Closed hyperelastic slant curves in the complex projective plane

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Abstract. We study slant curves in $\mathbb{CP}^2(4)$ which are critical points of the generalized elastic energy. In particular, we classify closed hyperelastic proper slant curves in $\mathbb{CP}^2(4)$ and show that they form a one-parameter family of helices.

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§ 1. Introduction

Regular curves in Riemannian manifolds which are critical points of the elastic energy $\int_\gamma \kappa^2$ (κ being the curvature of the curve γ) have been intensively investigated under various points of view. If the Riemannian manifold is $\mathbb{R}^3$ they can be used also to model stiff rods, stiff polymers, vortices in fluids, superconductors, membranes and mechanical properties of DNA molecules (for more details see [14], [15] and the references therein).

In a more general setting, we consider the variational problem associated with

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^{2m}\,, \tag{1.1}$$

where $\gamma : I \to \mathbb{M}^n$ is a curve (satisfying suitable boundary conditions) immersed in a Riemannian manifold $\mathbb{M}^n$, κ denotes its geodesic curvature and m is a natural number. From now on, critical points of (1.1) will be refer to as *free hyperelastic curves*, or simply, as $2m$-*elastic curves*. As we shall see, geodesics are trivial examples of $2m$-elastic curves.

Critical points of (1.1) have strong connections with higher dimensional variational problems in the theory of submanifolds and string theories. In this respect, we should mention that $2m$-*elastic curves* in spheres and hyperbolic spaces have been employed to provide reduction methods in constructing Chen-Willmore submanifolds [3, 2, 6, 7] and to study related conformal string theories [5]. The role of compactness in Chen-Willmore submanifolds and the fundamental importance of closed critical curves in Geometry and Relativity, makes boundary conditions leading to closed $2m$-elastic curves a natural choice.

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Assume first that $\mathbb{M}^n(c)$ is a real space form of constant sectional curvature c. Then it can be proved that $2m$-elastic curves must lie in a totally geodesic 3-dimensional submanifold of $\mathbb{M}^n(c)$. If $m = 1$ critical points of (1.1) correspond to the model for classical elasticae proposed by D. Bernoulli around 1740 which have been widely studied (see for instance [4], [9], [17], [16], [18], [13]). In particular, if $\mathbb{M}^n(c) = \mathbb{R}^2$ their possible shapes where discovered by L. Euler. There are no closed plane free elasticae. Closed classical elasticae in 2-dimensional round spheres and in the hyperbolic plane, have been classified in [17]. Closed classical elastic curves in $\mathbb{R}^3$ and $\mathbb{S}^3$ have been analyzed in [16] and [4], respectively.

If m is greater than 1, explicit determination of solutions and their classification presents serious difficulties even in real space forms. Moreover, existence of closed non-constant curvature $2m$-elastic curves should not be taken for granted. In this respect it is a surprising fact that if $m \in \mathbb{N} - \{1\}$, there are no closed $2m$-elastic curves in 2-dimensional spheres other than geodesics (for details see [4]). On the contrary, for any given natural number m there are closed $2m$-elastic non-constant curvature curves in the hyperbolic plane $\mathbb{H}^2(-1)$ [1, 2].

Hyperelastic curves in real space forms are reasonably well understood because they are highly symmetric spaces. On the contrary, very little is known about $2m$-elastic curves in Riemannian manifolds of non-constant sectional curvature. Classical $2m$-elastic curves ($m = 1$) in $\mathbb{CP}^2(4)$ with a penalty on the length can be used in combination with the Palais' symmetric criticality principle and the conformal invariance of the Chen-Willmore functional [12] to construct Chen-Willmore tori in the 5-sphere. This study was made by M. Barros, D. Singer and the second author in [8]. We adapt the method that was introduced there to investigate closed free $2m$-elastic curves in $\mathbb{CP}^2(4)$.

Slant submanifolds in Kälher manifolds where first studied by B-Y. Chen in [11] (for more information, see [10]). A curve γ in $\mathbb{CP}^2(4)$ is said to be a *slant curve*, if the angle between the complex tangent plane and the osculating plane of γ is constant along the curve. In particular, curves with osculating plane either holomorphic or Lagrangian are slant curves.

In this paper we classify the closed $2m$-elastic slant curves in $\mathbb{CP}^2(4)$. We shall prove that we have two alternatives (see details in §):

(1) They have zero Frenet torsion. *Then $m = 1$ and they are classical elasticae living in either a totally geodesic, holomorphic surface $\mathbb{S}^2(4)$ in $\mathbb{CP}^2(4)$ (this case corresponds to slant angle zero), or living in a totally geodesic, Lagrangian surface $\mathbb{RP}^2(1)$ in $\mathbb{CP}^2(4)$ (this case corresponds to slant angle $\frac{\pi}{2}$)*

(2) They have non-zero Frenet torsion. *Then for every $m \geq 1$ they form a 1-parameter family of $2m$-elastic Frenet helices living fully in $\mathbb{CP}^2(4)$.* This family is described in Theorem 3.7.

