A note on constrained complex Hamiltonian mechanics

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Abstract. In this paper, it is generalized the concept of Hamiltonian dynamics with constraints to complex case. Firstly, it is considered a Kählerian manifold as a phase space. Then a non-holonomic constraint is given by 1-form on it. If the form is closed, it is found the constraint is (locally) holonomic. Finally, complex analogous of some topics in Hamiltonian mechanical system with constraints is given.

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1 Introduction

Modern differential geometry provides a fundamental framework for studying Hamiltonian mechanics. In recent years, there are many studies as some articles in [1, 2, 3, 4] and books in [5, 6] about differential geometric methods in mechanics. It is well known that the dynamics of Hamiltonian formalisms is characterized by a suitable vector field defined on cotangent bundles which are phase-spaces of momentum of a given configuration manifold. If $H : T^*Q \to \mathbf{R}$ is a regular Hamiltonian function then there is a unique vector field Z_H on cotangent bundle T^*Q such that dynamical equations

$$(1.1) \qquad i_{Z_H}\Phi = dH,$$

where Φ is the symplectic form and H stands for Hamiltonian function. The paths of the Hamiltonian vector field Z_H are the solutions of the Hamiltonian equations shown by

$$(1.2) \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i},$$

where q^i and $(q^i, p_i), 1 \leq i \leq m$, are coordinates of Q and T^*Q. The triple, either (T^*Q, Φ, Z_H) or (T^*Q, Φ, H), is called *Hamiltonian system* on the cotangent bundle T^*Q with symplectic form Φ. Let T^*Q be symplectic manifold with closed symplectic form Φ. Similar to constraints on TQ, by a *constraint* on T^*Q it is said a non-zero 1-form $\omega = \wedge^a \omega_a$ on T^*Q. A set $\overline{\omega} = \{\omega_1, ..., \omega_s\}$ of s linearly independent 1- forms

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on T^*Q it may be named to be a *system of constraints* on T^*Q. We say that a curve α in T^*Q satisfies the constraints if $\omega_a(\dot{\alpha}\,(t)) = 0$, $1 \leq a \leq s$.

Let (T^*Q, Φ, H) be a Hamiltonian system on symplectic manifold T^*Q with closed symplectic form Φ. Consider a Hamiltonian system (T^*Q, Φ, H) together with a system $\overline{\omega}$ of constraints on T^*Q. So, it may be called $(T^*Q, \Phi, H, \overline{\omega})$ to be a *Hamiltonian system with constraints.* Generally a curve α satisfying the Hamiltonian equations for energy H will not satisfy the constraints. For a curve α to satisfy the constraints, some additional forces must act on the system in addition to the *force* dH. So, the dynamical equations of motion become

$$i_Z \Phi = dH + \wedge^a \omega_a, \omega_a(Z) = 0, \tag{1.3}$$

where Z is a vector field on T^*Q. From (1.3), Hamiltonian equations with constraints is given the following as:

$$\begin{aligned} \frac{dq^i}{dt} &= (\frac{\partial H}{\partial p_i} + \wedge^a (B_a)_i), \\ \frac{dp_i}{dt} &= -(\frac{\partial H}{\partial q_i} + \wedge^a (A_a)_i), \\ (A_a)_i \frac{dq^i}{dt} + (B_a)_i \frac{dp_i}{dt} &= 0, \end{aligned} \tag{1.4}$$

where $1 \leq i \leq m$, $1 \leq a \leq s$.

The purpose of this study is to make a contribution to the modern development of Hamiltonian formalisms of classical mechanics in terms of differential-geometric methods on differentiable manifolds. So, we obtain complex Hamiltonian equations with constraints on the Kählerian manifold. In the discussion section, geometrical and mechanical results of mechanical system with constraints have been given. The first of them is that if the distribution on T^*M is integrable, a system of constraints is holonomic. The second is that Hamiltonian energy with constraints is conserved.

The present paper is structured as follows. In section 2, it is recalled complex and Kählerian manifolds, and also Hamiltonian equations on Kählerian manifolds. In section 3, complex Hamiltonian equations with constraints on Kählerian manifold are deduced. In the final section, the geometrical and mechanical theory of complex mechanical system with constraints was discussed.

2 Preliminaries

In this letter, all geometric objects are assumed to be differentiable and the sum is taken over repeated indices. Now then it is assumed $1 \leq i \leq m$.

2.1 Complex manifolds

Let M be configuration manifold of real dimension m. A tensor field J on TM is called an *almost complex structure* on TM if at every point p of TM, J is endomorphism of the tangent space $T_p(TM)$ such that $J^2 = -I$. A manifold TM with fixed almost complex structure J is called *almost complex manifold.* Assume that (x_i) be coordinates of M and $(x_i,\, y_i)$ be a real coordinate system on a neighborhood U of any point p of TM. Also, let us to be $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{(dx_i)_p, (dy_i)_p\}$ to natural bases over $\mathbf{R}$ of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively.

