On weak symmetries of almost $r-$paracontact Riemannian manifold of $P-$Sasakian type

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Abstract. In this paper, we consider weakly symmetric and weakly Ricci-symmetric almost $r-$paracontact Riemannian manifolds of $P-$Sasakian type. We find necessary conditions in order that an almost $r-$paracontact Riemannian manifold of $P-$Sasakian type be weakly symmetric and weakly Ricci-symmetric.

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§ 1. Introduction

Weakly symmetric Riemannian manifolds are generalizations of locally symmetric manifolds and pseudo-symmetric manifolds. These are manifolds in which the covariant derivative DR of the curvature tensor R is a linear expression in R. The appearing coefficients of this expression are called associated $1-$forms. They satisfy in the specified types of manifolds gradually weaker conditions.

Firstly, the notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamássy and T.Q. Binh in 1992 (see [4], [5]). In [4], the authors considered weakly symmetric and weakly projective-symmetric Riemannian manifolds. In 1993, the authors considered weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds [5]. In 2000, U. C. De, T.Q. Binh and A.A. Shaikh gave necessary conditions for the compatibility of several $K-$contact structures with weak symmetry and weakly Ricci-symmetry [1]. In 2002, C. Özgür, considered weakly symmetric and weakly Ricci-symmetric Lorentzian para-Sasakian manifolds [6]. Recently in [7], C. Özgür studied weakly symmetric Kenmotsu manifolds.

In this study, we consider weakly symmetric and weakly Ricci-symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type.

§ 2. Preliminaries

Let (M^n, g) be an $n-$dimensional Riemann manifold. We denote by D the covariant differentiation with respect to the Riemann metric g. Then we have

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.$$

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The Riemannian curvature tensor is defined by

$$R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right).$$

The Ricci tensor of M is defined by

$$Ric(X,Y) = trace\left\{Z \to R(X,Z)Y\right\}.$$

Locally, Ric is given by

$$Ric(X,Y) = \sum_{i=1}^{n} R(X,E_i,Y,E_i),$$

where $\{E_1, E_2, ..., E_n\}$ is a local orthonormal frames field on M and X, Y, Z, W are vector fields on M.

The Ricci operator Q is a tensor field of type $(1,1)$ on M^n defined by

$$g(QX,Y) = Ric(X,Y),$$

for all vector field on M^n.

A non-flat differentiable manifold (M^n, g), $(n > 2)$, is called weakly symmetric if there exist $1-$forms α, β, γ, δ and σ on M such that

$$(2.1) \qquad \begin{aligned}(D_X R)(Y,Z,U,V) = \ & \alpha(X)R(Y,Z,U,V) + \beta(Y)R(X,Z,U,V) \\ & +\gamma(Z)R(Y,X,U,V) + \delta(U)R(Y,Z,X,V) \\ & +\sigma(V)R(Y,Z,U,X)\end{aligned}$$

holds for any vector fields X, Y, Z, U, V on M. A weakly symmetric manifold is said to be *proper* if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case [1].

A differentiable manifold (M^n, g), $(n > 2)$, is called *weakly Ricci-symmetric* if there exist $1-$forms ρ, μ, ν such that the relation

$$(2.2) \qquad \left(D_X Ric\right)(Y,Z) = \rho(X)Ric(Y,Z) + \mu(Y)Ric(X,Z) + \upsilon(Z)Ric(X,Y)$$

holds for all vector fields X, Y, Z, U, V on M. A weakly Ricci-symmetric manifold is said to be proper if $\rho = \mu = \nu = 0$ is not the case [1].

