Proper affine vector fields in spherically symmetric static Space-Time

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Abstract. An approach is adopted to study proper affine vector fields in spherically symmetric static space-times by using the rank of the $6X6$ Rieman matrix and holonomy. Studing proper affine vector field in each case it is shown that the special class of the above space-times admit proper affine vector fields.

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1 Introduction

The aim of this paper is to find proper affine vector fields in spherically symmetric static space-times by using holonomy and decomposability, the rank of the $6X6$ Rieman matrix and direct integration techinques. Through out M is representing the four dimensional, connected, hausdorff space-time manifold with Lorentz metric g of signature $(-,+,+,+)$ The curvature tensor associated with g through Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed nonflat in the sense that the Riemann tensor does not vanish over any nonempty open subset of M.

A vector field X on M is called an affine vector field if it satisfies

$$X_{a;bc} = R_{abcd}X^d. \tag{1.1}$$

If one decomposes $X_{a;b}$ on M into its symmetric and skew-symmetric parts

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab} \quad (h_{ab} = h_{ba}, \quad F_{ab} = -F_{ba}) \tag{1.2}$$

then equation (1.2) is equivalent to

$$(i) \quad h_{ab;c} = 0 \quad (ii) \quad F_{ab;c} = R_{abcd}X^d \quad (iii) \quad F_{ab;c}X^c = 0. \tag{1.3}$$

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If $h_{ab} = 2cg_{ab}$, $c \in R$ then the vector field X is called homothetic (and Killing if $c = 0$.) The vector field X is said to be proper affine if it is not homothetic vector field and also X is said to be proper homothetic vector field if it is not Killing vector field on M [2].

2 Affine vector fields

In this section we will briefly discuss when the space-times admit proper affine vector fields for further details see [4].

Suppose that M is a simple connected space-time. Then the holonomy group of M is a connected Lie subgroup of the idenity component of the Lorentz group and is thus characterized by its subalgebra in the Lorentz algebra. These have been labeled into fifteen types $R_1 - R_{15}$ [6], [3]. It follows from [4] that the only such space-times which could admit proper affine vector fields are those which admit nowhere zero covariantly constant second order symmetric tensor field h_{ab} and it is known that this forces the holonomy type to be either R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13}. Here, we will only discuss the space-times which has the holonomy type R_2, R_4, R_7, R_{10} or R_{13}.

First consider the case when M has type R_{13}. Then one can always set up local coordinates (t, x^1, x^2, x^3) on an open set $U = U_1XU_2$, where U_1 is a one dimensional timelike submanifold of U coordinatized by t and U_2 is a three dimensional spacelike submanifold of U coordinatized by x^1, x^2, x^3 and where the above product is a metric product and the metric on U is given by [2]

$$ds^2 = -dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta, \quad (\alpha, \beta = 1, 2, 3) \tag{2.1}$$

where $g_{\alpha\beta}$ depends on x^γ, $(\gamma = 1, 2, 3)$. The above space-time is clearly $1 + 3$ decomposable. The curvature rank of the above space-time is atmost three and there exists a unique nowhere zero vector field $t_a = t_{,a}$ satisfying $t_{a;b} = 0$ and also $t^a t_a = -1$. From the Ricci Identity $R^a{}_{bcd}t^d = 0$. It follows from [4] that affine vector fields in this case are

$$X = (c_1 t + c_2)\frac{\partial}{\partial t} + Y \tag{2.2}$$

where $c_1, c_2 \in R$ and Y is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant t.

Now consider the situation when M has type R_{10}. The situation is similar to that of previous R_{13} case except that now we have local decomposition is $U = U_1XU_2$, where U_1 is a one dimensional spacelike submanifold of U and U_2 is a three dimensional timelike submanifold of U. The space-time metric on U is given by [2]

$$ds^2 = dx^2 + g_{\alpha\beta}dx^\alpha dx^\beta, \quad (\alpha, \beta = 0, 2, 3) \tag{2.3}$$

where $g_{\alpha\beta}$ depends on x^γ, $(\gamma = 0, 2, 3)$. The above space-time is clearly $1 + 3$ decomposable. The curvature rank of the above space-time is atmost three and there exists a unique nowhere zero vector field $x_a = x_{,a}$ satisfying $x_{a;b} = 0$ and also $x^a x_a = 1$. From the Ricci Identity $R^a{}_{bcd}x^d = 0$. It follows from [4] that affine vector fields in this case are

$$X = (c_1 x + c_2)\frac{\partial}{\partial x} + Y \tag{2.4}$$

where $c_1, c_2 \in R$ and Y is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant x.

