On (k, μ)-contact metric manifolds

A.A. Shaikh and Kanak Kanti Baishya

Abstract. The object of the present paper is to study (k, μ)-contact metric manifolds.

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1 Introduction

In 1995 Blair, Koufogiorgos and Papantoniou [4]introduced the notion of (k, μ)-contact metric manifolds and a full classification of such a manifold is given by E. Boeckx [6]. In this paper we study (k, μ)-contact metric manifolds satisfying certain conditions.
In 1969 Matsumoto and Chuman [9] introduced the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties. In a Sasakian manifold, the Ricci operator Q commutes with the structure tensor ϕ, but in general $Q\phi \neq \phi Q$. Hence the definition of Matsumoto and Chuman is not sufficient in the study of contact metric manifolds as it does not include all the non-Sasakian cases. Keeping this view in mind and owing to H. Endo [7], in section 2 of our paper, we define C-Bochner curvature tensor in a contact metric manifold and studied in the context of contact geometry. Since as a particular case, we can obtain the definition of Matsumoto and Chuman [9], hence our definition of C-Bochner curvature tensor in the contact metric manifold will be called *generalized C-Bochner* (briefly GC-Bochner) curvature tensor.

After preliminaries, in section 3 of our paper, we study (k, μ)-contact metric manifolds with vanishing GC-Bochner curvature tensor and proved that such a manifold is either an η-Einstein manifold or a Sasakian manifold.

Section 4 is devoted to the study of (k, μ)-contact metric manifolds satisfying $R(X, Y) \cdot B = 0$, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Then it is shown that such a (k, μ)-contact metric manifold is locally isometric to one of the following: (i) the Riemannian product $E^{n+1}(0) \times S^n(4)$, (ii) a Sasakian manifold, (iii) an η-Einstein manifold. The last section deals with *generalized C-Bochner* recurrent (k, μ)-contact metric manifolds.

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2 Preliminaries

A contact manifold is a C^∞-(2n+1)-dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field of η, satisfying $\eta(\xi)$=1 and $d\eta(X,\xi)$=0 for any vector field X on M^{2n+1}. A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1,1) such that $\eta(X) = g(X,\xi)$, $d\eta(X,Y) = g(X,\phi Y)$ and $\phi^2 X = -X + \eta(X)\xi$ for every vector fields X, Y on M^{2n+1}. Then the structure (ϕ,ξ,η,g) on M^{2n+1} is called a contact metric structure and the manifold M^{2n+1} equipped with such a structure is said to be a contact metric manifold [2]. In a contact metric manifold, the following relations hold:

(2.1) $$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y on M^{2n+1}. Given a contact metric manifold $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)$ we define a tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the operator of Lie differentiation.Then h is symmetric and satisfies

(2.2) $$h\xi = 0, \quad h\phi = -\phi h, \quad Tr.h = 0, Tr.\phi h = 0.$$

If ∇ is the Riemannian connection of g, then we have the relation

$$\nabla_X \xi = -\phi X - \phi h X.$$

A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which ξ is a Killing vector field is called a K-contact manifold. A contact metric manifold is Sasakian if and only if the relation

(2.3) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

holds, where R is the Riemainnian curvature tensor of type (1, 3).

The (k,μ)-nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution [4]

$$N(k,\mu) : p \rightarrow N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]$$
$$+\mu[g(Y,Z)hX - g(X,Z)hY]\},$$

where k, μ are real constants. Hence if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, then we have

(2.4) $$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].$$

The interest of the nullity condition (2.4) is that it is preserved under a D-homothetic deformation. A contact metric manifold satisfying (2.4) is called a (k,μ)-contact metric manifold and the examples (for both Sasakian and non-Sasakian cases) of a such a manifold is given in [4] and also such a manifold is studied by K. Arslan and his coauthors[1]. In this connection we can mention the work of C. Udriste [11], S. Ianus and others [5], [8]. In particular, if $\mu = 0$, then we obtain the condition of k-nullity distribution introduced by Tanno [10].