§ 2. Slant curves in $\mathbb{CP}^2(4)$

Let $\mathbb{CP}^2(4)$ be the 2-dimensional complex projective space of constant holomorphic sectional curvature 4, endowed with complex structure J, Fubini-Study metric $\langle , \rangle$ and Levi-Civita connection ∇. Given a curve $\gamma\colon [0,1] \to \mathbb{CP}^2(4)$ smoothly immersed in $\mathbb{CP}^2(4)$, let $\{T(s), \xi_2(s), \xi_3(s), \xi_4(s)\}$ denote the Frenet frame along $\gamma(s)$, where s is the arclength parameter and $T(s) = \gamma'(s)$ is the unit tangent vector field. The standard Frenet equations of $\gamma(s)$ are

$$\begin{aligned} \nabla_T T &= \kappa\xi_2\,, & \nabla_T\xi_2 &= -\,\kappa T + \tau\xi_3\,, \\ \nabla_T\xi_3 &= -\,\tau\xi_2 + \delta\xi_4\,, & \nabla_T\xi_4 &= -\,\delta\xi_3\,. \end{aligned} \tag{2.2}$$

These equations define the Frenet curvatures $\{\kappa(s),\, \tau(s),\, \delta(s)\}$. As usual, the second Frenet curvature $\tau(s)$ is referred to as *torsion.* A curve $\gamma(s)$ of $\mathbb{CP}^2(4)$ is said to be a *Frenet helix* (or and W-curve) if it has constant Frenet curvatures $\{\kappa(s), \tau(s), \delta(s)\}$. By using the notation of (2.2) we can write

$$JT = A_2\xi_2 + A_3\xi_3 + A_4\xi_4\,, \tag{2.3}$$

where $A_i(s) = \cos\phi_i(s) = \langle JT, \xi_i\rangle$; $0 \le \phi_i \le \frac{\pi}{2}$; $2 \le i \le 4$ and $\sum_{i=2}^{i=4} A_i^2 = 1$.

On the other hand, it shall be useful for our purposes to consider what we shall call a *Complex Frenet reference frame* along the curve $\gamma(s)$. This can described as follows (for details see [8]). Let us denote by $\Pi : \mathbb{S}^5(1) \to \mathbb{CP}^2(4)$ the Hopf map and let $\pi : \mathbb{SU}(3) \to \mathbb{CP}^2(4)$ be the canonical projection, that is $\pi(\mathtt{H}) = \Pi(\mathtt{H} \times \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right))$, where $\times$ represents the matrix product. First, we lift the curve $\gamma(s)$ to a horizontal curve $Y(s)$ in $\mathbb{S}^5(1)$ via Π. Different horizontal lifts differ by a factor of the form e^{ri}, r being a constant. Since Π is a Riemannian submersion, the tangent vector $T(s)$ lifts to a unit vector $\bar{T}(s) = Y'(s)$. Now we may uniquely choose a vector U orthogonal to T so that its horizontal lift $\bar{U}$ gives the third vector in a special unitary frame $\sigma(s) = \{Y, \bar{T}, \bar{U}\}$ in $\mathbb{C}^3$. Hence, $\sigma(s)$ is a lifting of the curve $\gamma(s)$ to a curve in $\mathbb{SU}(3)$. Since the curve $Y(s)$ is horizontal, one can check by direct computation that $\sigma(s)$ satisfies the following differential equation:

$$\frac{d\sigma(s)}{ds} = \sigma(s) \times \mathcal{A}(s)\,, \tag{2.4}$$

where $\mathcal{A}(s)$ is a matrix in $\mathfrak{su}(3)$ given by

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & \kappa_1 i & -\kappa_2 + \kappa_3 i \\ 0 & \kappa_2 + \kappa_3 i & -\kappa_1 i. \end{pmatrix},$$

and κ_i, $i = 1, 2, 3$ are smooth functions. By projecting down (2.4) via Π, we obtain a new frame $\{\Pi_*(\bar{T}) = T,\ \Pi_*(i\bar{T}) = JT,\ \Pi_*(\bar{U}) = U,\ \Pi_*(i\bar{U}) = JU\}$ along the curve $\gamma(s)$ satisfying:

$$\nabla_T T = \kappa_1 JT + \kappa_2 U + \kappa_3 JU\,, \tag{2.5}$$

$$\nabla_T JT = -\kappa_1 T - \kappa_3 U + \kappa_2 JU\,, \tag{2.6}$$

$$\nabla_T U = -\kappa_2 T + \kappa_3 JT - \kappa_1 JU\,, \tag{2.7}$$

$$\nabla_T JU = -\kappa_3 T - \kappa_2 JT + \kappa_1 U. \tag{2.8}$$

One can find some relations between the Frenet curvatures and the new "curvatures" $\kappa_i(s)$, $i = 1, 2, 3.$ associated to the *Complex Frenet frame.* From (2.2) and (2.5) we have