From (2.1), an easy calculation shows that if M is weakly symmetric then we obtain (see [4], [5])

$$(2.3) \qquad \begin{aligned}\left(D_X Ric\right)(Z,U) = \ & \alpha(X)Ric(Z,U) + \beta(Z)Ric(X,U) + \delta(U)Ric(Z,X \\ & +\beta(R(X,Z)U) + \delta(R(X,U)Z).\end{aligned}$$

§ 3. Almost $r-$paracontact Riemannian manifolds

We need the following definition ([3])

Let (M, g) be a Riemannian manifold with $dim(M) = 2n+r$ and denote by $T(M)$ the tangent space of M. Then M is said to be an almost $r-$paracontact Riemannian manifold if there exist on M a tensor field ϕ of type $(1,1)$ and r global vector fields $\xi_1, ..., \xi_s$ (called structure vector fields) such that

(i) If $\eta_1, ..., \eta_r$ are dual $1-$forms of $\xi_1, ..., \xi_r$,then:

$$\eta_i(\xi_j) = \delta^i_j, \quad g(\xi_i, X) = \eta_i(X) \quad \phi^2 = I - \sum_{i=1}^{r} \xi_i \otimes \eta_i \tag{3.4}$$

(ii)

$$g(\phi X, \phi Y) = g(X,Y) - \sum_{i=1}^{r} \eta_i(X)\eta_i(Y), \tag{3.5}$$

for any $X, Y \in T(M)$, $i = 1, ..., r$.

In an almost $r-$paracontact Riemannian manifold M, besides the relations (3.4) and(3.5) the following also hold

$$\phi\xi_\alpha = 0 \tag{3.6}$$

$$\eta^a \circ \phi = 0. \tag{3.7}$$

An almost $r-$paracontact Riemannian manifold M is said to be of $P-$Sasakian type if

$$D_X \xi_i = \phi X \tag{3.8}$$

$$(D_X\phi)Y = -\sum_{i=1}^{r} \left[g(\phi X, \phi Y)\xi_i + \eta_i(Y)\phi^2 X\right] \tag{3.9}$$

for all $X, Y \in T(M)$.

In an almost $r-$paracontact Riemannian manifold of $P-$Sasakian type M, the following hold

$$Ric(\xi_i, X) = -2n \sum_{\beta=1}^{r} \eta_\beta(X) \tag{3.10}$$

$$R(\xi_i, X)\xi_\beta = X - \sum_{\gamma=1}^{r} \eta^\gamma(X)\xi_\gamma \tag{3.11}$$

$$g(R(\xi_i, X)Y, \xi_\beta) = -g(X,Y) + \sum_{\gamma=1}^{r} \eta_\gamma(X)\eta_\gamma(Y) \tag{3.12}$$

for any vector fields $X, Y \in T(M)$.

Since ϕ and the Ricci operator Q are symmetric in an almost $r-$paracontact Riemannian manifold of $P-$Sasakian type M, $Q\phi + \phi Q = 0$ and the Lie derivative of Ric vanishes, i.e.

$$L_{\xi_\alpha} Ric = 0, \tag{3.13}$$

for any $\alpha = 1, ..., r$.

§ 4. Weakly symmetric almost $r-$paracontact Riemannian manifolds of $P-$Sasakian type

In this chapter we investigate weakly symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type. We assume that the weakly symmetric manifold is almost $r-$paracontact Riemannian manifold of $P-$Sasakian type. Then we have,

Theorem 1. *Any weakly symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type M satisfies $\alpha+\delta+\beta=0$.*

Proof. Since the manifold is weakly symmetric, from (2.3), by putting $X=\xi_\alpha$ yields

$$\begin{aligned}(D_{\xi_\alpha}Ric)(Z,U) &= \alpha(\xi_\alpha)Ric(Z,U)+\beta(Z)Ric(\xi_\alpha,U)\\ &\quad+\delta(U)Ric(Z,\xi_\alpha)+\beta(R(\xi_\alpha,Z)U)+\delta(R(\xi_\alpha,U)Z)\end{aligned} \tag{4.14}$$

From (3.13), it follows that

$$(D_{\xi_\alpha}Ric)(Z,U)=-Ric(D_Z\xi_\alpha,U)-Ric(Z,D_U\xi_\alpha).$$

By virtue of (3.8), we get from

$$(D_{\xi_\alpha}Ric)(Z,U)=-Ric(\phi Z,U)-Ric(Z,\phi U).$$

Now, since ϕ is skew symmetric and Ricci operator is symmetric, we obtain

$$(D_{\xi_\alpha}Ric)(Z,U)=0. \tag{4.15}$$

From (4.14) and (4.15), we have

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(Z,U)+\beta(Z)Ric(\xi_\alpha,U)\\ &+\delta(U)Ric(Z,\xi_\alpha)+\beta(R(\xi_\alpha,Z)U)+\delta(R(\xi_\alpha,U)Z)=0\end{aligned} \tag{4.16}$$