Let TM be a almost complex manifold with fixed almost complex structure J. The manifold TM is called *complex manifold* if there exists an open covering $\{U\}$ of TM satisfying the following condition: There is a local coordinate system $(x_i,\, y_i)$ on each U, such that

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\; J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}. \tag{2.1}$$

for each point of U. Let $z_i = x_i + \mathbf{i}\, y_i$, $\mathbf{i} = \sqrt{-1}$, be a complex local coordinate system on a neighborhood U of any point p of TM. We define the vector fields by

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{i}(\frac{\partial}{\partial y^i})_p\},\; (\frac{\partial}{\partial \overline{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{i}(\frac{\partial}{\partial y^i})_p \tag{2.2}$$

and the dual covector fields

$$\left(dz^i\right)_p = \left(dx^i\right)_p + \mathbf{i}(dy^i)_p,\; \left(d\overline{z}^i\right)_p = \left(dx^i\right)_p - \mathbf{i}(dy^i)_p \tag{2.3}$$

which represent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then the endomorphism J is shown as

$$J(\frac{\partial}{\partial z_i}) = \mathbf{i}\frac{\partial}{\partial z_i},\; J(\frac{\partial}{\partial \overline{z}_i}) = -\mathbf{i}\frac{\partial}{\partial \overline{z}_i}. \tag{2.4}$$

The dual endomorphism J^* of the cotangent space $T_p^*(TM)$ at any point p of manifold TM satisfies $J^{*2} = -I$, and is defined by

$$J^*(dz_i) = \mathbf{i}dz_i,\; J^*(d\overline{z}_i) = -\mathbf{i}d\overline{z}_i. \tag{2.5}$$

2.2 Hermitian and Kählerian manifolds

A *Hermitian metric* on an almost complex manifold with almost complex structure J is a Riemannian metric g on TM such that

$$g(JX, JY) = g(X, Y), \tag{2.6}$$

for any vector fields X, Y on TM. An almost complex manifold TM with a Hermitian metric is called an *almost Hermitian manifold*. If TM is a complex manifold, then TM is called a *Hermitian manifold*.

Let further TM be a 2m-dimensional real almost Hermitian manifold with almost complex structure J and Hermitian metric g. The triple (TM, J, g) may be named an *almost Hermitian structure*. We denote by $\chi(TM)$ the set of complex vector fields on TM and by $\wedge^1(TM)$ the set of complex 1-forms on TM. Let (TM, J, g) be an almost Hermitian structure. The 2-form defined by

$$\Phi(X, Y) = g(X, JY), \quad \forall X, Y \in \chi(TM) \tag{2.7}$$

is called the *Kählerian form* of (TM, J, g).

An almost Hermitian manifold is called *almost Kählerian* if its Kählerian form Φ is closed. If, moreover, TM is Hermitian, then TM is called a Kählerian manifold.

2.3 Complex Hamiltonian Equations

In this subsection, we remind complex Hamiltonian equations for classical mechanics structured on Kählerian manifold obtained in [4].

Let T^*M be the Kählerian manifold and $(z_i, \overline{z}_i)$ its complex coordinates. Let almost complex structure J^* and Liouville form λ given by $J^*(dz_i) = \mathrm{i}dz_i$, $J^*(d\overline{z}_i) = -\mathrm{i}d\overline{z}_i$ and by $\lambda = (J^*(\omega)) = \frac{1}{2}\mathrm{i}(-z_i d\overline{z}_i + \overline{z}_i dz_i)$ such that $\omega = \frac{1}{2}(z_i d\overline{z}_i + \overline{z}_i dz_i)$ complex 1-form on T^*M. If $\Phi = -d\lambda$ is closed Kählerian form, then Φ is also a symplectic structure on T^*M.

Proposition 1: Let T^*M be Kählerian manifold with closed Kählerian form Φ. Hamiltonian vector field Z_H on T^*M with closed form Φ is given by

$$Z_H = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z_i} - \frac{1}{\mathbf{i}}\frac{\partial H}{\partial z_i}\frac{\partial}{\partial \overline{z}_i}. \tag{2.8}$$

Suppose that the curve

$$\alpha : I \subset \mathbf{C} \rightarrow T^*M \tag{2.9}$$

be an integral curve of Hamiltonian vector field Z_H, i.e., $Z_H(\alpha(t)) = \dot{\alpha}$.

Then we infer the following equations

$$\frac{dz_i}{dt} = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}_i}, \frac{d\overline{z}_i}{dt} = -\frac{1}{\mathbf{i}}\frac{\partial H}{\partial z_i}. \tag{2.10}$$

which are called *complex Hamiltonian equations* on Kählerian manifold T^*M. The triple, either (T^*M, Φ, Z_H) or (T^*M, Φ, H), is called a *Hamiltonian system* on Kählerian manifold T^*M with closed Kählerian form Φ.

3 Complex Hamiltonian Equations with Constraints

In this part, we obtain complex analogous of some topics in the geometric theory of constraints given in [2].

