}^i = 0$ ,  $(\alpha = \overline{1,2})$  in Theorem 4.3. we obtain the set of all  $\overset{0}{N}$ -linear connections compatible with the conformal structure  $\hat{g}$ , corresponding to the 1-form  $\omega$ , having the same nonlinear connection  $\overset{0}{N}$ , result given in [11].

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