Putting $Z=U=\xi_\alpha$ in(4.16), we get

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(\xi_\alpha,\xi_\alpha)+\beta(\xi_\alpha)Ric(\xi_\alpha,\xi_\alpha)\\ &+\delta(\xi_\alpha)Ric(\xi_\alpha,\xi_\alpha)+\beta(R(\xi_\alpha,\xi_\alpha)\xi_\alpha)+\delta(R(\xi_\alpha,\xi_\alpha)\xi_\alpha)=0.\end{aligned} \tag{4.17}$$

And using (3.11), we have

$$(\alpha(\xi_\alpha)+\beta(\xi_\alpha)+\delta(\xi_\alpha))\,Ric(\xi_\alpha,\xi_\alpha)=0.$$

which gives us

$$\alpha(\xi_\alpha)+\beta(\xi_\alpha)+\delta(\xi_\alpha)=0. \tag{4.18}$$

So the vanishing of the 1-form $\alpha+\beta+\delta$ over the vector field ξ_α is necessary in order that M be a almost $r-$paracontact Riemannian manifold of $P-$Sasakian type.

Now we will show that $\alpha+\beta+\delta=0$ holds for all vector fields on M.

In (2.3), taking $X=Z=\xi_\alpha$, we get

$$\begin{aligned}(D_{\xi_\alpha}Ric)(\xi_\alpha,U) &= \alpha(\xi_\alpha)Ric(\xi_\alpha,U)+\beta(\xi_\alpha)Ric(\xi_\alpha,U)\\ &\quad+\delta(U)Ric(\xi_\alpha,\xi_\alpha)+\beta(R(\xi_\alpha,\xi_\alpha)U)\\ &\quad+\delta(R(\xi_\alpha,U)\xi_\alpha).\end{aligned} \tag{4.19}$$

From (4.19) and (3.11), We get

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(\xi_\alpha,U)+\beta(\xi_\alpha)Ric(\xi_\alpha,U)\\&+\delta(U)Ric(\xi_\alpha,\xi_\alpha)+\delta(R(\xi_\alpha,U)\xi_\alpha)=0\end{aligned}$$

Replacing U by X in (4.19) we have

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(\xi_\alpha,X)+\beta(\xi_\alpha)Ric(\xi_\alpha,X)\\&+\delta(X)Ric(\xi_\alpha,\xi_\alpha)+\delta(R(\xi_\alpha,X)\xi_\alpha)=0.\end{aligned}\tag{4.20}$$

In (2.3), taking $X=U=\xi_\alpha$, we get

$$\begin{aligned}(D_{\xi_\alpha}Ric)(Z,\xi_\alpha) &= \alpha(\xi_\alpha)Ric(Z,\xi_\alpha)+\beta(Z)Ric(\xi_\alpha,\xi_\alpha)\\&+\delta(\xi_\alpha)Ric(Z,\xi_\alpha)+\beta(R(\xi_\alpha,Z)\xi_\alpha)+\delta(R(\xi_\alpha,\xi_\alpha)Z).\end{aligned}$$

From (4.15) and (3.11), We get

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(Z,\xi_\alpha)+\beta(Z)Ric(\xi_\alpha,\xi_\alpha)\\&+\delta(\xi_\alpha)Ric(Z,\xi_\alpha)+\beta(R(\xi_\alpha,Z)\xi_\alpha)=0.\end{aligned}\tag{4.21}$$

Replacing Z by X in (4.21) we have

$$\begin{aligned}&\alpha(\xi_\alpha)Ric(X,\xi_\alpha)+\beta(X)Ric(\xi_\alpha,\xi_\alpha)\\&+\delta(\xi_\alpha)Ric(X,\xi_\alpha)+\beta(R(\xi_\alpha,X)\xi_\alpha)=0.\end{aligned}\tag{4.22}$$