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-contact metric manifold. Then the following relations hold[4]:

(2.5) $$h^2 = (k-1)\phi^2, \quad k \leq 1,$$

$$Q\phi - \phi Q = 2[2n-2+\mu]h\phi, \tag{2.6}$$

$$Q\xi = 2nk\xi, \tag{2.7}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.8}$$

$$\begin{aligned} QX &= [2n-2-n\mu]X + [2n-2+\mu]hX \\ &\quad + [2-2n+2nk+n\mu]\eta(X)\xi, \quad n \geq 1, \end{aligned} \tag{2.9}$$

$$r = 2n(2n-2+k-n\mu), \tag{2.10}$$

where Q is the Ricci-operator and r is the scalar curvature of the manifold.

In view of (2.5)-(2.7), it can be easily seen that in a (k,μ)-contact metric manifold, the following relations hold:

$$Tr.h^2 = 2n(1-k), \tag{2.11}$$

$$S(X,\phi Y) + S(\phi X, Y) = 2(2n-2+\mu)g(h\phi X, Y), \tag{2.12}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y), \tag{2.13}$$

$$Q\phi + \phi Q = 2\phi Q + 2(2n-2+\mu)h\phi, \tag{2.14}$$

$$\phi Q\phi X = 2(2n-2+\mu)hX - QX + 2nk\eta(X)\xi, \tag{2.15}$$

$$S(\phi X,\xi) = 0, \tag{2.16}$$

$$Tr.(Q\phi) = Tr.(\phi Q) = 0, \tag{2.17}$$

where S is the Ricci tensor of type (0, 2), i.e., $g(QX,Y) = S(X,Y)$.

A contact metric manifold is said to be η-Einstein if its Ricci tensor is of the form

$$QX = aX + b\eta(X)\xi,$$

where a, b are the smooth functions on the manifold.

In a contact metric manifold $M^{m+1}(\phi,\xi,\eta,g)(m=2n)$, the *generalized contact Bochner* (briefly *GC-Bochner*) curvature tensor $B(X,Y)Z$ of type (1, 3) is defined by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + g(\phi Y, hZ)h\phi X - g(\phi X, hZ)h\phi Y \\ &+ \tfrac{1}{m+4}\{\tfrac{m}{2} + \alpha + \tfrac{Tr.h^2}{m+2}\}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &- \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] - \tfrac{1}{m+4}\{\alpha + m + \tfrac{Tr.h^2}{m+2}\}[g(\phi X,Z)\phi Y \\ &- g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] - \tfrac{1}{m+4}\{\alpha - 4 + \tfrac{Tr.h^2}{m+2}\}[g(X,Z)Y \\ &- g(Y,Z)X] + \tfrac{1}{2(m+4)}[g(X,Z)QY - g(Y,Z)QX - S(Y,Z)X \\ &+ S(X,Z)Y - g(X,Z)\phi Q\phi Y + g(Y,Z)\phi Q\phi X - S(\phi Y,\phi Z)X \\ &\quad + S(\phi X,\phi Z)Y - S(\phi Y,Z)\phi X + S(Y,\phi Z)\phi X + S(\phi X,Z)\phi Y \\ &\quad - S(X,\phi Z)\phi Y + 2S(\phi X,Y)\phi Z - 2S(X,\phi Y)\phi Z \\ &\quad + g(\phi X,Z)(\phi Q + Q\phi)Y - g(\phi Y,Z)(\phi Q + Q\phi)X \\ &\quad + 2g(\phi X,Y)(\phi Q + Q\phi)Z - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX \\ &\quad + \eta(X)\eta(Z)\phi Q\phi Y - \eta(Y)\eta(Z)\phi Q\phi X + S(Y,Z)\eta(X)\xi \\ &\quad - S(X,Z)\eta(Y)\xi + S(\phi Y,\phi Z)\eta(X)\xi - S(\phi X,\phi Z)\eta(Y)\xi], \end{aligned} \tag{2.18}$$

where $\alpha = \frac{r+m}{m+2}$, $m = 2n$, r is the scalar curvature of the manifold.