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \kappa^2\,, \tag{2.9}$$

$$\kappa A_2 = \kappa\cos\phi_2 = \kappa_1\,, \tag{2.10}$$

Moreover, using (2.2) and (2.5) and comparing both expressions for $\nabla_T^2 T$ one has

$$(\kappa_1')^2 + (\kappa_2')^2 + (\kappa_3')^2 = (\kappa')^2 + \kappa^2\tau^2 \,, \tag{2.11}$$

$$\kappa' A_2 + \kappa\tau A_3 = \kappa_1'. \tag{2.12}$$

Here $()'$ denotes derivative with respect to the arclength parameter s. Using (2.9) we can write $\kappa_i(s) = \varepsilon_i(s)\kappa(s)$, $i = 1, 2, 3$ where $\varepsilon_i(s)$, $i = 1, 2, 3$ are functions satisfying $\sum_{i=1}^3 \varepsilon_i^2 = 1$. Thus, we get from (2.11)

$$\tau^2(s) = \sum_{i=1}^{3} (\varepsilon_i'(s))^2. \tag{2.13}$$

Therefore, (2.11) and (2.13) give the following

Lemma 2.1. *Let $\gamma(s)$ a smooth immersed curve in $\mathbb{CP}^2(4)$, then*

1. *$\gamma(s)$ has constant "curvatures" $\kappa_i(s)\, i = 1, 2, 3$, if and only if, it is a curve with constant curvature $\kappa(s)$ and zero torsion $\tau(s)$.*
2. *The torsion $\tau(s)$ of $\gamma(s)$ vanishes, if and only if, there exist three real constant $\varepsilon_i \in \mathbb{R}$, $i = 1, 2, 3$, satisfying $\sum_{i=1}^3 \varepsilon_i^2 = 1$ such that $\kappa_i(s) = \varepsilon_i\kappa(s)$, $i = 1, 2, 3$.*

A curve $\gamma\colon [0, 1] \to \mathbb{CP}^2(4)$ is said to be a *slant curve* if $A_2 = \langle JT, \xi_2\rangle = \cos\phi_2$ is constant along $\gamma(s)$, that is if the angle between the complex tangent plane and the osculating plane is constant along $\gamma(s)$. A slant curve is *proper* if $A_2 \in (0, 1)$. The following proposition give us some properties of slant curves that will be useful later

Proposition 2.2. *Let $\gamma(s)$ be a non-geodesic curve immersed in $\mathbb{CP}^2(4)$. Let us denote by $\{T(s), \xi_2(s), \xi_3(s), \xi_4(s)\}$ and by $\{\kappa(s),\, \tau(s),\, \delta(s)\}$ the Frenet frame and Frenet curvatures along $\gamma(s)$ respectively and write $JT(s)$ as in (2.3). Then $\gamma(s)$ is a slant curve in $\mathbb{CP}^2(4)$, if and only if, either it has vanishing torsion or $A_3 = 0$. Assuming that $\gamma(s)$ is a slant curve (i.e. A_2 is constant), we have*

1. *if $\tau(s) = 0$, then there are two real constants ε_2, ε_3 satisfying $\varepsilon_2^2 + \varepsilon_3^2 = 1 - A_2^2$, such that $\kappa_1(s) = A_2\kappa(s)$, $\kappa_2(s) = \varepsilon_2\kappa(s)$, and $\kappa_3(s) = \varepsilon_3\kappa(s)$.*
2. *if $\tau(s) \neq 0$, then*
   - *(i) $A_2 \in [0, 1)$,*
   - *(ii) $\langle J\xi_3, \xi_4\rangle = \langle JT, \xi_2\rangle = A_2$,*
   - *(iii) $\langle J\xi_2, \xi_4\rangle = \langle JT, \xi_3\rangle = A_3 = 0$, and*
   - *(iv) $\langle J\xi_2, \xi_3\rangle = \langle JT, \xi_4\rangle = A_4 = \sqrt{1 - A_2^2}\,$,*
   - *(v) $(\kappa(s) + \delta(s))\sqrt{1 - A_2^2} = A_2\tau(s)\,$.*

Proof. Given an immersed non-geodesic curve $\gamma(s)$ in $\mathbb{CP}^2(4)$, we have from (2.10) and (2.12) that $A_2' = \tau(s)A_3$. This proves the first assertion. Now, assume that $\gamma(s)$ is a slant curve of $\mathbb{CP}^2(4)$, then either $\tau(s) = 0$ or $A_3 = 0$. If $\tau(s) = 0$, we get (1) from the second part of Lemma 2.1. If $\tau(s) \neq 0$, then, we obtain $A_3 = \langle JT, \xi_3\rangle = 0$ and $A_2^2 + A_4^2 = 1$. If $A_4 = 0$, then $JT = \xi_2$, but using the Frenet equations we have that this is impossible since $\tau(s) \neq 0$. Thus, we may assume that $A_4 > 0$ and therefore $A_4 = \sqrt{1 - A_2^2}$ is also constant and $A_2 \in [0, 1)$. From $A_3 = \langle JT, \xi_3 \rangle = 0$ and (2.2) we get

$$-\tau A_2 + \delta A_4 + \kappa \langle J\xi_2, \xi_3 \rangle = 0 \,. \tag{2.14}$$

