Finsler structure on the tangent supermanifold

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Abstract. The theory of locally anistropic superspaces (supersymmetric generalizations of various types of Lagrange and Finsler spaces) is given [16]. By using the global definition of supermanifolds proposed by Leites and Kostant ([5],[8]), we stablish a super version of a Finsler structure on a supermanifold.

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1 Introduction

There are two different definitions of supermanifolds proposed by various people. The first attempt to provide a mathematically framework for supergeometry was the Leites-Berzin-Kostant theory [5], [8]). The tools which used in this theory come from algebraic geometry. A different approach was initiated by Dewitt and Rogers ([7]). The basic idea is to enlarge the space over which the manifold is modeled by replacing the real field by a large set, containing both commuting and anti-commuting quantities.

In [15]-[17], Vacaru defined the Finsler structure on the Rogers supermanifold. But we are going to use the global properties of the supermanifolds, so we will use global definition of a supermanifold and define the Finsler structure on a supermanifold. An important tool in this structure is linear connection. It is very convenient for treating many global questions of Finsler geometry. In [1] the author gave a brief, very clear and self-contained setting for Finsler geometry. Several types of Finsler connections such as the Chern-Rund connection are introduced. In this paper we try to construct a linear (super) connection of Chern type.

Any smooth supermanifold $(M, \mathcal{A}_M)$ of dimension (m, n) is isomorphic (although not canonically) to $(M, \wedge\mathcal{E})$, where $\wedge\mathcal{E}$ is the sheaf of sections of the exterior algebra bundle $\wedge\mathcal{E} \mapsto M$ of a smooth vector bundle $E \mapsto M$ which is defined by $\mathcal{A}_M$. $(M, \wedge\mathcal{E})$ is then called the Batchelor trivialization of $(M, \mathcal{A}_M)$ (see [4], [3]).

In [14], it is showed that the underlying smooth manifold of the Batchelor trivialization of the super tangent bundle $\mathcal{STM}$ of $(M, \mathcal{A}_M)$ is not TM but $TM \oplus E^*$. In particular $dim\mathcal{STM} = (2m + n, m + 2n)$.

In this paper, with using algebraic geometry apparatus, in order to generalize our considerations for Finsler superspaces, in section 1, we shall begin our study

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with tangent superbundles. We construct the structure of the tangent supermanifold $ST(M,\mathcal{A}_M)$ as a subsupermanifolds of $S\mathcal{T}M$. There is a super tangent bundle structure of dimension $(4m+2n, 2m+4n)$ on the supermanifold $ST(M,\mathcal{A}_M)$ denoted by $S\mathcal{T}^2M$, so we will construct two subsupermanifolds of $S\mathcal{T}^2M$ denoted by $ST^2(M,\mathcal{A}_M)$ and the pull-back tangent pseudobundle $\pi^*ST(M,\mathcal{A}_M)$. In section 2, we generalize the concept of Finsler structure on supermanifolds. The concepts of (super) metric, Finsler metric, Nonlinear connection, linear connection and an example of a Finsler metric introduced in section 2.

Supermanifold and Supervector bundle.

Definition 1 *A (r,s)-Supervector bundle on a (m,n)-supermanifold $(M,\mathcal{A}_M)$ is the quadruple $\big((E,\mathcal{A}_E),\tilde{\tau},(M,\mathcal{A}_M),V_{\mathcal{S}}\big)$ where $\tilde{\tau}:(E,\mathcal{A}_E)\mapsto(M,\mathcal{A}_M)$ is a submersion of supermanifolds, V is a real (p,q)-dimensional supervector space, and each point, $x\in M$ lies in a coordinate neighborhood $U\subset M$, for which an isomorphism, Ψ_U, exists making the following diagram to commute:*

$$\begin{array}{ccc}\Big(\tau^{-1}(U),\mathcal{A}_E\mid_{\tau^{-1}(U)}\Big) & \xrightarrow{\Psi_U} & (U,\mathcal{A}_M\mid_U)\times V_{\mathcal{S}} \\ \tilde{\tau}\Big\downarrow & & \Big\downarrow P_1 \\ \big(U,\mathcal{A}_M\mid_U\big) & = & \big(U,\mathcal{A}_M\mid_U\big).\end{array}$$

$(E,\mathcal{A}_E)$ is called a supervector bundle over $(M,\mathcal{A}_M)$ of rank (r,s) (c.f.[14]).

Remark 1 *Let $(E,\mathcal{A}_E)$ be a supervector bundle over $(M,\mathcal{A}_M)$ of rank (r,s). Then there is an open covering $\{U_\alpha\}$ where $(U_\alpha,\mathcal{A}_\alpha(U_\alpha))$ is a splitting neighborhood for each α, a locally free sheaf $\mathcal{L}$ of $\mathcal{A}_M$-supermodules of rank (r,s) and sheaf isomorphism*

$$g_\alpha:\mathcal{L}|_{U_\alpha}\mapsto(\mathcal{A}_M(U_\alpha))^r\oplus(\mathcal{A}_M(U_\alpha))^s$$

such that on each $U_\alpha\cap U_\beta\neq\varnothing$, $g_{\alpha\beta}=g_\alpha o g_\beta^{-1}$,

$$g_{\alpha\beta}:\mathcal{A}_M(U_\alpha\cap U_\beta)^r\oplus\mathcal{A}_M(U_\alpha\cap U_\beta)^s\mapsto\mathcal{A}_M(U_\alpha\cap U_\beta)^r\oplus\mathcal{A}_M(U_\alpha\cap U_\beta)^s$$

where $g_{\alpha\beta}$ is an $\mathcal{A}_M(U_\alpha\cap U_\beta)$-supermodule isomorphism. Thus $g_{\alpha\beta}$ is of the form

$$g_{\alpha\beta} = \begin{pmatrix} A_{\alpha\beta} & \Theta_{\alpha\beta} \\ \Gamma_{\alpha\beta} & D_{\alpha\beta}\end{pmatrix},$$

where

$$A_{\alpha\beta}\in(\mathcal{A}_M(U_\alpha\cap U_\beta)_0)^{r\times r}, \Theta_{\alpha\beta}\in(\mathcal{A}_M(U_\alpha\cap U_\beta)_1)^{r\times s}$$

$$\Gamma_{\alpha\beta}\in(\mathcal{A}_M(U_\alpha\cap U_\beta)_1)^{s\times r}, D_{\alpha\beta}\in(\mathcal{A}_M(U_\alpha\cap U_\beta)_0)^{s\times s}$$

and $g_{\alpha\beta}$ is invertible and $E\simeq E_r\oplus E_s$ where E_r and E_s are smooth vector bundles over M of rank r and s which gives by cocycles $A_{\alpha\beta}$ and $D_{\alpha\beta}$ respectively.