Next suppose M has type R_7. Then each $p \in M$ has a neighborhood U which decomposes metrically as $U = U_1 X U_2$, where U_1 is a two dimensional submanifold of U with an induced metric of Lorentz signature and U_2 is a two dimensional submanifold of U with positive definite induced metric. The space-time metric on U is given by [2]

$$ds^2 = P_{AB} dx^A dx^B + Q_{\alpha\beta} dx^\alpha dx^\beta, \tag{2.5}$$

where $P_{AB} = P_{AB}(x^C)$, for all $A, B, C = 0, 1$ and $Q_{\alpha\beta} = Q_{\alpha\beta}(x^\gamma)$, for all $\alpha, \beta, \gamma = 2, 3$ and the above space-time is clearly $2+2$ decomposable. The space-time (2.5) admits two recurrent vector fields [1] l and n i.e. $l_{a;b} = l_a p_b$ and $n_{a;b} = n_a p_b$, where p_b is the recurrent 1-form. It also admits two covariantly constant second order symmetric tensors which are $2l_{(a}n_{b)}$ and $(x_a x_b + y_a y_b)$. The rank of the $6X6$ Riemann matrix is two. It follows from [4] that if X is an affine vector field on M then X decomposes as

$$X = X_1 + X_2, \tag{2.6}$$

where the vector fields X_1 and X_2 are tangent to the two dimensional timelike and spacelike submanifolds, respectively. It also follows from [4] that X_1 and X_2 are homothetic vector fields in their respective submanifolds with their induced geometry. Conversely, every pair of affine vector fields, one in the timelike submanifolds and one spacelike submanifolds give rise to a affine vector field in space-time.

Now suppose that M has type R_4. Then each $p \in M$ has a neighborhood U which decomposes metrically as $U = U_1 X U_2 X U_3$, where U_1 and U_2 are one dimensional submanifold of U and U_3 is a two dimensional submanifold of U. The space-time metric on U is given by [4]

$$ds^2 = -dt^2 + dx^2 + g_{AB} dx^A dx^B, \quad (A, B = 2, 3) \tag{2.7}$$

where g_{AB} depends only on x^C $(C = 2, 3)$. The above space-time is clearly $1+1+2$ decomposable. The curvature rank of the above space-time is one and there exist two independent nowhere zero unit timelike and spacelike covariantly constant vector field $t_a = t_{,a}$ and $x_a = x_{,a}$ satisfying $t_{a;b} = 0$ and $x_{a;b} = 0$. From the Ricci identity $R^a{}_{bcd} t_a = 0$ and $R^a{}_{bcd} x_a = 0$. It follows from [4] that affine vector fields in this case are

$$X = (c_1 t + c_2 x + c_3)\frac{\partial}{\partial t} + (c_4 t + c_5 x + c_6)\frac{\partial}{\partial x} + Y \tag{2.8}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R$ and Y is a homothetic vector field in the induced geometry on each of the two dimensional submanifolds of constant t and x.

Now suppose that M has type R_2. Here each $p \in M$ admits a neighborhood U which decomposes metrically as $U = U_1 X U_2 X U_3$, where U_1 and U_2 are one dimensional submanifold of U and U_3 is a two dimensional submanifold of U. The space-time metric on U is given by [4]

$$ds^2 = dy^2 + dz^2 + g_{AB} dx^A dx^B, \quad (A, B = 0, 1) \tag{2.9}$$

where g_{AB} depends only on x^C $(C = 0, 1)$. The above space-time is clearly $1+1+2$ decomposable. The curvature rank of the above space-time is one and there exist two independent nowhere zero unit spacelike covariantly constant vector fields $y_a = y_{,a}$ and $z_a = z_{,a}$ satisfying $y_{a;b} = 0$ and $z_{a;b} = 0$. From the Ricci identity $R^a{}_{bcd}y_a = 0$ and $R^a{}_{bcd}z_a = 0$. It follows from [4] that affine vector fields in this case are

$$X = (c_1y + c_2z + c_3)\frac{\partial}{\partial y} + (c_4y + c_5z + c_6)\frac{\partial}{\partial z} + Y \tag{2.10}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R$ and Y is a homothetic vector field in the induced geometry on each of the two dimensional submanifolds of constant y and z.

