# Tensor product surfaces of a Euclidean space curve and a Lorentzian plane curve

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**Abstract.** In this paper, we study the surfaces $f(t,s) = \alpha(t) \otimes \beta(s)$ in the semi-Euclidean space $R^6_3$, which are the tensor product of a Euclidean space curve $\alpha(t)$ and a Lorentzian plane curve $\beta(s)$. In particular, we classify all minimal and totally real tensor product surfaces $\alpha(t) \otimes \beta(s)$ and prove that there are no complex tensor product surfaces $\alpha(t) \otimes \beta(s)$ in $R^6_3$.

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## § 1. Introduction

In the Euclidean space $\mathbb{E}^n$, the tensor product immersion of two immersions of a given Riemannian manifold was firstly defined and studied by Chen in [1]. In particular, the direct sum and the tensor product maps of two immersions of two different Riemannian manifolds, are introduced in [2] in the following way:

Let $M$ and $N$ be two differentiable manifolds and let us assume that $f : M \to \mathbb{E}^m$ and $h : N \to \mathbb{E}^n$ are two immersions. Then the direct sum map and tensor product map are defined respectively by

$$f \oplus h : M \times N \to \mathbb{E}^{m+n},$$

$$(f \oplus h)(p,q) = (f^1(p), ..., f^m(p), h^1(q), ..., h^n(q)),$$

and

$$f \otimes h : M \times N \to \mathbb{E}^{mn},$$

$$(f \otimes h)(p,q) = (f^1(p)h^1(q), ..., f^1(p)h^n(q), ..., f^m(p)h^n(q)).$$

Under certain conditions obtained in [2], the tensor product map $f \otimes h$ is an immersion in the space $\mathbb{E}^{mn}$. Some geometric properties of tensor products of submanifolds are given in [5].

The simplest examples of the tensor product immersions are the tensor product surfaces. In the Euclidean space $\mathbb{E}^n$, the tensor product surfaces of two Euclidean

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planar curves are investigated in [4]. On the other hand, in the semi-Euclidean space $\mathbb{E}^n_\nu$, the tensor product surfaces of two Lorentzian planar curves are studied in [6].

In this paper, we study the tensor product surfaces of a Euclidean space curve and a Lorentzian plane curve and classify all minimal, totaly real and complex tensor product surfaces of such curves.

## § 2. Preliminaries

Let $\mathbb{R}^m_\mu$ and $\mathbb{R}^n_\nu$ be two pseudo-Euclidean spaces with metric matrices $G_1$ and $G_2$ respectively. We identify, in the usual way, the space $\mathbb{R}^{mn}$ with the space $\mathcal{M}$ of real $m \times n$ matrices. Let us consider the metric $g$ in $\mathcal{M}$ given by

$$g(A, B) = trace(G_1 A G_2 B^T),$$

where $B^T$ denotes the transpose of $B$. Then $(\mathcal{M}, g)$ is isometric to the pseudo-Euclidean space $\mathbb{R}^{mn}_r$ of index $r = \mu(n-\nu) + \nu(m-\mu)$. The metric product $\otimes$ : $\mathbb{R}^m_\mu \times \mathbb{R}^n_\nu \to \mathcal{M}$ can be defined as $P \otimes Q = P^T Q$. Accordingly, the metric $g$ in $\mathcal{M}$ is given by

$$g(X \otimes V, Y \otimes W) = g_1(X, Y) g_2(V, W), \tag{2.1}$$

where $g_1$ and $g_2$ are the metrics of $\mathbb{R}^m_\mu$ and $\mathbb{R}^n_\nu$, respectively.

In particular, if $\alpha : \mathbb{R} \to \mathbb{R}^3$ and $\beta : \mathbb{R} \to \mathbb{R}^2_1$ are the Euclidean space curve and a Lorentzian plane curve respectively, then their tensor product $f(t, s) = \alpha(t) \otimes \beta(s)$ is defined as

$$\otimes : \mathbb{R}^3 \times \mathbb{R}^2_1 \to \mathbb{R}^6_3, \qquad f(t, s) = \alpha(t)^T \beta(s), \tag{2.2}$$

where $\alpha(t)^T$ denotes the transpose of $\alpha(t)$. The pseudo-Riemannian metric $g$ in $\mathbb{R}^6_3$ is given by (2.1), where $g_1 = dx_1^2 + dx_2^2 + dx_3^2$ and $g_2 = -dx_1^2 + dx_2^2$ are the metrics of $\mathbb{R}^3$ and $\mathbb{R}^2_1$, respectively.