Remark 2 *The supertangent bundle over $(M, \mathcal{A}_M)$ is a supervector bundle over $(M, \mathcal{A}_M)$ where $g_{\alpha\beta}, A_{\alpha\beta}, \Theta_{\alpha\beta}, D_{\alpha\beta}, \Gamma_{\alpha\beta}$ are as in remark 1.*

$$A_{\alpha\beta} = \left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right), \quad \Theta_{\alpha\beta} = \left(\frac{\partial x^i_\beta}{\partial \xi^\lambda_\alpha}\right)$$

$$\Gamma_{\alpha\beta} = \left(\frac{\partial \xi^\lambda_\beta}{\partial x^i_\alpha}\right), \quad D_{\alpha\beta} = \left(\frac{\partial \xi^\lambda_\beta}{\partial \xi^\mu_\alpha}\right)$$

We denote this supervector bundle by $S\mathcal{T}M = (STM, ST\mathcal{A}_M)$ where $STM = TM \oplus E^*$ and E is the Batchelor vector bundle over M (c.f.[4]).

Let $(E, \mathcal{A}_E)$ be a supervector bundle over a supermanifold $(M, \mathcal{A}_M)$ of rank (r, s). Any subsupermanifold $(E', \mathcal{A}_{E'})$ of $(E, \mathcal{A}_E)$ of dimension $(m+p, n+q); p \leq r, q \leq s$ which, E' is subvector bundle of E and $\mathcal{A}_{E'}$ is obtain from the sheafification of the presheaf of $(\mathcal{A}_M)_0$-modules over $M, U \mapsto ((\mathcal{A}_M)_0(U))^p \oplus ((\mathcal{A}_M)_1(U))^q$, for each splitting neighborhood $(U, \mathcal{A}_M(U))$. This subsupermanifold is called *the pseudobundle of rank (p, q) over $(M, \mathcal{A}_M)$*. There are a one to one correspondence between $((\mathcal{A}_M)_0)^p \oplus ((\mathcal{A}_M)_1)^q$ and $\mathcal{M}aps((U, \mathcal{A}_M|_U), (R^p, R^{p|q}))$.

Proposition 1 *Let $f : (M, \mathcal{A}_M) \mapsto (N, \mathcal{A}_N)$ be a morphism of supermanifolds. There is a morphism*

$$S\mathcal{T}f : (STM, ST\mathcal{A}_M) \mapsto (STN, ST\mathcal{A}_N)$$

such that the following diagram is commutative

$$\begin{array}{ccc} \Big(STM, ST\mathcal{A}_M\Big) & \xrightarrow{S\mathcal{T}f} & (STN, ST\mathcal{A}_N) \\ \Big\downarrow \tau_{STM} & & \Big\downarrow \tau_{STN} \\ (M, \mathcal{A}_M) & \xrightarrow{f} & (N, \mathcal{A}_N). \end{array}$$

Proof. c.f. [14]. □

As in general case, in order to define a Finsler metric or a Finsler connection, we need to define the tangent supermanifold and the pull-back tangent pseudobundle. First we describe the tangent supermanifold $ST(M, \mathcal{A}_M) = (TM, T\mathcal{A}_M)$, so let $(M, \mathcal{A}_M)$ be a (m, n)-dimensional supermanifold. If U, U' are two open subsets of M that intersect and there are local coordinates over them, the transition functions of $(TM, T\mathcal{A}_M)$ are given by

(1.1.1) $$x'^i = x'^i(x^{,}..., x^m, \xi^1, ..., \xi^n), \quad \xi'^\alpha = \xi'^\alpha(x^{,}..., x^m, \xi^1, ..., \xi^n)$$

(1.1.2) $$y'^j = \sum_{i=1}^m y^i \frac{\partial x'^j}{\partial x^i} + \sum_{\alpha=1}^n \eta^\alpha \frac{\partial x'^j}{\partial \xi^\alpha}$$

$$\eta'^{\beta} = \sum_{i=1}^{m} y^i \frac{\partial \xi'^{\beta}}{\partial x^i} + \sum_{\alpha=1}^{n} \eta^{\alpha} \frac{\partial \xi'^{\beta}}{\partial \xi^{\alpha}}. \tag{1.1.3}$$

Moreover $(TM, T\mathcal{A}_M)$ is a subsupermanifold of $S\mathcal{T}M = (STM, ST\mathcal{A}_M)$ and called *the tangent supermanifold* with canonical projection map $(\tau_1, \tau_1^{\sharp}) : (TM, T\mathcal{A}_M) \mapsto (M, \mathcal{A}_M)$. Since $\dim(M, \mathcal{A}_M) = (m, n)$ it follows that $\dim(TM, T\mathcal{A}_M) = (2m, 2n)$.