In (2.3), taking $Z=U=\xi_\alpha$, we get

$$\begin{aligned}(D_XRic)(\xi_\alpha,\xi_\alpha) &= \alpha(X)Ric(\xi_\alpha,\xi_\alpha)+\beta(\xi_\alpha)Ric(X,\xi_\alpha)\\&+\delta(\xi_\alpha)Ric(\xi_\alpha,X)+\beta(R(X,\xi_\alpha)\xi_\alpha)\\&+\delta(R(X,\xi_\alpha)\xi_\alpha).\end{aligned}\tag{4.23}$$

We also have

$$(D_XRic)(\xi_\alpha,\xi_\alpha)=0\tag{4.24}$$

Using (4.24) in (4.23), we have

$$\begin{aligned}&\alpha(X)Ric(\xi_\alpha,\xi_\alpha)+\beta(\xi_\alpha)Ric(X,\xi_\alpha)+\delta(\xi_\alpha)Ric(\xi_\alpha,X)\\&+\beta(R(X,\xi_\alpha)\xi_\alpha)+\delta(R(X,\xi_\alpha)\xi_\alpha)=0.\end{aligned}\tag{4.25}$$

Adding (4.20), (4.22) and (4.25), we obtain

$$\begin{aligned}&2\left(\alpha(\xi_\alpha)+\beta(\xi_\alpha)+\delta(\xi_\alpha)\right)Ric(\xi_\alpha,X)\\&+\left(\alpha(X)+\delta(X)+\beta(X)\right)Ric(\xi_\alpha,\xi_\alpha)\\&+\delta(R(\xi_\alpha,X)\xi_\alpha)+\beta(R(\xi_\alpha,X)\xi_\alpha)\\&+\beta(R(X,\xi_\alpha)\xi_\alpha)+\delta(R(X,\xi_\alpha)\xi_\alpha)=0.\end{aligned}\tag{4.26}$$

Using (3.11) and (4.18) in (4.26) we have

$$\left(\alpha(X)+\delta(X)+\beta(X)\right)Ric(\xi_\alpha,\xi_\alpha)=0\tag{4.27}$$

Hence from (4.27), we obtain

$$\alpha(X) + \delta(X) + \beta(X) = 0 \quad \text{for all } X. \tag{4.28}$$

Thus

$$\alpha + \delta + \beta = 0.$$

Our theorem is thus proved.

§ 5. Weakly Ricci-symmetric almost $r-$paracontact Riemannian manifolds of $P-$Sasakian type

In this chapter we investigate weakly Ricci-symmetric almost $r-$paracontact Riemannian manifolds of $P-$Sasakian type. We suppose that the considered weakly Ricci-symmetric manifold is almost $r-$paracontact Riemannian manifold of $P-$Sasakian type. We have,

Theorem 2. *Any weakly Ricci-symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type M, satisfies $\rho + \mu + \upsilon = 0$.*

Proof. Suppose that M is a weakly Ricci-symmetric almost $r-$paracontact Riemannian manifold of $P-$Sasakian type. Replacing X with ξ_α in (2.2) we have

$$\left(D_{\xi_\alpha} Ric\right)(Y,Z) = \rho(\xi_\alpha) Ric(Y,Z) + \mu(Y) Ric(\xi_\alpha, Z) + \upsilon(Z) Ric(\xi_\alpha, Y). \tag{5.29}$$

By virtue of (4.15) and (5.29), we have

$$\rho(\xi_\alpha) Ric(Y,Z) + \mu(Y) Ric(\xi_\alpha, Z) + \upsilon(Z) Ric(\xi_\alpha, Y) = 0. \tag{5.30}$$

Putting $Y = Z = \xi_\alpha$ in (5.30), we get

$$\left(\rho(\xi_\alpha) + \mu(\xi_\alpha) + \upsilon(\xi_\alpha)\right) Ric(\xi_\alpha, \xi_\alpha) = 0, \tag{5.31}$$

which gives

$$\rho(\xi_\alpha) + \mu(\xi_\alpha) + \upsilon(\xi_\alpha) = 0, \tag{5.32}$$

In (2.2), taking $X = Y = \xi_\alpha$, and using (4.15), we get

$$\begin{aligned} \left(D_{\xi_\alpha} Ric\right)(\xi_\alpha, Z) = \rho(\xi_\alpha) Ric(\xi_\alpha, Z) + \mu(\xi_\alpha) Ric(\xi_\alpha, Z) \\ + \upsilon(Z) Ric(\xi_\alpha, \xi_\alpha) = 0. \end{aligned} \tag{5.33}$$

