# On a deformation of the Kropina metric

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**Abstract.** Considering the Kropina metric $K(\alpha, \beta) = \dfrac{\alpha^2}{\beta}$ we study its deformation given by $F(\alpha, \beta) = K(\alpha, \beta) + \varphi(x)\beta$, where $\varphi(x)$ depends on the position only. From the variational problem we obtain the Lorentz equations. Consequently, we determine the canonical nonlinear connection $N$ of the considered Finsler space $F^n = (M, F(x, y))$.

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## 1 Preliminaries

Let $M$ be a $n$-dimensional, real, differentiable manifold and $\pi : TM \longrightarrow M$ be the tangent bundle of $M$.

**Definition 1.1** *Let $F : \widetilde{TM} = TM \setminus \{0\} \longrightarrow \mathrm{R}$ be a Finsler metric. The Finsler space $F^n = (M, F(x, y))$ is called with $(\alpha, \beta)$-metric if the fundamental function $F$ is written in the form*

$$F(x, y) = \widetilde{F}(\alpha(x, y), \beta(x, y)), \tag{1.1}$$

*where*

$$\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j} \text{ with } a_{ij}(x) \text{ a Riemannian metric on } M \tag{1.2}$$

*and*

$$\beta(x, y) = b_i(x)y^i \quad \text{with } b_i(x)dx^i \quad \text{a } 1-\text{form field on } TM. \tag{1.3}$$

The theory of Finsler spaces with $(\alpha, \beta)$-metric was studied by M. Matsumoto [1], R. Miron [3], M. Roman [6], [7], [8], V.S. Sabău and H. Shimada [5], [9], and many others mathematicians.

Since $\alpha(x, y)$ and $\beta(x, y)$ are 1-homogeneous with respect to $y^i$:

$$\alpha(x, ty) = t\alpha(x, y), \qquad \beta(x, ty) = t\beta(x, y), \qquad \forall t \in \mathrm{R}_+,$$

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it follows

$$\widetilde{F}(t\alpha(x,y), t\beta(x,y)) = t\widetilde{F}(\alpha(x,y), \beta(x,y)), \quad \forall t \in \mathbf{R}_+, \tag{1.4}$$

that is the function $\widetilde{F}(\alpha, \beta)$ is positively 1-homogeneous in both of the arguments.

Taking into account the 2-homogeneity of the Lagrangian $L(\alpha, \beta) = \widetilde{F}^2(\alpha, \beta)$ we obtain:

$$\begin{aligned} &\alpha L_\alpha + \beta L_\beta = 2L, \qquad \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha, \\ &\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \end{aligned} \tag{1.5}$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}. \tag{1.6}$$

In order to determine the fundamental tensor $g_{ij}(x,y)$ of a Finsler space with $(\alpha, \beta)$-metric, we introduce the following invariants, [1],

$$\begin{aligned} &\rho = \frac{1}{2\alpha} L_\alpha, \qquad \rho_1 = \frac{1}{2} L_\beta, \\ &\rho_0 = \frac{1}{2} L_{\beta\beta}, \qquad \rho_{-1} = \frac{1}{2\alpha} L_{\alpha\beta}, \qquad \rho_{-2} = \frac{1}{2\alpha^2}\big(L_{\alpha\alpha} - \frac{1}{\alpha} L_\alpha\big), \end{aligned} \tag{1.7}$$

where the subscripts 1, 0, -1, -2 are the homogeneity degree of these invariants.

As we know, denoting by

$$p_i = \alpha \frac{\partial \alpha}{\partial y^i} = a_{ij} y^j, \tag{1.8}$$

and using the invariants from 1.7, the fundamental tensor is obtained in the following theorem, [1] :

**Theorem 1.1** *The fundamental tensor field $g_{ij}$ of the Finsler space $F^n$ with $(\alpha, \beta)$-metric is represented by*

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_{-1}(b_i p_j + b_j p_i) + \rho_{-2} p_i p_j. \tag{1.9}$$

# 2 The deformation of the Kropina metric $K(\alpha, \beta)$

Let us consider the Kropina metric $K(\alpha, \beta) = \dfrac{\alpha^2}{\beta}$ where $\alpha$ and $\beta$ are given by 1.2 and 1.3.