The supertangent bundle on the tangent supermanifold is defined using the same method as in Remark 2. We denote it by $S\mathcal{T}^2M$. It is of dimension $(4m+2n, 2m+4n)$. As before, we can construct the subsupermanifold $(T(TM), T(T\mathcal{A}_M))$ of $S\mathcal{T}^2M$. For the coordinate system $(x, y, u, v; \xi, \eta, \theta, \mu)$ the transition functions are given by (1.1.1) and

$$y'^{j} = \sum_{i=1}^{m} \frac{\partial x'^{j}}{\partial x^i} y^i + \sum_{\alpha=1}^{n} \frac{\partial x'^{j}}{\partial \eta^{\alpha}} \eta^{\alpha} \quad , \quad u'^{j} = y'^{j}, \tag{1.1.4}$$

$$v'^{j} = \sum_{i=1}^{m} \left(\sum_{k=1}^{m} \frac{\partial^2 x'^{j}}{\partial x^i \partial x^k} y^k + \sum_{\alpha=1}^{n} \frac{\partial^2 x'^{j}}{\partial \xi^{\alpha} \partial x^i} \eta^{\alpha} \right) y^i + \sum_{i=1}^{m} \frac{\partial x'^{j}}{\partial x^i} v^i$$

$$+ \sum_{\alpha=1}^{n} \left(\sum_{i=1}^{m} \frac{\partial^2 x'^{j}}{\partial \xi^{\alpha} \partial x^i} y^i + \sum_{\beta=1}^{n} \frac{\partial^2 x'^{j}}{\partial \xi^{\beta} \partial \xi^{\alpha}} \eta^{\beta} \right) \eta^{\alpha} - \sum_{\alpha=1}^{n} \frac{\partial x'^{j}}{\partial \xi^{\alpha}} \mu^{\alpha}, \tag{1.1.5}$$

$$\eta'^{\beta} = \sum_{i=1}^{m} \frac{\partial \xi'^{\beta}}{\partial x^i} y^i + \sum_{\alpha=1}^{n} \frac{\partial \xi'^{\beta}}{\partial \xi^{\alpha}} \eta^{\alpha} \quad , \quad \theta'^{\beta} = \eta'^{\beta}, \tag{1.1.6}$$

$$\mu'^{\beta} = \sum_{i=1}^{m} \left(\sum_{k=1}^{m} \frac{\partial^2 \xi'^{\beta}}{\partial x^k \partial x^i} y^k + \sum_{\alpha=1}^{n} \frac{\partial^2 \xi'^{\beta}}{\partial \xi^{\alpha} \partial x^i} \eta^{\alpha} \right) y^i + \sum_{i=1}^{m} \frac{\partial \xi'^{\beta}}{\partial x^i} v^i$$

$$+ \sum_{\alpha=1}^{n} \left(\sum_{i=1}^{m} \frac{\partial^2 \xi'^{\beta}}{\partial \xi^{\alpha} \partial x^i} y^i + \sum_{\gamma=1}^{n} \frac{\partial^2 \xi'^{\beta}}{\partial \xi^{\gamma} \partial \xi^{\alpha}} \eta^{\gamma} \right) \eta^{\alpha} + \sum_{\alpha=1}^{n} \frac{\partial \xi'^{\beta}}{\partial \xi^{\alpha}} \mu^{\alpha}. \tag{1.1.7}$$

These transition functions give rise to a supermanifold $(T(TM), T(T\mathcal{A}_M))$ that will be denoted by $ST^2(M, \mathcal{A}_M)$ and its dimension is $(4m, 4n)$.

Let $((E, \mathcal{A}_E), \bar{\pi}, (M, \mathcal{A}_M))$ be a supervector bundle then a section of $(E, \mathcal{A}_E)$ on $(M, \mathcal{A}_M)$ is a morphism

$$X : (M, \mathcal{A}_M) \mapsto (E, \mathcal{A}_E)$$

such that $Xo\bar{\pi} = id$ where

$$\bar{\pi} = (\pi, \pi^{\sharp}) : (E, \mathcal{A}_E) \mapsto (M, \mathcal{A}_M)$$

is the canonical projection map. The sections of this supervector bundle is denoted by $\Gamma((M,\mathcal{A}_M),(E,\mathcal{A}_E))$.

Let $((E,\mathcal{A}_E),\bar{\pi},(M,\mathcal{A}_M))$ be a supervector bundle of rank (r,s) and $(E',\mathcal{A}_{E'})$ a pseudobundle of rank (p,q) over $(M,\mathcal{A}_M)$. A section of pseudobundle $(E',\mathcal{A}_{E'})$ is a morphism

$$X:(E',\mathcal{A}_{E'})\mapsto(M,\mathcal{A}_M)$$

such that $Xo\pi'=id_{(M,\mathcal{A}_M)}$ where π' is a restriction of projection map $\bar{\pi}:(E,\mathcal{A}_E)\mapsto(M,\mathcal{A}_M)$ to $(E',\mathcal{A}_{E'})$.

Let $(M,\mathcal{A}_M)$ be a supermanifold of dimension (m,n). The correspondence, $U\mapsto Der\mathcal{A}_M(U)$, for each open subset $U\subset M$, defines a locally free sheaf of (left) $\mathcal{A}_M$-supermodules over M of rank (m,n). So

Theorem 1 *Let $(STM,ST\mathcal{A}_M),\tilde{\tau},(M,\mathcal{A}_M))$ be the supertangent bundle and $U\subset M$ a coordinate neighborhood. There is a one to one correspondence between $Der\mathcal{A}_M(U)$ and $\Gamma((U,\mathcal{A}_M(U)),(\tau^{-1}(U),ST\mathcal{A}_M|_{\tau^{-1}(U)}))$.*

Proof. This is a consequence of Proposition 2.7 of [14], . □

Theorem 2 *Let $(N,\mathcal{A}_N)$ be a subsupermanifold of $(M,\mathcal{A}_M)$. Then the inclusion morphism $i:(N,\mathcal{A}_N)\mapsto(M,\mathcal{A}_M)$ gives rise to a morphism $S\mathcal{T}i:(STN,ST\mathcal{A}_N)\mapsto(STM,ST\mathcal{A}_M)$ such that the following diagram is commutative*

$$\begin{array}{ccc}\Big(STN,ST\mathcal{A}_N\Big) & \xrightarrow{S\mathcal{T}i} & (STM,ST\mathcal{A}_M)\\ \Big\downarrow{\scriptstyle \tau_{STN}} & & \Big\downarrow{\scriptstyle \tau_{STM}}\\ (N,\mathcal{A}_N) & \xrightarrow{i} & (M,\mathcal{A}_M).\end{array}$$