Here, we conclude complex Hamiltonian equations with constraints on Kählerian manifold T^*M. Similar to (1.1), it may be indicated by Z_a the vector fields on the T^*M given by

$$i_{Za}\Phi = \omega_a,\ 1 \leq a \leq s. \tag{3.1}$$

Proposition 2: Let T^*M be Kählerian manifold with closed Kählerian form Φ. The vector field Z_a on T^*M is given by

$$Z_a = \frac{1}{\mathbf{i}}(B_a)_i\frac{\partial}{\partial z_i} - \frac{1}{\mathbf{i}}(A_a)_i\frac{\partial}{\partial \overline{z}_i}, \tag{3.2}$$

Proof: Let T^*M be Kählerian manifold with form Φ. Consider that vector field Z_a is given by

$$Z_a = (Z_a)_i\frac{\partial}{\partial z_i} + (\overline{Z}_a)_i\frac{\partial}{\partial \overline{z}_i}. \tag{3.3}$$

From (3.1), $i_{Z_a}\Phi$ is calculated by

$$-\mathbf{i}(\bar{Z}_a)_i dz_i + \mathbf{i}(Z_a)_i d\overline{z}_i. \tag{3.4}$$

On the other hand, we set as

$$\omega_a = (A_a)_i dz_i + (B_a)_i d\overline{z}_i \tag{3.5}$$

With respect to (3.1), if (3.4) and (3.5) is equalized, we find as

$$Z_a = \frac{1}{\mathbf{i}}(B_a)_i \frac{\partial}{\partial z_i} - \frac{1}{\mathbf{i}}(A_a)_i \frac{\partial}{\partial \overline{z}_i},$$

the vector field on Kählerian manifold T^*M with closed Kählerian form Φ. Hence, proof finishes.◊

Now, from (1.1) and (1.3) and (3.1), one may easily deduce

$$Z = Z_H + \wedge^a Z_a. \tag{3.6}$$

Hence, by means of (2.8), (3.2) and (3.6) we obtain the vector field

$$Z = \frac{1}{\mathbf{i}}\left(\frac{\partial H}{\partial \overline{z}_i} + \wedge^a (B_a)_i\right)\frac{\partial}{\partial z_i} - \frac{1}{\mathbf{i}}\left(\frac{\partial H}{\partial z_i} + \wedge^a (A_a)_i\right)\frac{\partial}{\partial \overline{z}_i}. \tag{3.7}$$

Suppose that the curve

$$\alpha : I \subset \mathbf{C} \to T^*M$$

be an integral curve of the complex vector field Z given by (3.7), i.e.,

$$Z(\alpha(t)) = \dot{\alpha}(t), \; t \in I. \tag{3.8}$$

In the local coordinates, for $\alpha(t) = (z_i(t), \overline{z}_i(t))$, we have

$$\dot{\alpha}(t) = \frac{dz_i}{dt}\frac{\partial}{\partial z_i} + \frac{d\overline{z}_i}{dt}\frac{\partial}{\partial \overline{z}_i}. \tag{3.9}$$

Then we reach the following equations

$$\begin{aligned} \frac{dz_i}{dt} &= \frac{1}{\mathbf{i}}\left(\frac{\partial H}{\partial \overline{z}_i} + \wedge^a (B_a)_i\right) \\ \frac{d\overline{z}_i}{dt} &= -\frac{1}{\mathbf{i}}\left(\frac{\partial H}{\partial z_i} + \wedge^a (A_a)_i\right) \\ (A_a)_i\frac{dz_i}{dt} + (B_a)_i\frac{d\overline{z}_i}{dt} &= 0, \end{aligned} \tag{3.10}$$

which are called *complex Hamiltonian equations with constraints* on Kählerian manifold T^*M, where $1 \leq a \leq s$. Then the quartet $(T^*M, \Phi, Z, \overline{\omega})$ is said *mechanical system with constraints.*

Discussion

Finally, considering the above, complex analogous of the geometrical and mechanical meaning of constraints given in [2, 6, 7] may be explained as follows.

1) Let $\overline{\omega}$ be a system of constraints on Kählerian manifold T^*M. Then it may be defined a distribution D^*on $\overline{\omega}$ as follows.

$$D^*(x) = \{ Z \in T_x T^*M) | \; \omega_a(Z) = 0, \; \text{for all} \; a, 1 \leq a \leq s\} \tag{3.11}$$

Thus D^* is $(2m-s)$-dimensional distribution on T^*M. In this case, a system of complex constraints $\overline{\omega}$ is called holonomic, if the distribution D^* is integrable; otherwise we call $\overline{\omega}$ anholonomic. Hence, $\overline{\omega}$ is holonomic if and only if the ideal ρ of $\wedge T^*M$ generated by $\overline{\omega}$ is a differential ideal. Obviously (3.10) holds for holonomic as well as anholonomic constraints. For a system of holonomic constraints, the motion lies on a specific leaf of the foliation defined by D^*.

2) From (1.3) it is obtained equalities of

$$0 = (i_Z\omega)(Z) = dH(Z) = Z(H). \tag{3.12}$$

Therefore, the Hamiltonian energy H on Kählerian manifold T^*M for a solution $\alpha(t)$ of (3.10) is conserved.

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