Since A_4 is constant and $A_3 = 0$, Differentiating $A_4 = \langle JT, \xi_4 \rangle$, we obtain $\langle J\xi_2, \xi_4 \rangle = 0$. Differentiating this identity again one has

$$-\kappa A_4 + \tau \langle J\xi_3, \xi_4 \rangle - \delta \langle J\xi_2, \xi_3 \rangle = 0 \,. \tag{2.15}$$

Since $\langle J\xi_3, JT \rangle = A_4 \langle J\xi_3, \xi_4 \rangle - A_2 \langle J\xi_2, \xi_3 \rangle$, we can combine (2.14) and (2.15) to see that $\langle J\xi_3, \xi_2 \rangle^2 = A_4^2$, $\langle J\xi_3, \xi_4 \rangle^2 = A_2^2$. Actually one has that $\langle J\xi_3, \xi_4 \rangle = A_2$ and $\langle J\xi_2, \xi_3 \rangle = A_4$. □

Remark 2.3. *Notice that (2.v) of Proposition 2.2 implies that a slant curve $\gamma(s)$ with non-zero torsion is a sort of generalized helix and that it is a Frenet helix, if and only if, any two of its Frenet curvatures are constant.*

Assume now that $\gamma(s)$ is a proper slant Frenet helix. Let us denote ϕ_2 simply by ϕ. It is clear from (2.9) and (2.10) that we can find a function $\psi(s)$ such that

$$\kappa_1 = \kappa \cos\phi \,, \quad \kappa_2 = \kappa \sin\phi \cos\psi(s) \,, \quad \kappa_3 = \kappa \sin\phi \sin\psi(s) \,. \tag{2.16}$$

We put $\omega = \psi'$, then it is immediate from (2.11) and (2.16) that $\tau(s)^2 = \omega^2 \sin^2\phi$ and then ω is constant. We may assume that $\psi(s) = w \cdot s$. The differential equation (2.4) giving the lift $\sigma(s)$ of $\gamma(s)$ to $\mathbb{SU}(3)$ may be now written as

$$\sigma' = \sigma \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & Ci & -Se^{-i\omega s} \\ 0 & Se^{i\omega s} & -Ci \end{pmatrix} ,$$

where C and S are real constants given by

$$\kappa_1 = C = \kappa \cos\phi \,, \quad S = \kappa \sin\phi \,, \quad S = \kappa \sqrt{1 - A_2^2} \,, \tag{2.17}$$

and

$$\kappa_2 = S \cos(\omega s) \,, \quad \kappa_3 = S \sin(\omega s) \,, \quad \kappa^2 = C^2 + S^2 \,. \tag{2.18}$$

We denote by $\bar{\sigma}(s)$ a new curve in $\mathbb{SU}(3)$ which is defined as

$$\bar{\sigma}(s) = \sigma(s) \cdot \begin{pmatrix} e^{-\frac{\omega s}{3}} & 0 & 0 \\ 0 & e^{-\frac{\omega s}{3}} & 0 \\ 0 & 0 & e^{\frac{2\omega s}{3}} \end{pmatrix} .$$

The new curve also projects to $\gamma(s)$ and it satisfies the following equation

$$\bar{\sigma}'(s) = \bar{\sigma}(s) \cdot M \,, \tag{2.19}$$

where $M \in \mathfrak{su}(3)$ is given by

$$M = \begin{pmatrix} -\frac{i\omega}{3} & -1 & 0 \\ 1 & \left(C - \frac{\omega}{3}\right) i & -S \\ 0 & S & \left(\frac{2\omega}{3} - C\right) i \end{pmatrix} . \tag{2.20}$$

Then one can solve (2.19) obtaining $\bar{\sigma}(s) = \bar{\sigma}(0)e^{Ms}$, $\gamma(s) = \bar{\sigma}(0)e^{Ms} \times \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$. This shows

Proposition 2.4. *A proper slant Frenet helix of $\mathbb{CP}^2(4)$ is the image of a one-parameter subgroup of $\mathbb{SU}(3)$ under the natural projection π.*