Proof. See proposition 1. □

Let $(M,\mathcal{A}_M)$ be a supermanifold and $S\mathcal{T}^2M$ the supertangent bundle on $(TM,T\mathcal{A}_M)$ with projection map $\bar{\pi}=(\pi,\pi^\sharp):S\mathcal{T}^2M\mapsto(TM,T\mathcal{A}_M)$. If $(x^i;\xi^\alpha)$ is a coordinate system on an open subset U of M and $(x^i,y^i,\xi^\alpha,\eta^\alpha)$ the coordinate system on $\tau_1^{-1}(U)$ of TM, as we mention before, the coordinate system $(x^i,y^i,u^i,v^i;\xi^\alpha,\eta^\alpha,\theta^\alpha,\mu^\alpha)$ gives rise to the subsupermanifold $(T(TM),T(T\mathcal{A}_M))$ of dimension $(4m,4n)$. By this method we can construct a subsupermanifold $\pi^*ST(M,\mathcal{A}_M)=(\pi^*TM,\pi^*T\mathcal{A})$ which is defined locally by local coordinate systems $(x^i,y^i,u^i,0;\xi^\alpha,\eta^\alpha,\theta^\alpha,0)$. For these coordinate systems, the transition functions are given by (1.1.1) and (1.1.4-1.1.7). Since $dim(M,\mathcal{A}_M)=(m,n)$ it follows that $dim(\pi^*TM,\pi^*T\mathcal{A})=(3m,3n)$. It also celled the *pull-back tangent pseudobundle* (see remark 2.10 of [14]).

In the local coordinate system $(x^i,y^i,u^i;\xi^\alpha,\eta^\alpha,\theta^\alpha)$ in $\pi^*ST(M,\mathcal{A}_M)$, the set of sections of the pull-back tangent pseudobundle over $ST(M,\mathcal{A}_M)$, $\Gamma(ST(M,\mathcal{A}_M),\pi^*ST(M,\mathcal{A}_M))$, is locally generated by

$$\left\{\frac{\partial}{\partial x^i},\frac{\partial}{\partial\xi^\alpha}:\quad i=1,...,m;\alpha=1,...,n\right\}.$$

Given the above assumption, if we choose the coordinate system $(x^i, y^i, 0, v^i;\ \xi^\alpha, \eta^\alpha, 0, \mu^\alpha)$, so there is another pseudobundle of dimension $(3m, 3n)$ denoted by $SVT(M, \mathcal{A}_M) = (VTM, VT\mathcal{A})$ and is called the *supervertical pseudobundle.*

In a local coordinate system $(x^i, y^i, v^i; \xi^\alpha, \eta^\alpha, \mu^\alpha)$ in $SVT(M, \mathcal{A}_M)$, the set of sections of the supervertical pseudobundle over $ST(M, \mathcal{A}_M)$, $\Gamma(ST(M, \mathcal{A}_M),\ SVT(M, \mathcal{A}_M))$, is locally generated by

$$\left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial \eta^\alpha} : \quad i = 1, ..., m; \alpha = 1, ..., n \right\}.$$

Now, we are ready to introduce the concept of a nonlinear connection. So let

$$\mathcal{E}T^2M = \Gamma((TM, T\mathcal{A}_M), (T(TM), T(T\mathcal{A}_M)))$$

and

$$\mathcal{V}TM = \Gamma((TM, T\mathcal{A}_M), (VTM, VT\mathcal{A}_M)).$$

Definition 2 *A nonlinear connection on $(TM, T\mathcal{A}_M)$ is a supplementary distribution $\mathcal{H}TM$ of the $\mathcal{V}TM$. Given an exact sequence*

$$0 \longrightarrow \mathcal{V}TM \xrightarrow{i} \mathcal{E}T^2M \longrightarrow \Gamma\big((TM, T\mathcal{A}_M), (\pi^*TM, \pi^*T\mathcal{A}_M)\big) \longrightarrow 0$$

if there is a morphism $N : \mathcal{E}T^2M \mapsto \mathcal{V}TM$ such that $Noi = id$ then we call N a nonlinear connection on $(TM, T\mathcal{A}_M)$, (see [11], [15]).

By the definition of a nonlinear connection, we have the decomposition

$$\mathcal{E}T^2M = \mathcal{V}TM \oplus \mathcal{H}TM$$

where

$$\mathcal{H}TM = \Gamma((TM, T\mathcal{A}_M), (\pi^*TM, \pi^*T\mathcal{A}_M))$$

and $\mathcal{H}TM$ is a complementary distribution to $\mathcal{V}TM$ in $\mathcal{E}T^2M$.

Now, let $(x^i; \xi^\alpha)$ be a coordinate system on an open subset U of M and $(x^i, y^i, \xi^\alpha, \eta^\alpha)$ a coordinate system on $\tau_1^{-1}(U)$ of TM. To give a nonlinear connection on $(TM, T\mathcal{A}_M)$ is equivalent to give a set of superfunctions $\{N_i^j(x, y, \eta, \theta), N_i^\beta(x, y, \eta, \theta), N_\alpha^j(x, y, \eta, \theta), N_\alpha^\beta(x, y, \eta, \theta)\}$ which under the local transformation (1.1.1) and (1.1.4-1.1.7) satisfy in

$$\frac{\partial x^k}{\partial x'^i} N_k^j = N'^k_i \frac{\partial x^j}{\partial x'^k} - N'^\alpha_i \frac{\partial x^j}{\partial \xi'^\alpha} - \frac{\partial^2 x^j}{\partial x'^i \partial x'^k} y'^k - \frac{\partial^2 x^j}{\partial x'^i \partial \xi'^\alpha} {\eta'}^\alpha,$$