In particular, if the manifold is Sasakian, then we have $h = 0$, $Q\phi = \phi Q$, $Tr.h^2 = 0$, $S(\phi X, \phi Y) = S(X,Y) - m\eta(X)\eta(Y)$ and hence (2.18) reduces to the definition of the C-Bochner curvature tensor in a Sasakian manifold M^{2n+1} defined by Matsumoto and Chuman [9]. This is why we call the tensor field B defined in (2.18) the *generalized C-Bochner* curvature tensor.

From (2.18), it can be easily verify that the *GC-Bochner* curvature tensor in a contact metric manifold satisfies the following:

$$B(X,Y)Z = -B(Y,X)Z, \tag{2.19}$$

$$B(X,Y)Z + B(Y,Z)X + B(Z,X)Y = 0, \tag{2.20}$$

$$g(B(X,Y)Z,W) = -g(B(X,Y)W,Z), \tag{2.21}$$

$$g(B(X,Y)Z,W) = g(B(Z,W)X,Y) \tag{2.22}$$

for any vector fields X, Y, Z, W on M^{2n+1}.

We shall now state a known result which will be needed later on.

Lemma 2.1 [3]: *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with $R(X,Y)\xi = 0$ for all vector fields X, Y. Then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

3 (k,μ)-Contact metric manifolds with vanishing GC-Bochner curvature tensor

This section deals with a (k,μ)-contact metric manifold $M^{2n+1}(n > 1)$ such that the *GC-Bochner* curvature tensor vanishes identically. Then we have $B(X,Y)Z = 0$ for all X, Y, Z.

Using (2.4)-(2.8) and (2.11)-(2.16) in (2.18), we obtain

$$\tilde{B}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + g(\phi Y, hZ)g(h\phi X, W) \tag{3.1}$$

$$-g(\phi X, hZ)g(h\phi Y, W) + \tfrac{1}{m+4}\{\tfrac{m}{2} + \alpha + \tfrac{m(1-k)}{m+2}\}$$
$$[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(Z)$$
$$+g(Y,W)\eta(X)\eta(Z)] - \tfrac{1}{m+4}\{\alpha + m + \tfrac{m(1-k)}{m+2}\}$$

$$[g(\phi X,Z)g(\phi Y,W)-g(\phi Y,Z)g(\phi X,W)+2g(\phi X,Y)g(\phi Z,W)]$$
$$-\tfrac{1}{m+4}\{\alpha-4+\tfrac{m(1-k)}{m+2}\}[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)]$$
$$+\tfrac{1}{2(m+4)}[2\{S(Y,W)g(X,Z)-S(X,W)g(Y,Z)-S(Y,Z)g(X,W)$$
$$+S(X,Z)g(Y,W)+S(X,W)\eta(Y)\eta(Z)+S(Y,Z)\eta(X)\eta(W)$$
$$-S(X,Z)\eta(Y)\eta(W)-S(Y,W)\eta(X)\eta(Z)+g(\phi QX,Z)g(\phi Y,W)$$
$$-g(\phi QY,Z)g(\phi X,W)+g(\phi QY,W)g(\phi X,Z)$$
$$-g(\phi QX,W)g(\phi Y,Z)+2g(\phi QX,Y)g(\phi Z,W)$$
$$+2g(\phi QZ,W)g(\phi X,Y)\}+2(m-2+\mu)\{g(hX,W)g(Y,Z)$$
$$-g(hY,W)g(X,Z)+g(hY,Z)g(X,W)-g(hX,Z)g(Y,W)$$
$$+g(hY,W)\eta(X)\eta(Z)-g(hX,W)\eta(Y)\eta(Z)-g(hY,Z)\eta(X)\eta(W)$$
$$+g(hX,Z)\eta(Y)\eta(W)+g(h\phi X,Z)g(\phi Y,W)-g(h\phi Y,Z)g(\phi X,W)$$
$$+g(h\phi Y,W)g(\phi X,Z)-g(h\phi X,W)g(\phi Y,Z)$$
$$+2g(h\phi X,Y)g(\phi Z,W)+2g(h\phi Z,W)g(\phi X,Y)\}$$
$$+mk\{g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)$$
$$+g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)\}],$$