§ 3. Hyperelastic slant curves

We consider the space of immersed curves $\gamma(t)$ in $\mathbb{CP}^2(4)$ with fixed endpoints $p, q \in \mathbb{CP}^2(4)$: $\Omega = \{\gamma\colon [0,1] \to \mathbb{CP}^2(4);\ \gamma \text{ is a } C^\infty\text{-immersion};\ \gamma(0) = p; \gamma(1) = q\}$. Let us define the following energy functional $\mathcal{F}^m\colon \Omega \to \mathbb{R}$

$$\mathcal{F}^m(\gamma) = \int_\gamma \kappa^{2m} ds\,, \tag{3.21}$$

where s is the arclength parameter and $\kappa(s)$ is the curvature of $\gamma(s)$. We say that $\gamma(s)$ is a *$2m$-elastica* in $\mathbb{CP}^2(4)$ if it is an extremal curve of $\mathcal{F}^m$. For a given $\gamma \in \Omega$, we consider a C^∞ variation by curves in Ω, that is a C^∞ function $\Gamma\colon (-\epsilon, \epsilon) \times [0,1] \to \mathbb{CP}^2(4)$ such that $\Gamma(0,s) = \gamma(s)$ and $\Gamma(z,s) = \gamma_z(s) \in \Omega$. By using standard arguments one can compute the first variation formula of $\mathcal{F}^m$. We obtain

$$\frac{\mathrm{d}\,\mathcal{F}^m}{\mathrm{d}\,z}(0) = \int_\gamma \langle \mathcal{E}(\gamma), W\rangle ds + \mathcal{B}[\gamma, W]\,, \tag{3.22}$$

where $W = \frac{\partial \Gamma}{\partial z}(0,s)$ is the variation field of Γ and $\mathcal{E}(\gamma)$, $\mathcal{B}[\gamma, W]$ are the Euler-Lagrange and boundary operators respectively. They are given by

$$\begin{aligned} \mathcal{E}(\gamma) = {} & 2m\nabla_T^2(\kappa^{2m-2}\nabla_T T) + \\ & + (4m-1)\nabla_T(\kappa^{2m} T) + 2m\kappa^{2m-2} R(\nabla_T T, T)T\,, \end{aligned} \tag{3.23}$$

with R denoting the curvature tensor, and

$$\begin{aligned} \mathcal{B}[\gamma, W] = -(4m-1)\langle W, \kappa^{2m} T\rangle|_0^1 - 2m\langle W, \nabla_T(\kappa^{2m-2}\nabla_T T)\rangle|_0^1 + \\ +2m\langle \nabla_T W, \kappa^{2m-2}\nabla_T T\rangle|_0^1\,. \end{aligned}$$

Then, from (3.22) one sees that a curve $\gamma \in \Omega$ satisfying suitable boundary conditions (for instance, closed curves) is a critical point of $\mathcal{F}^m$, if and only if,

$$\mathcal{E}(\gamma) = 0\,. \tag{3.24}$$

From now we consider $\mathcal{F}^m$ acting on the subspace $\widetilde{\Omega} \subset \Omega$ which is formed by closed curves. One sees from (3.23) and (3.24) that *every closed geodesic is a critical point* *of* $\mathcal{F}^m$ in $\widetilde{\Omega}$. Hence we may assume in addition that $\gamma \in \widetilde{\Omega}$ is a non-geodesic closed curve. Then $\mathcal{E}(\gamma) = 0$, therefore using (2.2), (3.23) and the well known expression for the curvature tensor R we see that, under suitable boundary conditions, $\gamma(s)$ is a $2m$-elastic curve of $\mathbb{CP}^2(4)$ if and only if $\{\kappa(s), \tau(s), \delta(s)\}$ are solutions of

$$\begin{aligned}(3.25)\qquad 0 &= (2m-1)\kappa^4 + 2m(1-\tau^2)\kappa^2 + 2m(2m-1)\kappa\kappa'' + \\ &\quad + 2m(2m-1)(2m-2)(\kappa')^2 + 6mk^2 A_2^2\,,\\ (3.26)\qquad 0 &= 2(2m-1)\tau\kappa' + \kappa\tau' + 3\kappa A_2 A_3\,,\\ (3.27)\qquad 0 &= \tau\delta + 3A_2A_4\,,\end{aligned}$$

where $A_i = \langle JT, \xi_i\rangle$, $2 \le i \le 4$, and ()' denotes derivative with respect to s.

These are the equations to be satisfied by the Frenet curvatures of a $2m$-elastic curve in $\mathbb{CP}^2(4)$. Now, by using equations (2.5)-(2.8) and making similar computations, we can express the Euler-Lagrange equations in terms of the *Complex Frenet reference frame* in more symmetric manner. Thus if $\gamma(s)$ is $2m$-elastic curve of $\mathbb{CP}^2(4)$, the new "curvatures" $\{\kappa_1, \kappa_2, \kappa_3\}$ must satisfy

$$(3.28)\qquad 0 = \kappa_1'' + \kappa_1\left(4 + \frac{2m-1}{2m}\kappa^2\right) + \kappa_3\kappa_2' - \kappa_2\kappa_3'\,,$$

$$(3.29)\qquad 0 = \kappa_2'' + \kappa_2\left(1 + \frac{2m-1}{2m}\kappa^2\right) + \kappa_1\kappa_3' - \kappa_3\kappa_1'\,,$$

$$(3.30)\qquad 0 = \kappa_3'' + \kappa_3\left(1 + \frac{2m-1}{2m}\kappa^2\right) + \kappa_2\kappa_1' - \kappa_1\kappa_2'\,.$$