$$\frac{\partial x^k}{\partial x'^i} N_k^\beta = N'^j_i \frac{\partial \xi^\beta}{\partial x'^j} + N'^\alpha_i \frac{\partial \xi^\beta}{\partial \xi'^\alpha} - \frac{\partial^2 \xi^\beta}{\partial x'^i \partial x'^k} y'^k - \frac{\partial^2 \xi^\beta}{\partial x'^i \partial \xi'^\alpha} {\eta'}^\alpha$$

$$\frac{\partial \xi^\gamma}{\partial \xi'^\alpha} N_\gamma^i = N'^j_\alpha \frac{\partial x^i}{\partial x'^j} - N'^\beta_\alpha \frac{\partial x^i}{\partial \xi'^\beta} - \frac{\partial^2 x^i}{\partial \xi'^\alpha \partial x'^j} y'^j - \frac{\partial^2 x^i}{\partial \xi'^\alpha \partial \xi'^\beta} {\eta'}^\beta$$

and

$$\frac{\partial \xi^\gamma}{\partial \xi'^\alpha} N_\gamma^\beta = N'^i_\alpha \frac{\partial \xi^\beta}{\partial x'^j} + N'^\gamma_\alpha \frac{\partial \xi^\beta}{\partial \xi'^\gamma} - \frac{\partial^2 \xi^\beta}{\partial \xi'^\alpha \partial x'^i} y'^i - \frac{\partial^2 \xi^\beta}{\partial \xi'^\alpha \partial \xi'^\gamma} \eta'^\gamma.$$

In the tangent supermanifold a local basis adapted to the given nonlinear connection N are introduced by

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta \xi^\alpha}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial \eta^\beta} \quad | \quad i, j = 1, ..., m; \alpha, \beta = 1, ..., n \right\},$$

where

(1.1.8) $$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - N_i^\alpha \frac{\partial}{\partial \eta^\alpha}$$

and

(1.1.9) $$\frac{\delta}{\delta \xi^\alpha} := \frac{\partial}{\partial \xi^\alpha} - N_\alpha^i \frac{\partial}{\partial y^i} - N_\alpha^\beta \frac{\partial}{\partial \eta^\beta}.$$

2 (Super) metrics and connections

Let $(M, \mathcal{A}_M)$ be a (m, n)-supermanifold. If U is an open subset of M, and $\{x^i, \xi^\alpha\}$ are local coordinates on it, then $\{dx^1, ..., dx^m; -d\xi^1, ..., -d\xi^n\}$ is the basis of the module $(Der\mathcal{A}(U))^*$ dual to the basis $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}; \frac{\partial}{\partial \xi^1}, ..., \frac{\partial}{\partial \xi^n}\}$ of $Der\mathcal{A}(U)$. In particular, if $\omega \in (Der\mathcal{A}_M(U))^*$ then

$$\omega = \sum_{i=1}^m \omega_i dx^i + \sum_{\alpha=1}^n \omega_\alpha d\xi^\alpha$$

where the superfunctions ω_i, ω_α are given by $\omega_i = \omega(\frac{\partial}{\partial x^i})$ and $\omega_\alpha = -\omega(\frac{\partial}{\partial \xi^\alpha})$.

Definition 3 *An even $\mathcal{A}_M$-bilinear map $g_M : Der\mathcal{A}_M \otimes Der\mathcal{A}_M \mapsto \mathcal{A}_M$ is called a (super) metric or simply a metric on $(M, \mathcal{A}_M)$ if*
(1) $g_M(X \otimes Y) = (-1)^{\tilde{X}.\tilde{Y}} g_M(Y \otimes X)$ for each $X, Y \in Der\mathcal{A}_M$, where $\tilde{X}$ is parity of X.
(2) The induced $\mathcal{A}_M$-map $g_M(x \otimes .) : Der\mathcal{A}_M \mapsto (Der\mathcal{A}_M)^*$ for each $X \in Der\mathcal{A}_M$ is an isomorphism between $\mathcal{A}_M$-modules $Der\mathcal{A}_M$ and $(Der\mathcal{A}_M)^*$.
(3) The restriction of g_M to $Der(C^\infty_M) \otimes Der(C^\infty_M)$ is a Riemannian metric on (M, C^∞_M).

Let U be an open subset of M, and $\{x^i, \xi^\alpha\}$ local coordinates on it, then g locally has a form

$$g = g_{ij} dx^i \otimes dx^j + g_{i\alpha} dx^i \otimes d\xi^\alpha + g_{\alpha i} d\xi^\alpha \otimes dx^i + g_{\alpha\beta} d\xi^\alpha \otimes d\xi^\beta$$

where $g_{ij}, g_{\alpha\beta} \in (\mathcal{A}_M(U))_0$ and $g_{i\alpha}, g_{\alpha j} \in (\mathcal{A}_M(U))_1$ and $g_{ij} = g_{ji}, g_{\alpha\beta} = -g_{\beta\alpha}, g_{i\alpha} = g_{\alpha i}$.

Let $ST(M, \mathcal{A}_M) = (TM, T\mathcal{A}_M)$ be the tangent supermanifold and $ST(M, \mathcal{A}_M)_0 = (TM_0, T_0\mathcal{A}_M)$ where $TM_0 = TM \backslash \{0\}$ and $T_0\mathcal{A}_M = T\mathcal{A}_M|_{TM_0}$

Definition 4 *Let* $(x^i;\xi^\alpha)$ *be a coordinate system on an open subset* U *of* M *and* $(x^i,y^i;\xi^\alpha,\eta^\alpha)$ *the coordinate system on* $\tau_1^{-1}(U)$ *of* TM. *A function* $F\in T_0\mathcal{A}_M$ *is called a* Finsler metric *if the following conditions are satisfied:*
(1) $F(x^i,\lambda y^i;\xi^\alpha,\lambda\eta^\alpha)=\lambda F(x^i,y^i;\xi^\alpha,\eta^\alpha),\quad \forall\lambda>0$
(2) If we denote by $\bar{F}$ *the value of* F *at* (x^i,y^i), *then* $\bar{F}$ *is a positive function.*
(3) If we put

$$g_{ij}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j},\quad g_{i\beta}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial\eta^\beta},\quad g_{\alpha j}=\frac{1}{2}\frac{\partial^2F^2}{\partial\eta^\alpha\partial y^j},\quad g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2F^2}{\partial\eta^\alpha\partial\eta^\beta}$$

then

$$g=\begin{bmatrix} g_{ij} & g_{i\beta}\\ g_{\alpha j} & g_{\alpha\beta}\end{bmatrix}$$

to be invertible.