where $\tilde{B}(X,Y,Z,W)=g(B(X,Y)Z,W)$ and $\tilde{R}(X,Y,Z,W)=g(R(X,Y)Z,W)$. Let $\{e_i : i=1,2,\ldots,2n+1\}$ be an orthonormal basis of the tangent space T_pM at any point $p\in M$. Then putting $X=W=e_i$ in (3.1) and taking summation over $1\le i\le m+1$, $m=2n$, and using (2.1), (2.2), (2.17), symmetry of h and the skew-symmetry of ϕ, we obtain

$$\sum_{i=1}^{m+1}\tilde{B}(e_i,Y,Z,e_i)=(m-2+\mu)g(hY,Z)+\frac{(1-k)(3m+8)}{2(m+4)}g(Y,Z)+\frac{(1-k)(1-m)(m+8)}{2(m+4)}\eta(Y)\eta(Z) \tag{3.2}$$

for all Y,Z. Since we have $B(X,Y)Z=0$, it follows from (3.2) that

$$(m-2+\mu)g(hY,Z)+\frac{(1-k)(3m+8)}{2(m+4)}g(Y,Z)+\frac{(1-k)(1-m)(m+8)}{2(m+4)}\eta(Y)\eta(Z)=0, \tag{3.3}$$

which implies that either $\mu=-(2n-2)$ or $h=0$ and hence $k=1$.

If $k=1$ then a (k,μ)-contact metric manifold is Sasakian. If $\mu=-(2n-2)$, then (2.9) yields that the manifold is η-Einstein. Hence we can state the following:

Theorem 1. *Let $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ be a (k,μ)-contact metric manifold with vanishing generalized C-Bochner curvature tensor. Then the manifold is either*
(i) a Sasakian manifold, or
(ii) an η-Einstein manifold.

4 (k, μ)-contact metric manifolds satisfying $R(X,Y)\cdot B = 0$

Let us consider a (k, μ)-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ satisfying the condition

$$R(X,Y)\cdot B = 0, \tag{4.1}$$

where $R(X,Y)$ is considered as a curvature operator. Then from (4.1) it follows that

$$R(\xi,Y)B(U,V)W - B(R(\xi,Y)U,V)W - B(U,R(\xi,Y)V)W \\ -B(U,V)R(\xi,Y)W = 0. \tag{4.2}$$

In view of (2.8), it follows from (4.2) that

$$k[g(B(U,V)W,Y)\xi - \eta(B(U,V)W)Y - g(Y,U)B(\xi,V)W \\ +\eta(U)B(Y,V)W - g(Y,V)B(U,\xi)W + \eta(V)B(U,Y)W \\ -g(Y,W)B(U,V)\xi + \eta(W)B(U,V)Y] + \mu[g(B(U,V)W,hY)\xi \\ -\eta(B(U,V)W)hY - g(hY,U)B(\xi,V)W + \eta(U)B(hY,V)W \\ -g(hY,V)B(U,\xi)W + \eta(V)B(U,hY)W - g(hY,W)B(U,V)\xi \\ +\eta(W)B(U,V)hY] = 0, \tag{4.3}$$

where (2.4) have been used.

Putting $Z = \xi$ in (2.18), we obtain by virtue of (2.1), (2.2), (2.4)-(2.8), (2.15) and (2.16) that

$$B(X,Y)\xi = \frac{(1-k)(m+8)}{2(m+4)}[\eta(X)Y - \eta(Y)X] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{4.4}$$