3 Main results

As mentioned in section 2, the space-times which can admit proper affine vector fields having holonomy type R_2, R_3, R_4, R_6, R_7, R_8, R_{10}, R_{11} or R_{13}. It also follows from [1] that the rank of the $6X6$ Riemann matrix is atmost three. Here in this paper we will consider the rank of the $6X6$ Riemann matrix to study affine vector fields in spherically symmetric static space-time. Consider a spherically symmetric static space-time in the usual coordinate system (t, r, θ, ϕ) with line elememt [7]

$$ds^2 = -e^{v(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.1}$$

It follows from [7], the above space-time admits four independent Killing vector fields which are

$$\frac{\partial}{\partial t}, \quad sin\phi\frac{\partial}{\partial\theta} + cos\phi cot\theta\frac{\partial}{\partial\phi}, \quad cos\phi\frac{\partial}{\partial\theta} - sin\phi cot\theta\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\phi}.$$

The Ricci tensor Segre type of the above space-time is $\{1, 1(11)\}$ or one of its degeneracies $\{1(1, 11)\}$, $\{(11)(11)\}$ or $\{(1, 111)\}$. The non-zero independent components of the Riemann tensor are

$$\begin{aligned}
R^{01}{}_{01} &= \frac{1}{e^{\lambda(r)}}(\frac{1}{4}\lambda^{'}(r)v^{'}(r) - \frac{1}{2}v^{''}(r) - \frac{1}{4}v^{'2}(r)) \equiv \alpha_1(say)\\
R^{02}{}_{02} &= R^{03}{}_{03} = -\frac{v^{'}(r)}{2re^{\lambda(r)}} \equiv \alpha_2(say)\\
R^{12}{}_{12} &= R^{13}{}_{13} = \frac{\lambda^{'}(r)}{2re^{\lambda(r)}} \equiv \alpha_3(say)\\
R^{23}{}_{23} &= \frac{e^{\lambda}(r) - 1}{r^2e^{\lambda(r)}} \equiv \alpha_4(say).
\end{aligned} \tag{3.2}$$

The curvature tensor of M can be described by components $R^{ab}{}_{cd}$, which can be written as a $6X6$ symmetric matrix in a well known way [8]

$$R^{ab}{}_{cd} = diag(\alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_4) \tag{3.3}$$

where $\alpha_1, \alpha_2, \alpha_2, \alpha_3$ and α_3 are are real functions of r only and 6-dimensional labeling is in the order 01, 02, 03, 12, 13, 23 with $x^0 = t$. We are only interested in those cases when the rank of the $6X6$ Riemann matrix is less than or equal to three. Thus there exist the following possibilities:

$(A) Rank = 3,\ when\ v \in R, \lambda = \lambda(r)$
$(B) Rank = 3,\ when\ v = v(r), \lambda = 0$
$(C) Rank = 3,\ when\ v = v(r), 0 \neq \lambda \in R, 2v^{''} + v^{'2} = 0$
$(D) Rank = 2,\ when\ v = v(r), \lambda = 0, 2v^{''} + v^{'2} = 0$
$(E) Rank = 1,\ v, \lambda \in R (\lambda \neq 0).$
We will consider each case in turn.