We omit the nilpotent elements of g_{ij} and denote it by $\tilde{g}_{ij}$. (n must be even, i.e. n=2k).

Example. Suppose $R^{2|2}=(R^2,C^\infty(R^2)\otimes\wedge R^2)$ is a $(2,2)$-supermanifold. Let $\tilde{g}_{ij}(x)$ be a usual riemannian metric on R^2, $(x;\xi)$ is coordinate system of $R^{2|2}$, (x,y,ξ,η) is coordinate system of $ST(R^2,C^\infty(R^2)\otimes\wedge R^2)$ and $g_{\alpha\beta}$ is a supermatrix of the form

$$g_{\alpha\beta}=\begin{bmatrix} 0 & \sqrt{h(x)}\xi^1\xi^2\\ \sqrt{h(x)}\xi^2\xi^1 & 0\end{bmatrix}$$

where $h(x)>0,\quad\forall x\in R^2$. Put

$$F(x,y;\xi,\eta)=\sqrt{\tilde{g}_{ij}(x)y^iy^j}+\sqrt{h(x)}(\xi^1\eta^2+\xi^2\eta^1)$$

so

$$F^2(x,y;\xi,\eta)=(\tilde{g}_{ij}(x)y^iy^j+2\sqrt{\tilde{g}_{ij}(x)h(x)y^iy^j}(\xi^1\eta^2+\xi^2\eta^1)+2h(x)\xi^1\xi^2\eta^1\eta^2)$$

then it is clear that the superfunction F and

$$g_{ij}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j},\quad g_{i\beta}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial\eta^\beta},\quad g_{\alpha j}=\frac{1}{2}\frac{\partial^2F^2}{\partial\eta^\alpha\partial y^j},\quad g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2F^2}{\partial\eta^\alpha\partial\eta^\beta}.$$

satisfy the definition in 4.

Let $(M,\mathcal{A})$ be a supermanifold and $Der\mathcal{A}$ be the sheaf of derivations of $\mathcal{A}$. We recall the notion of (super) connection. See [10] for the definitions of left and right (super) connections. Hear we consider only left (super) connection, which we simply call connections.

Definition 5 *Let* $\mathcal{E}$ *be a sheaf of* $\mathcal{A}$*-modules on* M. *A left (super) connection, or simply a connection on* $\mathcal{E}$ *is a morphism of sheaves of supervector spaces from* $Der\mathcal{A}\otimes\mathcal{E}$ *to* $\mathcal{E}$, *denoted* $D\otimes\alpha\mapsto\nabla_D\alpha$, *which satisfies the identity*

$$\nabla_{fD}\alpha=f\nabla_D\alpha,$$

and the leibniz rule,

$$\nabla_D f\alpha = D(f)\alpha + (-1)^{\tilde{D}.\tilde{f}} f\nabla_D\alpha,$$

for any section f of $\mathcal{A}$, any derivation D of $\mathcal{A}$, and any section α of $\mathcal{E}$.

A connection on the sheaf $Der\mathcal{A}$ of derivations of $\mathcal{A}$ is a linear connection on $(M, \mathcal{A})$.

The torsion and curvature tensors of a linear connection ∇ is defined by

$$T : DerA \otimes DerA \mapsto DerA$$

such that

$$T(X,Y) = \nabla_X Y - (-1)^{\tilde{X}.\tilde{Y}} \nabla_Y X - [X,Y]$$

and

$$\Omega : DerA \otimes DerA \otimes DerA \mapsto DerA$$

such that

$$\Omega(X,Y,Z) = \nabla_X(\nabla_Y Z) - (-1)^{\tilde{X}.\tilde{Y}} \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Definition 6 *Let g be a metric and ∇ is a left linear connection on a $(M, \mathcal{A}_M)$. ∇ is called metric compatible if for every $X, Y, Z \in Der\mathcal{A}$,*

$$\nabla_X (g(Y,Z)) = g(\nabla_X Y, Z)) + (-1)^{\tilde{X}.\tilde{Y}} g(Y, \nabla_X Z).$$

Theorem 3 *Let $(M, \mathcal{A}_M)$ be a supermanifold and g is a metric. Then there is a unique torsionfree linear connection ∇ on $(M, \mathcal{A}_M)$ such that ∇ is compatible with the given metric g.*

Proof. c.f. [18] and [13]. □

In 1943, S.S. Chern studied the equivalence problem for Finsler Spaces using the Cartan's exterior differentiation method [6],[2]. He discovered a very simple connection. We use this method and introduce a linear connection .

As we mention before, a Finsler metric F defines an important tensor on $(TM, T\mathcal{A}_M)$ which is the metric g. Another important tensor is the cartan tensor. Take a local coordinates system $(x^i, y^i, \xi^\alpha, \eta^\alpha)$ in $(TM, T\mathcal{A}_M)$. As we know, the coefficients of a metric g are superfunctions of $(x^i, y^i, \xi^\alpha, \eta^\alpha)$, thus by differentiating of these superfunctions with respect to y^i, η^α we will introduce two important tensors. So let

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \quad C_{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial \eta^\gamma},$$

$$A_{ijk} = F(x,y;\xi,\eta)C_{ijk}, \quad A_{\alpha\beta\gamma} = F(x,y;\xi,\eta)C_{\alpha\beta\gamma}$$

where g_{ij} and $g_{\alpha\beta}$ are defined in Def. 4. We obtain two tensors on $\pi^* ST(M, \mathcal{A}_M)$ defined by

$$C = C_{ijk}dx^i \otimes dx^j \otimes dx^k + C_{\alpha\beta\gamma}d\xi^\alpha \otimes d\xi^\beta \otimes d\xi^\gamma$$

and

$$A = A_{ijk}dx^i \otimes dx^j \otimes dx^k + A_{\alpha\beta\gamma}d\xi^\alpha \otimes d\xi^\beta \otimes d\xi^\gamma.$$

We call C the *Cartan tensor* of F.