Using (2.22), we get from (4.4) that

$$B(\xi,X)Y = \frac{(1-k)(m+8)}{2(m+4)}[\eta(Y)X - g(X,Y)\xi] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{4.5}$$

which implies

$$B(\xi,X)\xi = \frac{(1-k)(m+8)}{2(m+4)}[X - \eta(X)\xi] - \mu hX.$$

From (4.4), it follows that

$$\eta(B(X,Y)\xi) = 0, \text{ for all } X, Y. \tag{4.6}$$

Again (4.4) yields

$$B(hX,Y)\xi = -\frac{(1-k)(m+8)}{2(m+4)}\eta(Y)hX + \mu(k-1)[\eta(X)\eta(Y)\xi - \eta(Y)X], \tag{4.7}$$

which implies by (2.19) that

$$B(X,hY)\xi = \frac{(1-k)(m+8)}{2(m+4)}\eta(X)hY + \mu(k-1)[\eta(X)Y - \eta(X)\eta(Y)\xi]. \tag{4.8}$$

Putting $W=\xi$ in (4.3) and then using (4.4)-(4.8), we get

$$(4.9)\qquad k[\frac{(1-k)(m+8)}{2(m+4)}\{g(Y,V)U-g(Y,U)V\}+\mu\{g(hU,Y)\eta(V)\xi$$
$$-g(hV,Y)\eta(U)\xi+g(Y,U)hV-g(Y,V)hU\}+B(U,V)Y]$$
$$+\mu[\tfrac{(1-k)(m+8)}{2(m+4)}\{g(hY,V)U-g(hY,U)V\}+\mu\{g(hY,U)hV$$
$$-g(hY,V)hU\}+\mu(k-1)\{g(Y,V)\eta(U)\xi$$
$$-g(Y,U)\eta(V)\xi\}+B(U,V)hY]=0.$$

which implies that

$$(4.10)\qquad k[\frac{(1-k)(m+8)}{2(m+4)}\{g(Y,V)g(U,W)-g(Y,U)g(V,W)\}+\mu\{g(hU,Y)\eta(V)\eta(W)$$
$$-g(hV,Y)\eta(U)\eta(W)+g(Y,U)g(hV,W)-g(Y,V)g(hU,W)\}+\tilde{B}(U,V,Y,W)]$$
$$+\mu[\tfrac{(1-k)(m+8)}{2(m+4)}\{g(hY,V)g(U,W)-g(hY,U)g(V,W)\}+\mu\{g(hY,U)g(hV,W)$$
$$-g(hY,V)g(hU,W)\}+\mu(k-1)\{g(Y,V)\eta(U)$$
$$-g(Y,U)\eta(V)\}\eta(W)+\tilde{B}(U,V,hY,W)=0.$$

From (3.2), it follows that

$$(4.11)\qquad \sum_{i=1}^{m+1}\tilde{B}(e_i,Y,hZ,e_i)=\frac{(1-k)(3m+8)}{2(m+4)}g(hY,Z)$$
$$+(1-k)(m-2+\mu)\{g(Y,Z)-\eta(Y)\eta(Z)\}.$$

Putting $U=W=e_i$ in (4.10) and taking summation over i and then using (3.2) and (4.11) we obtain

$$(4.12)\qquad (1-k)[\frac{km(m+8)}{2(m+4)}+\frac{k(3m+8)}{2(m+4)}+\mu(m-2+\mu)]g(Y,V)$$
$$+[k(m-2+\mu)+\tfrac{m\mu(1-k)(m+8)}{2(m+4)}+\tfrac{\mu(1-k)(3m+8)}{2(m+4)}]g(hY,V)$$
$$+(1-k)[\tfrac{k(1-m)(m+8)}{2(m+4)}-\mu(m-2+\mu)]\eta(Y)\eta(V)=0$$

for all vector fields Y and V. The relation (4.12) gives us either $k=1$, or

$$(4.13)\qquad km(m+8)+k(3m+8)+2\mu(m+4)(m-2+\mu)=0;$$

either $h=0$, or

$$(4.14)\qquad \mu(1-k)(3m+8)+\mu m(1-k)(m+8)+2k(m+4)(m-2+\mu)=0;$$

either $k=1$, or

$$(4.15)\qquad k(1-m)(m+8)-2\mu(m+4)(m-2+\mu)=0.$$

If $k = 1$, then $h = 0$ and hence the manifold is Sasakian. Thus we have either the manifold is Sasakian (for $k = 1$) or (4.13)-(4.15) holds (non-Sasakian case).

