Dualistic structures on warped product manifolds

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Abstract. In this note we show that the projection of a dualistic structure defined on a warped product space induces dualistic structures on the base and the fiber manifolds. Conversely dualistic structures on the base and the fiber induces a dualistic structure on the warped product space. We prove that if the warped product (endowed with the induced dualistic sructure) is a dually flat space, then the base manifold is also dually flat and the fiber is a space form; i.e. of constant sectional curvature.

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1 Introduction

Let (M, g) a Riemannian manifold and ∇ an affine connection on M. An affine connection ∇' is said to be dual or conjugate of ∇ w.r.t. the metric g if,

$$X.g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla'_X Z) \ , \ \forall \ X, Y, Z \in TM \ .$$

Given an affine connection ∇ on a Riemannian manifold (M, g), there exists an unique affine connection dual of ∇ w.r.t. g, denoted by ∇^*.
The triple (g, ∇, ∇^*) is then called a dualistic structure on M.
The manifold M endowed with a dualistic structure (g, ∇, ∇^*) is called a dually flat space if both dual connections ∇ and ∇^* are torsion-free and flat; that is the curvature tensors w.r.t. ∇ and ∇^* respectively vanishe identically.
Dualistic structures are a fundamental mathematical concept of information geometry, specially in the investigation of the natural differential geometric structure possessed by families of probability distributions. For example it is important, when investigating properties of the Fisher metric g and the
α-connection $\nabla^{(\alpha)}$ on statistical manifolds (see [1] , [2] for more information) to consider them not individually, but rather as the triple $(g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$. Through the Fisher metric g, there exists a kind of duality between $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ which is of significance importance.
From a differential geometric point of view the dualistic structure first generalizes somehow the invariance of the inner product under parallel translation trough metric connections. Moreover the existence of a dually flat structure on a manifold points

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out some topological and geometrical properties of the manifold. For example if a manifold M admits a dually flat structure $(g, \nabla, \nabla*)$ and if one of the dual connections, say ∇, is complete, then only the first homotopy group of M is non trivial, and any two points in M can be joined by a ∇-geodesic (see [3]).
In [2] the author proved that a dualistic structure on a manifold M induces, through the canonical projection on a submanifold, a dualistic structure on any submanifold S of M, that is furthermore dually flat if M is a dually flat space. Thus for example the canonical divergence, that is an important concept defined on dually flat statistical manifolds, is always defined on curves (considered as submanifolds) of M even if M is not dually flat (see [2] and also [4] for more information about the canonical divergence on statistical manifolds).
When M is a warped product space, the base (resp. the fiber) manifold is isometric (resp. homothetic) to a submanifold of M, and moreover M is determined by the base and the fiber. Our aim is to establish not only analogous results to the previous one, but also to show that dualistic structures on the base and the fiber manifolds induce on the warped product space a dualistic structure, and we give then dual flatness conditions for this induced structure.

2 Some preliminaries on dualistic structures

Let (M, g) a Riemannian manifold and ∇ an affine connection on M. The dual or conjugate connection of ∇ w.r.t. g is the unique affine connection on M, denoted by ∇^*, such that:

$$X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)\ ,\ \forall\ X, Y, Z \in TM\ .$$

The duality wr.t. a given metric between affine connections on a manifold M is a symmetric relation; ie. $(\nabla^*)^* = \nabla$, and thus ∇ and ∇^* are dual of each other w.r.t the metric g. The triple (g, ∇, ∇^*) is then called a dualistic structure on M. The following result relates dual connections to the metric connection.
We have [2]:

Lemma 2.1
Let (g, ∇, ∇^*) a dualistic structure on manifold M.
Then $\frac{1}{2}(\nabla + \nabla^*)$ is the Levi-Civita connection on (M, g).
Conversely if an affine connection ∇' on M has the same torsion as ∇^* and if $\frac{1}{2}(\nabla + \nabla')$ is the metric connection, then $\nabla' = \nabla^*$.

The Levi-Civita connection is the only self-dual affine connection w.r.t.the given metric. Thus an affine connection on (M, g) is metric if and only if it is self-dual w.r.t. g.