We also define the map

$$\mu : \Gamma((TM, T\mathcal{A}_M), (T(TM), T(T\mathcal{A}_M))) \to \Gamma((TM, T\mathcal{A}_M), \pi^*ST(M, \mathcal{A}_M))$$

which satisfies

$$\mu(\frac{\partial}{\partial x^i}) = N_i^k \frac{\partial}{\partial x^k}, \quad \mu(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}, \quad \mu(\frac{\partial}{\partial \xi^\alpha}) = N_\alpha^\beta \frac{\partial}{\partial \xi^\beta}, \quad \mu(\frac{\partial}{\partial \eta^\alpha}) = \frac{\partial}{\partial \xi^\alpha}$$

and the map

$$\rho : \Gamma((TM, T\mathcal{A}_M), (T(TM), T(T\mathcal{A}_M))) \to \Gamma((TM, T\mathcal{A}_M), \pi^*ST(M, \mathcal{A}_M))$$

which satisfies

$$\rho(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}, \quad \rho(\frac{\partial}{\partial \xi^\alpha}) = \frac{\partial}{\partial \xi^\alpha}, \quad \rho(\frac{\partial}{\partial y^i}) = 0, \quad \rho(\frac{\partial}{\partial \eta^\alpha}) = 0.$$

Now, to construct a linear connection in $\pi^*ST(M, \mathcal{A}_M)$, we consider a special metric g which is defined by a Finsler metric F (see Def. 4.) and satisfies in the conditions $\frac{\partial}{\partial \xi^\alpha}(g_{ij}) = 0$ and $\frac{\partial}{\partial x^i}(g_{\alpha\beta}) = 0$.

Theorem 4 *Let (M, F) be a Finsler supermanifold. Let g be a metric given as above and C be the Cartan tensor on $\pi^*ST(M, \mathcal{A}_M)$. Then there is a unique linear connection D in $\pi^*ST(M, \mathcal{A}_M)$, which has the following properties:*

(i) D is torsion-free, i.e., for all $X, Y \in \Gamma((TM, T\mathcal{A}_M), (T(TM), T(T\mathcal{A}_M)))$

$$D_X\rho(Y) - (-1)^{\tilde{X}\tilde{Y}} D_Y\rho(X) - \rho([X, Y]) = 0. \tag{2.2.1}$$

(ii) D is compatible with the Finsler structure in the following sense: for all $X, Y \in \Gamma((TM, T\mathcal{A}_M), \pi^*ST(M, \mathcal{A}_M))$ and $Z \in \Gamma((TM, T\mathcal{A}_M),\ (T(TM), T(T\mathcal{A}_M)))$

$$Zg(X, Y) - g(D_Z X, Y) - (-1)^{\tilde{Z}.\tilde{X}} g(X, D_Z Y) = 2F^{-1}A(\mu(Z), X, Y). \tag{2.2.2}$$

Proof. In a standard local coordinate system $(x^i, y^i; \xi^\alpha, \eta^\alpha)$ in $ST(M, \mathcal{A}_M)$, we write

$$\begin{array}{rcl} D_{\frac{\partial}{\partial x^i}}\partial_j & = & \Gamma_{ij}^k \partial_k, \quad D_{\frac{\partial}{\partial y^i}}\partial_j = F_{ij}^k \partial_k \\ D_{\frac{\partial}{\partial \xi^\alpha}}\partial_i & = & \Gamma_{i\alpha}^j \partial_j, \quad D_{\frac{\partial}{\partial \eta^\alpha}}\partial_i = F_{i\alpha}^j \partial_j \\ D_{\frac{\partial}{\partial x^i}}\partial_\alpha & = & \Gamma_{\alpha i}^\beta \partial_\beta, \quad D_{\frac{\partial}{\partial y^i}}\partial_\alpha = F_{\alpha i}^\beta \partial_\beta \\ D_{\frac{\partial}{\partial \xi^\alpha}}\partial_\beta & = & \Gamma_{\beta\alpha}^\gamma \partial_\gamma, \quad D_{\frac{\partial}{\partial \eta^\alpha}}\partial_\beta = F_{\beta\alpha}^\gamma \partial_\gamma. \end{array}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}$.

Clearly, (2.2.1) and (2.2.2) are equivalent to the following

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \Gamma_{\alpha i}^\beta = \Gamma_{i\alpha}^\beta, \Gamma_{\alpha i}^j = \Gamma_{i\alpha}^j, \Gamma_{\beta\alpha}^\gamma = -\Gamma_{\alpha\beta}^\gamma \tag{2.2.3}$$

(2.2.4) $$F^k_{ij} = F^\beta_{\alpha i} = F^j_{\alpha i} = F^\gamma_{\beta\alpha} = 0$$

(2.2.5) $$\frac{\partial}{\partial x^k}(g_{ij}) = \Gamma^l{}_{ki} g_{lj} + \Gamma^l_{kj} g_{li} + 2A_{ijl}\frac{N^l_k}{F}$$

(2.2.6) $$\frac{\partial}{\partial x^i}(g_{\alpha\beta}) = g_{\gamma\beta}\Gamma^\gamma{}_{\alpha i} + g_{\alpha\gamma}\Gamma^\gamma{}_{\beta i}$$