Case A

In this case $v \in R, \lambda = \lambda(r)$, the rank of the $6X6$ Riemann matrix is 3 and there exists a unique (up to a multiple) nowhere zero timelike vector field $t_a = t_{,a}$ satisfying $t_{a;b} = 0$ (and so, from the Ricci idenity $R^a{}_{bcd} t_a = 0$). The line element can, after a recaling of t, be written in the form

$$ds^2 = -dt^2 + (e^{\lambda(r)} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)). \tag{3.4}$$

The space-time is clearly $1+3$ decomposable. The affine vector fields in this case [4] are

$$X = (c_7 t + c_8)\frac{\partial}{\partial t} + X^{'} \tag{3.5}$$

where $c_7, c_8 \in R$ and $X^{'}$ is a homothetic vector field in the induced geometry on each of the three dimensional submanifolds of constant t. The completion of case A necessities finding an homothetic vector fields in the induced geometry of the submanifolds of constant t. The induced metric $g_{\alpha\beta}$ (where $\alpha, \beta = 1, 2, 3$) with nonzero components is given by

$$g_{11} = e^{\lambda(r)}, \quad g_{22} = r^2, \quad g_{22} = r^2 sin^2\theta. \tag{3.6}$$

A vector field $X^{'}$ is a homothetic vector field if it satisfies $L_{X^{'}} g_{\alpha\beta} = 2c g_{\alpha\beta}$, where $c \in R$. One can expand by using (3.6) to get

$$\lambda^{'} X^1 + 2X^1{}_{,1} = 2c \tag{3.7}$$

$$e^{\lambda} X^1{}_{,2} + r^2 X^2{}_{,1} = 0 \tag{3.8}$$

$$e^{\lambda} X^1{}_{,3} + r^2 sin^2\theta X^3{}_{,1} = 0 \tag{3.9}$$

$$\frac{1}{r} X^1 + X^2{}_{,2} = c \tag{3.10}$$

$$\lambda X^2{}_{,3} + sin^2\theta X^3{}_{,2} = 0 \tag{3.11}$$

$$\frac{1}{r} X^1 + cot\theta X^2 + X^3{}_{,3} = c. \tag{3.12}$$

Equation (3.7) gives

$$X^1 = c e^{-\frac{\lambda}{2}} \int e^{\frac{\lambda}{2}} dr + e^{-\frac{\lambda}{2}} A^1(\theta, \phi) \tag{3.13}$$

where $A^1(\theta, \phi)$ is a function of integration. Substituting the value of X^1 in (3.8) and (3.9) gives

$$\begin{aligned} X^2 &= -A^1_\theta(\theta, \phi) \int \frac{1}{r^2} e^{\frac{\lambda}{2}} dr + A^2(\theta, \phi) \\ X^3 &= \frac{A^1_\phi(\theta, \phi)}{sin^2\theta} \int \frac{1}{r^2} e^{\frac{\lambda}{2}} dr + A^3(\theta, \phi) \end{aligned} \tag{3.14}$$

where $A^2(\theta,\phi)$ and $A^3(\theta,\phi)$ are functions of integration. Considering equation (3.11) differentiating with respect to r one finds

$$A^1(\theta,\phi) = sin\theta B^1(\phi) + B^2(\theta)$$

where $B^1(\phi)$ and $B^2(\theta)$ are functions of integration. Substituting back into (3.13) and (3.14) gives

$$\begin{aligned} X^1 &= ce^{-\frac{\lambda}{2}}\int e^{\frac{\lambda}{2}}dr + e^{-\frac{\lambda}{2}}(sin\theta B^1(\phi) + B^2(\theta)) \\ X^2 &= -(cos\theta B^1(\phi) + B^2_{\theta}(\theta))\int \frac{1}{r^2}e^{\frac{\lambda}{2}}dr + A^2(\theta,\phi) \\ X^3 &= \frac{B^1_{\phi}(\phi)}{sin\theta}\int \frac{1}{r^2}e^{\frac{\lambda}{2}}dr + A^3(\theta,\phi) \end{aligned} \tag{3.15}$$

Now consider equation (3.10) and differentiate with respect to ϕ to get

$$sin\theta B^1_{\phi}(\phi)(\frac{e^{-\frac{\lambda}{2}}}{r} + \int \frac{1}{r^2}e^{\frac{\lambda}{2}}dr) + A^2_{\theta\phi}(\theta,\phi) = 0.$$