From (4.13) and (4.15), it follows that $k = 0$. If $k = 0$, then (4.13) or (4.15) implies that either $\mu = 0$ or $\mu = -(2n-2)$, $m = 2n$. Hence in the non-Sasakian case, we have either $k = 0 = \mu$, or $k = 0$ and $\mu = -(2n-2)$. If $k = 0 = \mu$, then by Lemma 2.1, the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$. Again, if $k = 0$ and $\mu = -(2n-2)$ then (2.9) implies that the manifold is η- Einstein.

Next, from (4.13) and (4.14) we get

$$(m^2 + 11m + 8)\{k - \mu(1-k)\} + 2(m+4)(m-2+\mu)(\mu - k) = 0. \tag{4.16}$$

Also from (4.14) and (4.15), it follows that

$$\mu(1-k)(m^2 + 11m + 8) - k(8 - 7m - m^2) + 2(m+4)(m-2+\mu)(k-\mu) = 0. \tag{4.17}$$

Adding (4.16) and (4.17), we get $k = 0$. If $k = 0$, then (4.13) or (4.15) implies that either $\mu = 0$ or $\mu = -(2n-2)$. Hence we arrive at the same conclusion.

Thus considering all the cases, we can state the following:

Theorem 2. *Let $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ be a (k, μ)-contact metric manifold satisfying the condition $R(X, Y) \cdot B = 0$. Then the manifold is locally isometric to one of the following:*
(i) the Riemannian product $E^{n+1}(0) \times S^n(4)$,
(ii) a Sasakian manifold,
(iii) an η-Einstein manifold.

5 Generalized C-Bochner recurrent (k, μ)-contact metric manifolds

Definition 5.1. A (k, μ)-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be *generalized C - Bochner* recurrent if its *generalized C - Bochner* curvature tensor B is non-zero and satisfies $\nabla B = A \otimes B$, where A is an everywhere non-zero 1-form.

Let us consider a *generalized C-Bochner* recurrent (k, μ)-contact metric manifold $M^{2n+1}(n > 1)$. We now define a function f on M^{2n+1} by $f^2 = g(B, B)$, where the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion. Then we have $f(Yf) = f^2 A(Y)$, which implies that $Yf = fA(Y)$ and hence $X(Yf) - Y(Xf) = \{XA(Y) - YA(X)\}f$. Since the left hand side of the above relation is identically zero and f $\neq$0 on M^{2n+1} by our assumption, we get $dA(X, Y) = 0$ for all X, Y, that is, the 1-form A is closed. Hence from

$$(\nabla_X B)(Y, Z)U = A(X)B(Y, Z)U,$$

it follows that

$$(\nabla_X \nabla_Y B)(U, V)W = \{XA(Y) + A(X)A(Y)\}B(U, V)W.$$

This yields

$$(R(X, Y) \cdot B)(U, V)W = [2dA(X, Y)]B(U, V)W = 0.$$

Hence by Theorem 2, we can state the following:

Theorem 3. *A generalized C-Bochner recurrent (k,μ)-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is locally isometric to one of the following:*
(i) the Riemannian product $E^{n+1}(0)\times S^n(4)$,
(ii) a Sasakian manifold,
(iii) an η-Einstein manifold.

In particular, if A vanishes identically then $\nabla B=0$ i.e., the manifold is *generalized C-Bochner* symmetric, which implies that $R(X,Y)\cdot B=0$. Then we can state the following:

Corollary. *A generalized C-Bochner symmetric (k,μ)-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is locally isometric to one of the following:*
(i) the Riemannian product $E^{n+1}(0)\times S^n(4)$,
(ii) locally isometric to a Sasakian manifold,
(iii) an η-Einstein manifold.

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Authors' addresses:

A. A. Shaikh and K. K. Baishya
Department of Mathematics, University of Burdwan,
Golapbag, Burdwan-713104, West Bengal, India.
email: aask2003@yahoo.co.in, aask@epatra.com and kanak2006@yahoo.com