In the following we will study the perturbation of this metric given by

$$F(\alpha, \beta) = K(\alpha, \beta) + \varphi(x)\beta, \tag{2.1}$$

where $\varphi : M \longrightarrow \mathbf{R}^*$ is a differentiable function.

We remark that the particular case $\varphi(x) = \pm 1$ was studied by R.Miron, H.Shimada and V.S. Sabău in [5].

In order for $F$ to be positive on $TM \setminus \{0\}$ we obtain the following proposition:

**Proposition 2.1** *The positivity of the metric $F(\alpha,\beta)$ given in 2.1 holds if and only if*

$$||b|| < \frac{1}{\sqrt{|\varphi|}} \quad \text{and} \quad \beta > 0. \tag{2.2}$$

*Proof.*
The positivity of $F$ on $TM \setminus \{0\}$ means that

$$K(\alpha,\beta) + \varphi(x)\beta > 0, \quad \forall y \neq 0.$$

It follows

$$\frac{\alpha^2 + \varphi\beta^2}{\beta} > 0, \quad \forall y \neq 0. \tag{2.3}$$

$\rightarrow$ Suppose positivity holds. Substitute $y^i = -b^i = -a^{ij}b_j$ in the previous inequality we obtain the relation $||b|| < \dfrac{1}{\sqrt{|\varphi|}}$ and as necessary condition $\beta > 0$.

$\leftarrow$ Conversely, using the relations 2.2 and starting with the Cauchy-Buniakowski-Schwarz inequality:

$$|a_{ij}b^iy^j| \leq \sqrt{a_{pq}b^pb^q}\sqrt{a_{rs}y^ry^s}$$

we obtain the positivity of $F$. □

Regarding to the fundamental tensor of the metric $F$ we state:

**Proposition 2.2** *The following formula holds good:*

$$det||g_{ij}|| = \frac{b^2A^{n-1}(B-C)}{\alpha^{n-1}\beta^{2(n+1)}}det||a_{ij}||, \tag{2.4}$$

*where*

$$\begin{cases} A = 2\alpha(\alpha^2 + \varphi\beta^2), \\ B = 3\alpha^4 + \varphi^2\beta^4, \\ C = -2\alpha^2(\alpha^2 - \varphi\beta^2). \end{cases}$$

Resuming the previous propositions,we have the following theorem:

**Theorem 2.1** *The pair $(M,F)$ with the fundamental function $F$ defined in 2.1 which satisfies the conditions 2.2 is a Finsler space.*

*Proof.* As we know, a Finsler metric $F : TM \longrightarrow \mathbf{R}$ satisfies the following conditions:

1°. $F$ is positive.

2°. $F$ is positive 1-homogeneous with respect to $y^i$.

3°. the fundamental tensor $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2F^2}{\partial y^i\partial y^j}$ is positively defined.

1°. The positivity of $F$ is given in *Proposition 2.1.* Indeed, from the positive 1-homogeneity of $\alpha$ and $\beta$ it follows 2°.

For 3°. we shall use the *Proposition 2.2.* We obtain

$$det||g_{ij}|| = \frac{2^{n-1}b^2F^{n-1}}{\beta^{n-1}}[5(\frac{\alpha}{\beta})^4 - 2\varphi(\frac{\alpha}{\beta})^2 + \varphi^2]det||a_{ij}||.$$

Since

$$5(\frac{\alpha}{\beta})^4 - 2\varphi(\frac{\alpha}{\beta})^2 + \varphi^2 = 4(\frac{\alpha}{\beta})^4 + [(\frac{\alpha}{\beta})^2 - \varphi]^2 > 0,$$

we obtain 3°. □

Using the homogeneous frame $\{b_i, l_i\}$, [6], with $l_i := \frac{1}{\alpha}p_i$ we express the fundamental tensor $g_{ij}$.