Let $\gamma : t \longmapsto \gamma(t)$ be a curve in M with boundary points p and q.
Denote by Π_γ and Π^*_γ the translation along γ w.r.t. ∇ and its dual ∇^* respectively.
It holds:

$$g(\Pi_\gamma(X), \Pi^*_\gamma(Y))_{|q} = g(X, Y)_{|p}\ ,\ \forall\ X, Y \in TM\ .$$

The previous equality can be traduced as a generalization of the invariance of the inner product under parallel translation through metric connections.

From this relation it follows that if one of Π_γ and Π^*_γ is independent of γ, then it holds also for the second. Thus curvature flatness w.r.t. ∇ is equivalent to curvature flatness w.r.t. ∇^*. This equivalence can be also observed from the relation between curvature tensors. If R and R^* denote the curvature tensors w.r.t. ∇ and ∇^* respectively, then it holds:

$$g(R(X,Y)Z,W) = -g(R^*(X,Y)W,Z)\ ,\ \forall\ X,Y,Z,W \in TM\ .$$

An affine connection is said to be flat if its curvature tensor vanishes identically. A dual flat space is a manifold M endowed with a dualistic structure (g,∇,∇^*) such that ∇ and ∇^* are both torsion-free and flat.

3 Dualistic structures on warped product spaces

Let (M,g) and (N,h) be two Riemannian manifolds of dimension m and n respectively and $f \in C^\infty(M)$ a positive function on M. The warped product of (M,g) and (N,h), with warping function f, is the $(m\times n)$-dimensional manifold $M\times N$ endowed with the metric G_f given by:

$$G_f =: \pi^* g + (f\circ\pi)^2\sigma^* h\ ,$$

where π^* and σ^* are the pull-backs of the projections π and σ of $M\times N$ on M and N respectively.
The tangent space $T_{(p,q)}(M\times N)$ at a point $(p,q)\in M\times N$ is isomorph to the direct sum $T_pM\oplus T_qN$. Let $\mathcal{L}_H(M)$ (resp. $\mathcal{L}_V(N)$) be the set of all vector fields on $M\times N$, each of which is the horizontal lift (resp. the vertical lift) of a vector field on M (resp. on N). We have:

$$T(M\times N) = \mathcal{L}_H(M)\oplus\mathcal{L}_V(N)\ ;$$

and thus a vector field A on $M\times N$ can be written as

$$A = X + U\ ,\ \text{with}\ X\in\mathcal{L}_H(M)\ \text{and}\ U\in\mathcal{L}_V(N)\ .$$

Obviously

$$\pi_*(\mathcal{L}_H(M)) = TM\ \ \text{and}\ \ \sigma_*(\mathcal{L}_V(N)) = TN\ .$$

For any vector field $X\in\mathcal{L}_H(M)$ we denote $\pi_*(X)$ by $\bar{X}$, and for any vector field $U\in\mathcal{L}_V(N)$ we denote $\sigma_*(U)$ by $\tilde{U}$. Furthermore we denote the horizontal lift on $M\times N$ of a vector field $x\in TM$ by $(x)^H$, and the vertical lift on $M\times N$ of a vector field $u\in TN$ by $(u)^V$.

Let (G_f,D,D^*) a dualistic structure on $M\times N$.
For $X,Y\in\mathcal{L}_H(M)$ and $U,V\in\mathcal{L}_V(N)$ we put:

$$\pi_*(D_XY) = {}^M\nabla_{\bar{X}}\bar{Y}\ \ \text{and}\ \ \pi_*(D^*_XY) = {}^M\nabla'_{\bar{X}}\bar{Y}\ ,$$

and

$$\sigma_*(D_UV) = {}^N\nabla_{\tilde{U}}\tilde{V}\ \ \text{and}\ \ \sigma_*(D^*_UV) = {}^N\nabla'_{\tilde{U}}\tilde{V}\ .$$

Since D and D^* are affine connections on $M \times N$ and π and σ are the projections of $M \times N$ on M and N respectively, ${}^M\nabla$ and ${}^M\nabla'$ are affine connections on M and, ${}^N\nabla$ and ${}^N\nabla'$ are affine connections on N. We have the following result:

Proposition 3.1
The triple $(g\ ,\ {}^M\nabla\ ,\ {}^M\nabla')$ is a dualistic structure on M and the triple $(h\ ,\ {}^N\nabla\ ,\ {}^N\nabla')$ is a dualistic structure on N; ie.

$${}^M\nabla' = ({}^M\nabla)^* \text{ w.r.t. } g \text{ and } {}^N\nabla' = ({}^N\nabla)^* \text{ w.r.t. } h\ .$$

Proof
Let $\bar{X}\ ,\ \bar{Y}\ ,\ \bar{Z} \in TM$ and $X\ ,\ Y\ ,\ Z \in \mathcal{L}_H(M)$ their corresponding horizontal lifts respectively. Denoting the inner product w.r.t. G_f by $<\ ,\ >$, we have:

$$\begin{aligned}
\bar{X}.g(\bar{Y},\bar{Z}) \circ \pi &= X. < Y\ ,\ Z > \\
&= < D_X Y\ ,\ Z > + < Y\ ,\ D^*_X Z > \\
&= g(\pi_*(D_X Y)\ ,\ \bar{Z}) \circ \pi + g(\bar{Y}\ ,\ \pi_*(D^*_X Z)) \circ \pi \\
&= [g({}^M\nabla_{\bar{X}}\bar{Y}\ ,\ \bar{Z}) + g(\bar{Y}\ ,\ {}^M\nabla'_{\bar{X}}\bar{Z})] \circ \pi\ .
\end{aligned}$$

Thus

$$\bar{X}.g(\bar{Y},\bar{Z}) = g({}^M\nabla_{\bar{X}}\bar{Y}\ ,\ \bar{Z}) + g(\bar{Y}\ ,\ {}^M\nabla'_{\bar{X}}\bar{Z})\ ,\ \forall\ \bar{X},\bar{Y},\bar{Z} \in TM\ .$$

Hence ${}^M\nabla'$ and ${}^M\nabla$ are dual w.r.t. g.
Let now be $\tilde{U}\ ,\ \tilde{V}\ ,\ \tilde{W} \in TN$ and $U\ ,\ V\ ,\ W \in \mathcal{L}_V(N)$ their corresponding vertical lifts respectively. We have:

$$\begin{aligned}
\tilde{U}.h(\tilde{V},\tilde{W}) \circ \sigma &= (f \circ \pi)^{-2} U. < V\ ,\ W > \\
&= (f \circ \pi)^{-2} [< D_U V\ ,\ W > + < V\ ,\ D^*_U W >] \\
&= (f \circ \pi)^{-2} \{(f \circ \pi)^2 h(\sigma_*(D_{\tilde{U}}\tilde{V})\ ,\ \tilde{W}) \circ \sigma + (f \circ \pi)^2 h(\tilde{V}\ ,\ \sigma_*(D_{\tilde{U}}\tilde{W})) \circ \sigma\} \\
&= [h({}^N\nabla_{\tilde{U}}\tilde{V}\ ,\ \tilde{W}) + h(\tilde{V}\ ,\ {}^N\nabla'_{\tilde{U}}\tilde{W})] \circ \sigma\ .
\end{aligned}$$

It follows then

$$\tilde{U}.h(\tilde{V},\tilde{W}) = h({}^N\nabla_{\tilde{U}}\tilde{V}\ ,\ \tilde{W}) + h(\tilde{V}\ ,\ {}^N\nabla'_{\tilde{U}}\tilde{W})\ ,\ \forall\ \tilde{U},\tilde{V},\tilde{W} \in TN\ .$$

Hence ${}^N\nabla$ and ${}^N\nabla'$ are dual w.r.t h. □

In this paragraph we construct a dualistic structure on the warped product space from those on its base and fiber manifolds by following the model of the relation between the metric connection of the warped product space and the Levi-Civita connections on its base and fiber manifolds (see [5]).
Let (g,∇,∇^*) and $(h,\tilde\nabla,\tilde\nabla^*)$ be dualistic structures on M and N respectively.
For $X,Y\in\mathcal{L}_H(M)$ and $U,V\in\mathcal{L}_V(N)$ we put:

$(i)\quad D_XY=(\nabla_{\bar X}\bar Y)^H$
$(ii)\quad D_XU=D_UX=\frac{X.f}{f}\,U$
$(iii)\quad D_UV=-\frac{<U\ ,\ V>}{f}\ \mathrm{grad}\,f+(\tilde\nabla_{\tilde U}\tilde V)^V.$

and

$(a)\quad D'_XY=(\nabla^*_{\bar X}\bar Y)^H$
$(b)\quad D'_XU=D'_UX=\frac{X.f}{f}\,U$
$(c)\quad D'_UV=-\frac{<U\ ,\ V>}{f}\ \mathrm{grad}\,f+(\tilde\nabla^*_{\tilde U}\tilde V)^V,$
where we simplify the notation by writing f for $f\circ\pi$ and $\mathrm{grad}\,f$ for $\mathrm{grad}(f\circ\pi)$, and we denote by $<\ ,\ >$ the inner product w.r.t. G_f.
Obviously D and D' define affine connections on $T(M\times N)$. Furthermore:

Proposition 3.2
The triple (G_f,D,D') is a dualistic structure on $M\times N$; that is

$$D'=D^*\ \text{ w.r.t. }\ G_f\ .$$

Proof
Let $X,Y,Z\in\mathcal{L}_H(M)$ and $U,V,W\in\mathcal{L}_V(N)$. We have:

$$\begin{aligned}X.<Y\ ,\ X> &= \bar X.g(\bar Y,\bar Z)\circ\pi\\ &= g(\nabla_{\bar X}\bar Y\ ,\ \bar Z)\circ\pi+g(\bar Y\ ,\ \nabla^*_{\bar X}\bar Y)\circ\pi\\ &= <D_XY\ ,\ Z>+<Y\ ,\ D'_XZ>\ .\end{aligned}$$

$$\begin{aligned}U.<V\ ,\ W> &= f^2\,\tilde U.h(\tilde V,\tilde W)\circ\sigma\\ &= f^2[h(\tilde\nabla_{\tilde U}\tilde V\ ,\ \tilde W)+h(\tilde V\ ,\ \tilde\nabla^*_{\tilde U}\tilde W)]\circ\sigma\\ &= <(\tilde\nabla_{\tilde U}\tilde V)^V\ ,\ W>+<V\ ,\ (\tilde\nabla^*_{\tilde U}\tilde W)^V>\\ &= <D_UV\ ,\ W>+<V\ ,\ D'_UW>\ ,\ \text{since } D_UV=-\frac{<U\ ,\ V>}{f}\ \mathrm{grad}\,f+(\tilde\nabla_{\tilde U}\tilde V)^V\ ,\\ &\quad\ D'_UW=-\frac{<U\ ,\ W>}{f}\ \mathrm{grad}\,f+(\tilde\nabla^*_{\tilde U}\tilde W)^V \text{ and } \mathrm{grad}\,f\in\mathcal{L}_H(M)\ .\end{aligned}$$

$$U.<Y\ ,\ Z>=0=<D_UY\ ,\ Z>+<Y\ ,\ D'_UZ>\ ,\ \text{since }\ <D_UY\ ,\ Z>=<Y\ ,\ D'_UZ>=0\ .$$

$$X.<V\ ,\ Z>=0=<D_XV\ ,\ Z>+<V\ ,\ D'_XZ>\ ,\ \text{since }\ <D_XV\ ,\ Z>=<V\ ,\ D'_XZ>=0\ .$$

$X. < Y\ ,\ W > = 0 = < D_X Y\ ,\ W > + < Y\ ,\ D'_X W >$, since $< D_X Y\ ,\ W > = < Y\ ,\ D'_X W > = 0$.