Differentiating with respect to r we get $B^1_{\phi}(\phi)((\frac{e^{-\frac{\lambda}{2}}}{r})' + \frac{1}{r^2}e^{\frac{\lambda}{2}}) = 0$ and there exists two possible cases:

$$(i) \quad B^1_{\phi}(\phi) \neq 0, ((\frac{e^{-\frac{\lambda}{2}}}{r})' + \frac{1}{r^2}e^{\frac{\lambda}{2}}) = 0 \quad (ii) \quad B^1_{\phi}(\phi) = 0, ((\frac{e^{-\frac{\lambda}{2}}}{r})' + \frac{1}{r^2}e^{\frac{\lambda}{2}}) \neq 0.$$

Case Ai

$((\frac{e^{-\frac{\lambda}{2}}}{r})' + \frac{1}{r^2}e^{\frac{\lambda}{2}}) = 0 \Rightarrow e^{\lambda} = \frac{a^2}{a^2-r^2}$, where $a(\neq 0) \in R$. The line element can, after a rescaling of t, be written in the form

$$ds^2 = -dt^2 + \frac{a^2}{a^2-r^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.16}$$

The above space-time is well known Einstein static space-time [5]. The affine vector fields in this case are known [2], which are

$$\begin{aligned} X^0 &= (c_7t + c_8), \\ X^1 &= \sqrt{(1-\frac{r^2}{a^2})}(sin\theta(c_2cos\phi + c_3sin\phi) + c_1cos\theta), \\ X^2 &= \frac{1}{r}\sqrt{(1-\frac{r^2}{a^2})}(cos\theta(c_2cos\phi + c_3sin\phi) - c_1sin\theta) + (c_4cos\phi + c_5sin\phi), \\ X^3 &= \frac{1}{rsin\theta}\sqrt{(1-\frac{r^2}{a^2})}(-c_2sin\phi + c_3cos\phi) + cot\theta(-c_4sin\phi + c_5cos\phi) + c_6, \end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \in R$.

Case Aii

$B^1_\phi(\phi) = 0 \Rightarrow B^1(\phi) = c_1$, where $c_1 \in R$ and so we have (from (3.15))

$$\begin{aligned} X^1 &= ce^{-\frac{\lambda}{2}} \int e^{\frac{\lambda}{2}} dr + e^{-\frac{\lambda}{2}} (c_1 sin\theta + B^2(\theta)), \\ X^2 &= -(c_1 cos\theta + B^2_\theta(\theta)) \int \frac{1}{r^2} e^{\frac{\lambda}{2}} dr + A^2(\theta, \phi), \\ X^3 &= A^3(\theta, \phi). \end{aligned}$$

A straightforward calculation now shows that a homothetic vector field exists if and only if

$$e^{-\frac{\lambda}{2}} \int e^{\frac{\lambda}{2}} dr = 1 \quad (\Rightarrow \int e^{\frac{\lambda}{2}} dr = re^{\frac{\lambda}{2}})$$

which upon differentiation with respect to r gives $\lambda = constant$. One then easily see from 3.3 that the rank of $6X6$ Riemann matix reduce to 1 or zero, giving a contradiction (since we are assuming that the rank of $6X6$ Riemann matrix is 3). Thus no proper homothetic vector field exist in the induced geometry. So it admits Killing vector fields which are

$$X^1 = 0, \quad X^2 = (c_1 sin\phi + c_2 cos\phi), \\ X^3 = cot\theta(c_1 cos\phi - c_2 sin\phi) + c_3, \tag{3.17}$$

where $c_1, c_2, c_3 \in R$. Thus, from (3.17) and (3.5) the affine vector fields in this case are

$$X^0 = (c_7 t + c_8), \quad X^1 = 0, \quad X^2 = (c_1 sin\phi + c_2 cos\phi), \\ X^3 = cot\theta(c_1 cos\phi - c_2 sin\phi) + c_3, \tag{3.18}$$

where $c_7, c_8 \in R$. This completes case (A).

Case B

In this case $v = v(r), \lambda = 0, 2v^{''} + v^{'2} \neq 0$ and and the rank of the $6X6$ Riemann matrix is 3. After recaling r the line element takes the form

$$ds^2 = -e^v(r)dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.19}$$

The affine vector fields in this case are

$$X^0 = c_4, \quad X^1 = 0, \quad X^2 = (c_1 sin\phi + c_2 cos\phi), \\ X^3 = cot\theta(c_1 cos\phi - c_2 sin\phi) + c_3, \tag{3.20}$$

where $c_1, c_2, c_3, c_4 \in R$. The affine vector fields admitted by (3.19) in this case are Killing vector fields.