(2.2.7) $$\frac{\partial}{\partial \xi^\gamma}(g_{\alpha\beta}) = g_{\mu\beta}\Gamma^\mu{}_{\alpha\gamma} + g_{\alpha\mu}\Gamma^\mu{}_{\beta\gamma} - 2A_{\mu\alpha\beta}\frac{N^\mu_\gamma}{F}$$

(2.2.8) $$\frac{\partial}{\partial \xi^\alpha}(g_{ij}) = g_{kj}\Gamma^k{}_{i\alpha} + g_{ik}\Gamma^k{}_{j\alpha}$$

where $g_{ij} = g_{ij}(x, y; \xi, \eta), A_{ijk} = A_{ijk}(x, y; \xi, \eta)$. We must compute Γ^k_{ij}, $\Gamma^\beta_{\alpha i}$, $\Gamma^j_{\alpha i}$, $\Gamma^\gamma_{\beta\alpha}$ from $(2.2.5 - 2.2.8)$. Putting $\Gamma^\beta_{\alpha i} = 0, \Gamma^j_{i\alpha} = 0$ and making a permutation to i, j, k in (2.2.5) and simple calculation similar to general case, will have

$$\begin{aligned}\frac{\partial}{\partial x^i}(g_{jk}) &+ \frac{\partial}{\partial x^j}(g_{ik}) - \frac{\partial}{\partial x^k}(g_{ij}) \\ &= \{\Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{lk} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{li}\} \\ &+ 2\left\{A_{jkl}\frac{N^l_i}{F} + A_{kil}\frac{N^l_j}{F} - A_{ijl}\frac{N^l_k}{F}\right\}.\end{aligned}$$

If we put

$$\gamma^k_{ij} = \frac{1}{2} g^{kl}\left[\frac{\partial}{\partial x^i}(g_{jk}) + \frac{\partial}{\partial x^j}(g_{ik}) - \frac{\partial}{\partial x^k}(g_{ij})\right]$$

then

(2.2.9) $$\Gamma^k_{ij} = \gamma^k_{ij} - g^{kl}\left\{A_{jlt}\frac{N^t_k}{F} + A_{ilt}\frac{N^t_j}{F} - A_{ijt}\frac{N^t_l}{F}\right\}.$$

Similarly, making a permutation to α, β, γ in (2.2.7), we will have

$$\begin{aligned}\frac{\partial}{\partial \xi^\beta}(g_{\gamma\alpha}) &- \frac{\partial}{\partial \xi^\gamma}(g_{\alpha\beta}) - \frac{\partial}{\partial \xi^\alpha}(g_{\gamma\beta}) \\ &= g_{\mu\alpha}\Gamma^\mu{}_{\gamma\beta} + g_{\gamma\mu}\Gamma^\mu{}_{\alpha\beta} + g_{\mu\gamma}\Gamma^\mu{}_{\beta\alpha} + g_{\beta\mu}\Gamma^\mu{}_{\gamma\alpha} - g_{\mu\beta}\Gamma^\mu{}_{\alpha\gamma} - g_{\alpha\mu}\Gamma^\mu{}_{\beta\gamma} \\ &- 2\left\{A_{\mu\gamma\alpha}\frac{N^\mu_\beta}{F} - A_{\mu\alpha\beta}\frac{N^\mu_\gamma}{F} + A_{\mu\beta\gamma}\frac{N^\mu_\alpha}{F}\right\}.\end{aligned}$$

If we put

$$\gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}\left[\frac{\partial}{\partial \xi^\beta}(g_{\gamma\alpha}) - \frac{\partial}{\partial \xi^\alpha}(g_{\gamma\beta}) - \frac{\partial}{\partial \xi^\gamma}(g_{\alpha\beta})\right]$$

so we have

$$\Gamma^{\gamma}_{\alpha\beta} = \gamma^{\gamma}_{\alpha\beta} + g^{\mu\gamma}\left\{A_{\nu\mu\alpha}\frac{N^{\nu}_{\beta}}{F} - A_{\nu\alpha\beta}\frac{N^{\nu}_{\mu}}{F} + A_{\nu\beta\mu}\frac{N^{\nu}_{\alpha}}{F}\right\}. \tag{2.2.10}$$

This proves the uniqueness of D. The set $\{\Gamma^k_{ij}, F^k_{ij} = 0, \Gamma^{\gamma}_{\alpha\beta}, F^{\gamma}_{\alpha\beta} = 0\}$ where $\{\Gamma^k_{ij}, \Gamma^{\gamma}_{\alpha\beta}\}$ are given by (2.2.9-2.2.10), can be defining a linear connection D satisfying (i) and (ii). □

Given the curvature tensor Ω of the linear connection D. It has three components that we call them the h-curvature component (R), the hv-curvature (P) and the vv-curvature (Q). Let $\dot{X}, \dot{Y} \in \mathcal{V}TM$ and $\bar{X}, \bar{Y} \in \mathcal{H}TM$, the h-curvature tensor

$$R : \mathcal{H}TM \otimes \mathcal{H}TM \otimes \mathcal{H}TM \mapsto \mathcal{H}TM$$

is defined by

$$R(\bar{X}, \bar{Y})Z = \Omega(\bar{X}, \bar{Y})Z. \tag{2.2.11}$$

The hv-curvature tensor

$$P : \mathcal{H}TM \otimes \mathcal{V}TM \otimes \mathcal{H}TM \mapsto \mathcal{H}TM$$

is defined by

$$P(\bar{X}, \dot{Y})Z = \Omega(\bar{X}, \dot{Y})Z. \tag{2.2.12}$$

The v-curvature tensor

$$Q : \mathcal{V}TM \otimes \mathcal{V}TM \otimes \mathcal{H}TM \mapsto \mathcal{H}TM$$

is defined by $Q(\dot{X}, \dot{Y})Z = \Omega(\dot{X}, \dot{Y})Z$.

It follows from theorem 4 that $Q = 0$. Thus for the linear connection D in theorem 4, the v-curvature Q always vanishes.

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