Now, we can prove the following

Proposition 3.5. *Let $\gamma(s)$ be a non-geodesic $2m$-elastic curve of $\mathbb{CP}^2(4)$. Then $\tau(s)$ vanishes identically if and only if the osculating plane of $\gamma(s)$ is either holomorphic or Lagrangian. Moreover in this case, we have: (a) The osculating plane of $\gamma(s)$ is holomorphic everywhere in $\mathbb{CP}^2(4)$ if and only if $\gamma(s)$ lies (as a $2m$-elastic curve) in some $\mathbb{S}^2(4)$ complex and totally geodesic in $\mathbb{CP}^2(4)$; (b) The osculating plane of $\gamma(s)$ is Lagrangian everywhere in $\mathbb{CP}^2(4)$ if and only if $\gamma(s)$ lies (as a $2m$-elastic curve) in some $\mathbb{RP}^2(1)$ Lagrangian and totally geodesic in $\mathbb{CP}^2(4)$.*

Proof. If $\tau(s) = 0$, then by Proposition 2.2.(1) and (3.28)-(3.30), we see that $\gamma(s)$ is a slant curve with $A_2 = 0$ or 1. Conversely, if $A_2 = 1$ then it is easy to see that $\tau(s) = 0$. If $A_2 = 0$ and $\tau(s) \neq 0$, then combining Proposition 2.2.(2) and (3.27) we get $\kappa(s) = -\delta(s) = 0$ which is impossible. Now if, $A_2 = 1$ (respectively, $A_2 = 0$), the normal subbundle span by $\{\xi_3, \xi_4\}$ is holomorphic (respectively, totally real), parallel and totally geodesic subbundle, hence using (3.25)-(3.27), we have finished. □

But it is known (see [4]) that there are no closed $2m$-elastic curves in either $\mathbb{S}^2(4)$ or $\mathbb{RP}^2(1)$ for $m \neq 1$. Therefore, since we are interested mainly in closed critical points, we only have to take care of the case $\tau(s) \neq 0$. So we can consider $\gamma(s)$ as a $2m$-elastic curve whose torsion does not vanish. Then we have

Proposition 3.6. *Let $\gamma(s)$ be a $2m$-elastic curve with non-zero torsion in $\mathbb{CP}^2(4)$. Then $\gamma(s)$ is a (proper) slant curve if and only if $\gamma(s)$ is a Frenet helix.*

Proof. By multiplying (3.28) by κ_1', (3.29) by κ_2', (3.30) by κ_3' and summing up we get

$$\kappa_1'\kappa_1'' + \kappa_2'\kappa_2'' + \kappa_3'\kappa_3'' + \frac{1}{2}(\kappa^2)'(\frac{2m-1}{2m}\kappa^2 + 1) + 3\kappa_1'\kappa_1 = 0\,. \tag{3.31}$$

Then, if $\gamma(s)$ is a helix $\kappa(s), \tau(s)$ are constant, therefore from (2.11) and (3.31) we have that $\kappa_1(s)$ is also constant and, consequently, from (2.10) that $\cos\phi_2 = A_2$ is constant.

Conversely, if $\gamma(s)$ is a proper slant curve, then Proposition 2.2 says that A_2, A_4 are non-zero constants and $A_3 = 0$. Thus (3.26) and (3.27) implies that $\kappa^{4m-2}(s)\tau(s)$ and $\tau(s)\delta(s)$ are constant. Combining these with (v) of Proposition 2.2 we obtain $\kappa(s), \tau(s)$ and $\delta(s)$ are constant. □

Next, let us assume that $\gamma(s)$ is an $2m$-elastic proper slant curve in $\mathbb{CP}^2(4)$. Then, it is a non-zero torsion helix with constant angle $\cos\phi_2 = A_2$. Thus, we may use (3.25), (3.26), (3.27) and (v) of Proposition 2.2 to obtain,

$$\tau^2 = \frac{A_2^2 - 1}{(4m-1)A_2^2 - 2m}\ \theta(m, A_2)\,, \tag{3.32}$$

$$\kappa = \frac{A_2}{A_4}\tau + 3\,, \frac{A_2A_4}{\tau}\,, \tag{3.33}$$

$$\delta = -3\,\frac{A_2A_4}{\tau}\,. \tag{3.34}$$

where $\theta(m, A_2) = 3(3m-1)A_2^2 + m + \sqrt{m(m + 9mA_2^4 + 6(9m-4)A_2^2)}$. Since $\theta(m, A_2) > 0$ and $A_2^2 < 1$, (3.32) implies that $A_2^2 < \frac{2m}{4m-1}$.