**Proposition 2.3** *The fundamental tensor $g_{ij}$ of the Finsler space $(M, F)$ is given by*

$$g_{ij} = \frac{2F}{\beta}a_{ij} + 4(\frac{\alpha}{\beta})^2 l_i l_j - 4(\frac{\alpha}{\beta})^3(b_i l_j + b_j l_i) + [\varphi^2 + 3(\frac{\alpha}{\beta})^4]b_i b_j, \tag{2.5}$$

*where $l_i = \frac{1}{\alpha}a_{ij}y^j = \frac{1}{\alpha}p_i$.*

*The contravariant tensor $g^{ij}$ is expressed in the following form:*

(2.6)

$$g^{ij} = \frac{\beta}{2F}a^{ij} - \Theta[\beta^2F^2b^ib^j - 2\alpha\beta^2F(b^il^j + b^jl^i) - 2\alpha^2(b^2(\alpha^2 - \varphi\beta^2) - 2\beta^2)l^il^j]$$

*where $b^2 = a^{ij}b_ib_j$ and*

$$\Theta = \frac{\alpha\beta^2}{b^2(AB - C)}.$$

*Proof.* Taking into account the invariants:

$$\rho = 2(\frac{\alpha}{\beta})^2 + 2\varphi, \quad \rho_0 = 3(\frac{\alpha}{\beta})^4 + \varphi^2, \quad \rho_{-1} = -\frac{4\alpha^2}{\beta^3}, \quad \rho_{-2} = \frac{4}{\beta^2},$$

we obtain 2.5. Next we have $g_{ij}g^{jk} = \delta_i^k$ and 2.6 holds true. □

By straightforward calculation, we get:

**Proposition 2.4** *If $k_{ij}$ is the fundamental tensor of the Kropina space $(M, K)$, then the fundamental tensor field $g_{ij}$ of the deformation space $(M, F)$ is expressed by:*

$$g_{ij} = k_{ij} + 2\varphi a_{ij} + \varphi^2 b_i b_j. \tag{2.7}$$

# 3 The variational problem

We shall study the variational problem of the space $F^n = (M, F(x, y))$ started from the integral of action for the Lagrangian $L(\alpha, \beta) = F^2(x, y)$.

Let $c : t \in [0, 1] \longmapsto c(t) \in M$ be a smooth curve on $M$ having the imagine in a local chart domain of manifold $M$. The curve $c$ can be express analiticaly by $x^i = x^i(t),\ t \in [0, 1]$.

The integral of action for the nondegenerate Lagrangian $F^2(x,y) = L(\alpha(x,y), \beta(x,y))$ is given by the functional

$$I(c) = \int_0^1 F^2(x, \frac{dx}{dt})dt. \tag{3.1}$$

The variational problem for the functional $I(c)$, (the points $x_0 = (x^i(0))$ and $x_1 = (x^i(1))$ being fixed) follows to the Euler-Lagrange equations, [2]:

$$E_i(F^2) := \frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\frac{\partial F^2}{\partial y^i} = 0,\ y^i = \frac{dx^i}{dt}. \tag{3.2}$$

The curves $c$, solutions of the differential equations above-mentioned is calling *extremals.*

**Proposition 3.1** *The Euler-Lagrange equations 2.2 are equivalent with the following differential equations:*

$$E_i(\alpha^2) + 2\frac{\rho_1}{\rho}E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i} = \frac{1}{\rho}\{\frac{dL_\alpha}{dt}\frac{\partial\alpha}{\partial y^i} + \frac{dL_\beta}{dt}\frac{\partial\beta}{\partial y^i}\},\quad y^i = \frac{dx^i}{dt}. \tag{3.3}$$

*Proof.* We remark the relation between $E_i(\alpha)$ and $E_i(\alpha^2)$ :

$$2\alpha E_i(\alpha) = E_i(\alpha^2) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i}.$$

Consequently, we have

$$E_i(L) = \frac{1}{2\alpha}L_\alpha E_i(\alpha^2) + \frac{1}{\alpha}L_\alpha\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i} + L_\beta E_i(\beta) - \{\frac{dL_\alpha}{dt}\frac{\partial\alpha}{\partial y^i} + \frac{dL_\beta}{dt}\frac{\partial\beta}{\partial y^i}\}.$$

and $E_i(L) = 0$ is exactly 3.3 . □

Let fix now a parameterization on curve $c$ and let consider the arc length of curve given by $ds^2 = \alpha^2(x, \frac{dx}{dt})dt^2$. In this case $\alpha^2(x, \frac{dx}{ds}) = 1$. Then, $s$ will be calling *canonical parameter*. Along the curve $c$ we have

$$\frac{d\alpha}{ds} = 0. \tag{3.4}$$

**Proposition 3.2** *In the canonical parameterization, along the extremal curves c, we have*

$$\frac{d\beta}{ds} = 0,\ \frac{dL}{ds} = 0,\ \frac{dL_\alpha}{ds} = 0,\ \frac{dL_\beta}{ds} = 0. \tag{3.5}$$