Replacing Z by X in (5.33) we have

$$\begin{aligned} \left(D_{\xi_\alpha} Ric\right)(\xi_\alpha, X) = \rho(\xi_\alpha) Ric(\xi_\alpha, X) + \mu(\xi_\alpha) Ric(\xi_\alpha, X) \\ + \upsilon(X) Ric(\xi_\alpha, \xi_\alpha) = 0. \end{aligned} \tag{5.34}$$

In (2.2), taking $X = Z = \xi_\alpha$, and using (4.15), we get

$$\begin{aligned} \left(D_{\xi_\alpha} Ric\right)(Y, \xi_\alpha) = \rho(\xi_\alpha) Ric(Y, \xi_\alpha) + \mu(Y) Ric(\xi_\alpha, \xi_\alpha) \\ + \upsilon(\xi_\alpha) Ric(\xi_\alpha, Y) = 0. \end{aligned} \tag{5.35}$$

Replacing Y by X in (5.35) we have

$$(D_{\xi_\alpha}Ric)(X,\xi_\alpha) = \rho(\xi_\alpha)Ric(X,\xi_\alpha) + \mu(X)Ric(\xi_\alpha,\xi_\alpha) + \upsilon(\xi_\alpha)Ric(\xi_\alpha,X) = 0. \tag{5.36}$$

In (2.2), taking $Y = Z = \xi_\alpha$, and using (4.24), we get

$$(D_X Ric)(\xi_\alpha,\xi_\alpha) = \rho(X)Ric(\xi_\alpha,\xi_\alpha) + \mu(\xi_\alpha)Ric(X,\xi_\alpha) + \upsilon(\xi_\alpha)Ric(X,\xi_\alpha) = 0. \tag{5.37}$$

Adding (5.34), (5.36) and (5.37) and then using (4.18), we obtain

$$2(\rho(\xi_\alpha) + \mu(\xi_\alpha) + \upsilon(\xi_\alpha))Ric(\xi_\alpha,X) + (\rho(X) + \mu(X) + \upsilon(X))\,Ric(\xi_\alpha,\xi_\alpha) = 0$$

From this, it follows that

$$(\rho(X) + \mu(X) + \upsilon(X))Ric(\xi_\alpha,\xi_\alpha) = 0. \tag{5.38}$$

Hence from (5.38), we have,

$$\rho(X) + \mu(X) + \upsilon(X) = 0, \text{ for all } X.$$

Thus

$$\rho + \mu + \upsilon = 0.$$

Hence our theorem is proved.

References

[1] U. C. De, T.Q. Binh and A.A. Shaikh, *On weakly symmetric and weakly Ricci-symmetric K-contact manifolds*, Acta Mathematica Acadeiae Paedagogicae Nyíregyháziensis, 16 (2000), 65-71.

[2] D. E., Blair, *Contact manifold in Riemannian Geometry*, Lecture Notes in Mathematics, 509, Springer Verlag, 1976.

[3] A. Bucki, *Representation of the Lie group of automorphisms of an almost $r-$paracontact Riemannian manifold of $P-$ Sasakian type*, Differential Geom. and Applications, Proceedings of the 6th international conference, Brno, Czech Republic, August 28–September 1, 1995. Brno: Masaryk University 1996, 19-28.

[4] L. Tamássy and T.Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc. J. Bolyai, 56 (1992), 663-670.

[5] L. Tamássy and T.Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor N.S. 53 (1993), 140-148.

[6] C. Özgür, *On weak symmetries of Lorentzian para Sasakian manifolds*, Radovi Matamatički 11 (2002), 263-270.

[7] C. Özgür, *On weakly symmetric Kenmotsu manifolds*, Differ. Geom. Dyn. Syst. 8 (2006), 204–209.

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