Now, by Proposition 3.6 we know that $\gamma(s)$ is a helix, so by using the arguments right before Proposition 2.4, we can find two constants C and S satisfying (2.17) and (2.18). Using these relations in (3.28)-(3.30) and simplifying the resulting equations, we get

$$3C = (C^2 + S^2)\,\omega - C\omega^2\,, \tag{3.35}$$

$$\omega^2 = 1 + C\omega + \frac{2m-1}{2m}\,(C^2 + S^2)\,, \tag{3.36}$$

which give

$$C = \frac{2m(\omega^2 - 1)\,\omega}{(4m-1)\,\omega^2 + 3(2m-1)}\,. \tag{3.37}$$

Combining (2.17), (2.18), (3.32)-(3.37), we obtain that the other constants can also be expressed in terms of ω:

$$\kappa^2 = \frac{2m(\omega^2+3)(\omega^2-1)}{(4m-1)\,\omega^2+3(2m-1)}\,, \tag{3.38}$$

$$S^2 = \frac{2m(\omega^2-1)\left(2m(9+10\,\omega^2+\omega^4)-(3+\omega^2)^2\right)}{\left(3+\omega^2-2m(3+2\omega^2)\right)^2}\,, \tag{3.39}$$

$$A_2^2 = \cos^2\phi = \frac{2m(\omega^2-1)\,\omega^2}{(\omega^2+3)\,(6m-3+(4m-1)\,\omega^2)}\,, \tag{3.40}$$

$$\tau^2 = \frac{\omega^2 S^2}{\kappa^2} = \frac{\omega^2\left(2m(9+10\,\omega^2+\omega^4)-(3+\omega^2)^2\right)}{(\omega^2+3)\,(6m-3+(4m-1)\,\omega^2)}\,, \tag{3.41}$$

$$\delta = -\frac{3A_2\sqrt{1-A_2^2}}{\tau}\,. \tag{3.42}$$

Since $A_2 \in (0,1)$, equation (3.40) implies that $\omega > 1$. Hence, for any real number $\omega > 1$ we can use (3.38)-(3.42) to construct a one-parameter family of proper slant Frenet helices in $\mathbb{CP}^2(4)$ which are $2m$-elastic curves under suitable boundary conditions. However, we are looking for closed $2m$-elastic curves. From Proposition 2.4 we know that $\gamma(s)$ is the image of a one-parameter subgroup of $\mathbb{SU}(3)$. Hence, to determine the closed $2m$-elastic helices it is enough to find out the condition for the lifted curve $\bar\sigma(s)$ to be a closed geodesic in $\mathbb{SU}(3)$. In order to do so, we must find a positive number, say L, so that $\bar\sigma(s+L) = \bar\sigma(s)$. Since $\bar\sigma(s) = e^{sM}$ where M is given in (2.20), this reduces to the problem of finding L such that the eigenvalues of $L\cdot M$ are all integer multiples of $2\pi i$. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of M. We have that $\lambda_1+\lambda_2+\lambda_3 = 0$ because $M \in \mathfrak{su}(3)$. It follows that the required condition for the roots is that $\frac{\lambda_2}{\lambda_1}$ be rational.

Moreover, by using (3.35) we see that the characteristic polynomial of M is

$$\chi_M(\lambda) = \lambda^3 + \frac{3\omega+9C+\omega^3}{3\omega}\lambda + i\frac{\left(18\omega-54C-2\omega^3\right)\omega}{27} = 0\,. \tag{3.43}$$

Replacing λ by ir and using (3.37) the above polynomial equation reduces to

$$r^3 + d\,r + e = 0\,, \tag{3.44}$$

where

$$d = -\frac{\left(3+w^2\right)h(m,\omega)+18\,m\,\left(w^2-1\right)}{3\,h(m,\omega)}\,, \tag{3.45}$$

$$e = \frac{2\,w\,\left((w^2-9)\,h(m,\omega)+54\,m\,\left(w^2-1\right)\right)}{27\,h(m,\omega)}\,, \tag{3.46}$$

$$h(m,\omega) = 3(2m-1)+(4m-1)w^2\,. \tag{3.47}$$

Hence (3.43) has either one or three pure imaginary roots. The discriminant of (3.44) $\Delta = \frac{e^2}{4}+\frac{d^3}{27}$ is given by

$$\Delta = -\frac{(w^2-1)}{27\,h(m,\omega)}\left(27+\sum_{i=1}^{4}p_{2i}(m)\omega^{2i}\right), \tag{3.48}$$

where $h(m,\omega)$ is defined in (3.47) and

$$p_2(m) = 162m(3+16\,m(m-1))\,, \qquad p_4(m) = 18(-1+23\,m-112\,m^2+176\,m^3\,,$$
$$p_6(m) = -8+114\,m-480\,m^2+608\,m^3\,, \quad p_8(m) = -1+10\,m+32\,m^2(m-1)\,.$$