Case C

In this case $v = v(r), \lambda(\neq 0) \in R, 2v^{''} + v^{'2} = 0$. The equation $2v^{''} + v^{'2} = 0 \Rightarrow v = ln(\frac{a}{2}r + b)^2$, where $a, b \in R (a \neq 0)$. The rank of the $6X6$ Riemann matrix is 3 and there exist a unique (up to a multiple) solution $r_a = r_{,a}$ of equation $R^a{}_{bcd} r_a = 0$ and $r_{a;b} \neq 0$. The line element is

$$ds^2 = -(\frac{a}{2}r + b)^2 dt^2 + kdr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{3.21}$$

where $k(=e^{\lambda}) \in R(k \neq 0 \ or \ 1)$. Substituting the above information into affines equations and after a strightforward calculation one find affine vector fields in this case are Killing vector fields which are given in equation (3.20).

Case D

In this case $v = v(r), \lambda(=0) \in R, 2v^{''} + v^{'2} = 0$. The equation $2v^{''} + v^{'2} = 0 \Rightarrow v = ln(\frac{a}{2}r + b)^2$, where $a, b \in R(a \neq 0)$. The rank of the $6X6$ Riemann matrix is 2 and there exists a unique (up to a multiple) spcelike vector field $r_a = r_{,a}$ of equation $R^a{}_{bcd}r_a = 0$ and $r_{a;b} \neq 0$. The line element in this case is

$$ds^2 = -(\frac{a}{2}r + b)^2 dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.22}$$

Substituting the above information into affines equations and after a strightforward calculation one find affine vector fields in this case are Killing vector fields which are given in equation (3.20).

Case E

In this case $v, \lambda \in R(\lambda \neq 0)$. The rank of the $6X6$ Riemann matrix is one and there exist two independent nowhere zero solutions of equations, $R^a{}_{bcd}t_a = R^a{}_{bcd}r_a = 0$, where $r_a = r_{,a}$ and $t_a = t_{,a}$ are the spacelike and timelike vector field, respectively. The sapce-time (3.1) admits only one independent nowhere zero timelike covariantly constant vector field t_a satisfying $t_{a;b} = 0$. After rescaling of t the line element takes the form

$$ds^2 = -dt^2 + kdr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{3.23}$$

where $k(=e^{\lambda}) \in R(k \neq 0 \ or \ 1)$. The above space-time is clearly $1+3$ decomposable but the rank of $6X6$ Riemann matrix is one. Substituting the above information into affine equations and one finds affine vector fields in this case are

$$\begin{gathered} X^0 = c_5 t + c_1, \quad X^1 = c_6 r, \quad X^2 = (c_2 sin\phi + c_3 cos\phi), \\ X^3 = cot\theta(c_2 cos\phi - c_3 sin\phi) + c_4, \end{gathered} \tag{3.24}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R$.

Summary

In this paper a study of spherically symmetric static space-times according to their proper affine vector fields is given. An approach is developed to study proper affine vector fields in the above space-time by using the rank of the $6X6$ Riemann matrix and holonomy. From the above study we obtain the following results:

(i) The case when the rank of the $6X6$ Riemann matrix is one and there exist two nowhere zero independent solutions of equation $R^a{}_{bcd}t_a = 0$ and $R^a{}_{bcd}r_a = 0$ but only one independent nowhere zero covariantly constant vector field. This is the space-time (3.23) and it admits the proper affine vector fields (see Case E).

(ii) The case when the rank of the $6X6$ Riemann matrix is two or three and there exist a nowhere zero independent spacelike vector field which is a solution of equation $R^a{}_{bcd}r_a = 0$ and is not covariantly constant. These are the space-time (3.21) and (3.22) and they admit affine vector fields which are Killing vector fields (see for details Cases C and D).

(iii) The case when the rank of the $6X6$ Riemann matrix is two or three and there exists a nowhere zero independent timelike vector field which is a solution of equation $R^a{}_{bcd}t_a = 0$ and is covariantly constant. This is the space-time (3.4) and it admits the proper affine vector fields (see for details Cases Ai and Aii).

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