It follows from the relations above that:

$$A. < B\ ,\ C > = < D_A B\ ,\ C > + < B\ ,\ D'_A C > ,\ \forall\ A, B, C \in T(M \times N)\ .$$

Hence D and D' are dual w.r.t. G_f. □

We call $(G_f, D, D*)$ the dualistic structure on $M \times N$ induced from (g, ∇, ∇^*) on M and $(h, \tilde{\nabla}, \tilde{\nabla}^*)$ on N.
Note that if the connections ∇, ∇^*, $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are symmetric, and therefore torsion-free, then the induced connections D and D' are also symmetric.
From now on all connections are assumed to be symmetric.
Let ${}^gR$, ${}^hR$ and R be the Riemannian curvature operators w.r.t. ∇, $\tilde{\nabla}$ and D respectively. It holds:

Lemma 3.1
For $X, Y, Z \in \mathcal{L}_H(M)$ and $U, V, W \in \mathcal{L}_V(N)$,

(i) $\quad R_{XY} Z = ({}^g R_{\bar{X}\bar{Y}} \bar{Z})^H$.

(ii) $\quad R_{VY} Z = \frac{1}{f} H^f(X, Z) V$.

(iii) $\quad R_{XY} U = R_{VW} Z = 0$.

(iv) $\quad R_{XV} W = \frac{<V\ ,\ W>}{f}\ D_X(\mathrm{grad}\, f)$.

(v) $\quad R_{VW} U = ({}^h R_{\tilde{V}\tilde{W}} \tilde{U})^V - \frac{<\mathrm{grad}\, f\ ,\ \mathrm{grad}\, f>}{f^2}\ (< V\ ,\ U > W - < W\ ,\ U > V)$.

Proof
By straightforward computations as in the proof of proposition 42 , p. 210 in [2], where such relations are proved in the case of metric connections. But excepted the symmetry, the metric property of the connections is not involved in the proof. □

When the warping function f is constant it follows from the previous lemma that the warped product space $(M \times N, G_f, D, D^*)$ is dually flat if and only if (M, g, ∇, ∇^*) and $(N, h, \tilde{\nabla}, \tilde{\nabla}^*)$ are dually flat spaces.
In the case of non-constant warping function f we get:

Corollary 3.1
Assume M to be connected and let $f \in C^\infty(M)$ be a non-constant positive function on M.
If $(M \times N, G_f, D, D^*)$ is a dually flat space then,
(M, g, ∇, ∇^*) is also dually flat and (N, h) is a Riemannian manifold of constant sectional curvature.

Proof
Assume $(M \times N, G_f, D, D^*)$ to be dually flat.
Then from the relation (i) of Lemma 3.1 we have

$$^g R \equiv 0 \ .$$

Thus the connection ∇ is flat, and since connections are assumed to be symmetric, its dual ∇^* is also flat. Hence (M, g, ∇, ∇^*) is dually flat.
For $X \in \mathcal{L}_H(M)$ and $V, W \in \mathcal{L}_V(N)$,

$$R_{XV}W = \frac{< V \ , \ W >}{f} \ D_X(\operatorname{grad} f) = 0 \ , \ \text{by the relation } (iv) \text{ of Lemma 3.1} \ .$$

So we get

$$D_X(\operatorname{grad} f) = 0 \ , \ \forall \ X \in \mathcal{L}_H(M) \ .$$

Thus $\operatorname{grad} f$ is a constant vector field since M is assumed to be connected.
Moreover for $U, V, W \in \mathcal{L}_V(N)$, we have by the relation (v) of lemma 3.1:

$$R_{VW}U = (^h R_{\tilde{V}\tilde{W}}\tilde{U})^V - \frac{< \operatorname{grad} f \ , \ \operatorname{grad} f >}{f^2} \ (< V \ , \ U > W - < W \ , \ U > V) = 0 \ .$$

Hence

$$^h R_{\tilde{V}\tilde{W}}\tilde{U} = \| \operatorname{grad} f \|^2 \{ h(\tilde{V} \ , \ \tilde{U})\tilde{W} - h(\tilde{W} \ , \ \tilde{U})\tilde{V} \} \ , \ \forall \ \tilde{U}, \tilde{V}, \tilde{W} \in TN \ .$$

Since $\| \operatorname{grad} f \|^2$ is constant, it follows from the previous equality that (N, h) has a constant sectional curvature. □

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