It is easy to prove that $p_i(m)$, $i = 2,4,6,8$ are increasing positive polynomials for $m > 1$. Therefore $\Delta < 0$ and (3.44) has three real roots which are given by $r_1 = a\cos\left(\frac{\theta}{3}\right)$, $r_2 = a\cos\left(\frac{\theta+2\pi}{3}\right)$ and $r_3 = a\cos\left(\frac{\theta+4\pi}{3}\right)$, where $a = \sqrt{-d/3}$, $\cos\theta = -\frac{e}{2a^3}$ and d, e are defined in (3.45) and (3.46) respectively. In particular

$$\cos\theta = w\,\left((4\,m-1)\,(w^4-27)+6\,(4\,m+1)\,w^2\right)\sqrt{\frac{h(m,\omega)}{q(m,\omega)^3}}\,, \tag{3.49}$$

where $q(m,\omega) = (4\,m-1)\;w^4+6\;(6\,m-1)\;w^2-9$ and $h(m,\omega)$ is defined in (3.47). Hence the $2m$-elastic helix is closed if $\frac{\lambda_2}{\lambda_1} = \frac{1}{2}\sqrt{3}\tan\frac{\omega}{3}$ is rational.

On the other hand,

$$\frac{\partial\cos\theta}{\partial\omega} = \frac{27}{\sqrt{q(m,\omega)^5\,h(m,\omega)}}\sum_{i=0}^{4}s_{2i}(m)\omega^{2i}\,, \tag{3.50}$$

where

$$s_0(m) = -27+162\,m-216\,m^2\,, \qquad s_2(m) = -324\,m+1440\,m^2-1728\,m^3\,,$$
$$s_4(m) = 18-264\,m+1032\,m^2-1056\,m^3\,, \quad s_6(m) = 8-76\;m+272\;m^2-384\;m^3\,,$$
$$s_8(m) = 1-10\;m+32\;m^2-32\;m^3\,.$$

One can prove that $s_i(m)$, $i = 0,2,4,6,8$, are decreasing negative polynomials for $m \geq 1$. Also $\lim_{\omega\to 1^+}\cos\theta = 1$, $\lim_{\omega\to\infty}\cos\theta = -1$. Therefore, for any fixed $m \in \mathbb{N}$, we have that $\cos\theta$ decreases monotonically from 1 to -1 for $\omega > 1$. Finally, combining the above facts we obtain:

Theorem 3.7. *Fix a natural number $m \in \mathbb{N}$ and choose a rational number q such that $0 < q < 3$. Define an angle θ, $0 < \theta < \pi$, by $q = \sqrt{3}\,\tan\frac{\theta}{3}$ and take ω as the only solution of (3.49) greater than 1. Then the helix of $\mathbb{CP}^2(4)$ with curvatures given by (3.38), (3.41) and (3.42) is a closed $2m$-elastic proper slant curve in $\mathbb{CP}^2(4)$. His slant angle is determined by (3.40) and satisfies $A_2^2 < \frac{2m}{4m-1}$. Any closed $2m$-elastic slant curve of $\mathbb{CP}^2(4)$ with non-vanishing torsion can be obtained in this way.*

Notice that no matter the value of $m \in \mathbb{N}$, $\tau^2(s) > 1$ and that the family of $2m$-elastic closed slant curves in $\mathbb{CP}^2(4)$ can be parameterized in the same interval.

One interesting issue is how to determine, for a given angle θ satisfying the conditions in the above statement, a solution $\omega > 1$ of the equation (3.49). If $\cos\theta = 0$, then (3.49) has only one solution greater than 1: $\omega = \sqrt{3\,t_2(m)}$, where $t_2(m)$ is shown below. If $\cos\theta \neq 0$, we can write

$$(\cos\theta)^2 = \frac{w^2\,\left(w^2+3\,t_1\right)^2\,\left(w^2-3\,t_2\right)^2\,h(m,\omega)}{(4\,m-1)\,\left(w^2+3\,t_3\right)^3\,\left(w^2-3\,t_4\right)^3}\,, \tag{3.51}$$

where we already know $h(m,\omega)$ from (3.47) and

$$t_1(m) = \frac{1+4\,m+2\,\sqrt{1-4\,m+16\,m^2}}{4\,m-1}\,,\quad t_2(m) = \frac{-1-4\,m+2\,\sqrt{1-4\,m+16\,m^2}}{4\,m-1}\,,$$
$$t_3(m) = \frac{-1+6\,m+2\,\sqrt{-2\,m+9\,m^2}}{4\,m-1}\,,\quad t_4(m) = \frac{1-6\,m+2\,\sqrt{-2\,m+9\,m^2}}{4\,m-1}\,.$$

For a given θ, there are only two solutions of (3.51) which satisfy $\omega > 1$. Then, one can choose the right one by using (3.49).

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