*Proof.* Indeed, $\frac{dL}{ds} = 0$ follows from the previous proposition. Taking into account that along the extremals $c$, (2.11) hold, it follows $\frac{dL}{ds} = L_\alpha\frac{d\alpha}{ds} + L_\beta\frac{d\beta}{ds} = L_\beta\frac{d\beta}{ds} = 0.$ Remarking that $L_\beta \neq 0$, we obtain $\frac{d\beta}{ds} = 0.$

Finally, $\dfrac{dL_\alpha}{ds} = L_{\alpha\alpha}\dfrac{d\alpha}{ds} + L_{\alpha\beta}\dfrac{d\beta}{ds} = 0$. Analogously for $\dfrac{dL_\beta}{ds} = 0$. □

Let $F_{ij}(x)$ be the electromagnetic tensorial field which correspond to the electromagnetic function $\beta = b_i(x)y^i$ :

$$F_{ij}(x) = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}. \tag{3.6}$$

Taking into account the expression of $E_i(\beta)$, we can write:

$$E_i(\beta) = F_{ij}(x)\frac{dx^j}{ds}. \tag{3.7}$$

Finally, we obtain the Lorentz equations of the space $(M, F)$

**Theorem 3.1** *The Lorentz equations of the Finsler space $F^n = (M, F)$ are given in the form:*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = [\varphi\beta - \frac{1}{2}F]F^i_j(x)\frac{dx^j}{ds} \tag{3.8}$$

*where*

$$F^i_j(x) = a^{is}(x)F_{sj}(x) \tag{3.9}$$

*and $\gamma^i_{jk}$ are the Christoffel symbols of the Riemannian metric tensor $a_{ij}(x)$.*

# 4 The Lorentz nonlinear connection of the Finsler space $(M, F)$.

The variational problem allows to introduce the Lorentz nonlinear connection, [4], [7], having the coefficients:

$$N^i{}_j(x,y) = \gamma^i{}_{jk}(x)y^k + [\frac{1}{2}F(x,y) - \varphi(x)\beta(x,y)]F^i_j(x). \tag{4.1}$$

The system of functions $N^i{}_j$ from 4.1 determines a canonical nonlinear connection $N$, which depends only on the fundamental function $F(x,y)$ of the Finsler spaces $F^n$.

The Lorentz equations 3.8 can be written in the form

$$\frac{d^2x^i}{ds^2} + N^i{}_j(x,y)\frac{dx^j}{ds} = 0, \quad y^i = \frac{dx^i}{ds}. \tag{4.2}$$

It follows that 4.2 gives us the autoparallel curves of the nonlinear connection $N$.

The canonical nonlinear connection $N$ determines a differentiable distribution which is supplementary to the vertical distribution $V$ on the manifold $TM$ :

$$T_u(TM) = N_u \oplus V_u, \quad \forall u \in TM. \tag{4.3}$$

Let $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ be the local basis adapted to $N$ and $V$, and $(dx^i, \delta y^i)$ its dual basis:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j},$$
$$\delta y^i = dy^i + N_j^i(x,y)dx^j. \tag{4.4}$$

While the vertical distribution $V$ is integrable, the horizontal distribution $N$ has not this property.

The tensor of integrability of $N$, [2] is:

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}. \tag{4.5}$$

Let $F_{j|k}^i$ be the covariant derivative of $F_j^i$ with respect to the Levi-Civita connection $\gamma^i{}_{jk}$.

We have, [7]:

**Proposition 4.1** *The tensor of integrability $R_{jk}^i$ of the nonlinear connection $N$ is given by*

$$R_{jk}^i = -\rho_j{}^i{}_{km}y^m + \sigma(F_{j|k}^i - F_{k|j}^i) + (F_j^i\frac{\delta\sigma}{\delta x^k} - F_k^i\frac{\delta\sigma}{\delta x^j}) \tag{4.6}$$

*where $\sigma = \frac{1}{2}F - \varphi\beta$ and $\rho_j{}^i{}_{km}$ is the curvature tensor of Levi-Civita connection.*

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