By using equation (2.2), the canonical tangent vectors of $f(t, s)$ can be easily computed as

$$\frac{\partial f}{\partial t} = \alpha'(t) \otimes \beta(s) = \alpha'(t)^T \beta(s), \quad \frac{\partial f}{\partial s} = \alpha(t) \otimes \beta'(s) = \alpha(t)^T \beta'(s). \tag{2.3}$$

Hence relations (2.1), (2.2) and (2.3) imply that the coefficients of the pseudo-Riemannian metric, induced on $f(t, s)$ by the pseudo-Riemannian metric $g$ of $\mathbb{R}^6_3$, are

$$g_{11} = g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}) = g_1(\alpha', \alpha') g_2(\beta, \beta),$$

$$g_{12} = g(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}) = g_1(\alpha, \alpha') g_2(\beta, \beta'),$$

$$g_{22} = g(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}) = g_1(\alpha, \alpha) g_2(\beta', \beta').$$

In the following, we will assume that $\beta$ is a spacelike or a timelike curve, with a spacelike or a timelike position vector, and we will assume that $\alpha$ is a regular curve not passing through the origin. Hence $g_{11} \neq 0 \neq g_{22}$. We will also assume that the tensor product surface $f(t,s)$ is a regular surface, i.e. $g_{11}g_{22} - g_{12}^2 \neq 0$. Consequently, an orthonormal basis for the tangent space of $f(t,s)$ is given by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{|g_{11}|}} \frac{\partial f}{\partial t}, \\ e_2 &= \frac{1}{\sqrt{|g_{11}(g_{11}g_{22} - g_{12}^2)|}} (g_{11} \frac{\partial f}{\partial s} - g_{12} \frac{\partial f}{\partial t}). \end{aligned}$$

Recall that the mean curvature vector field $H$ is defined by

$$H = \frac{1}{2}(\epsilon_1 h(e_1, e_1) + \epsilon_2 h(e_2, e_2)),$$

where $h$ is second fundamental form of $\alpha \otimes \beta$ and $\epsilon_i = g(e_i, e_i)$, $i = 1, 2$. In particular, by Beltrami's formula we have

$$H = -\frac{1}{2}\Delta f.$$

Next, recall that a surface $M$ in $\mathbb{R}^6_3$ is said to be minimal, if its mean curvature vector field $H$ vanishes identically.

A basis of the normal space of $f(t,s)$ can be calculated as follows. Let $J_i : \mathbb{R}^3 \to \mathbb{R}^3$, $i = 1, 2$ and $J : \mathbb{R}^2_1 \to \mathbb{R}^2_1$ be the following maps:

$$\begin{aligned} J_1(x,y,z) &= (-y, x, 0), \\ J_2(x,y,z) &= (0, z, -y), \\ J(x,y) &= (y, x). \end{aligned} \tag{2.4}$$

Observe that $g_1(X, J_i X) = 0$ for $X \in \mathbb{R}^3$, $i = 1, 2$ and $g_2(Y, JY) = 0$ for $Y \in \mathbb{R}^2_1$. Then a basis $\{n_1, n_2, n_3, n_4\}$ of the normal space is the following:

$$\begin{aligned} n_1(t,s) &= J_1(\alpha(t)) \otimes J(\beta(s)), \\ n_2(t,s) &= J_2(\alpha(t)) \otimes J(\beta(s)), \\ n_3(t,s) &= J_1(\alpha'(t)) \otimes J(\beta'(s)), \\ n_4(t,s) &= J_2(\alpha'(t)) \otimes J(\beta'(s)). \end{aligned} \tag{2.5}$$

Note that also the vector $n_5$ given by

$$n_5(t,s) = J_3(\alpha'(t)) \otimes J(\beta'(s)) \tag{2.6}$$

is a normal vector, where $J_3 : \mathbb{R}^3 \to \mathbb{R}^3$ is the map defined by $J_3(x,y,z) = (-z, 0, x)$. It will be convenient later on to also consider this vector.

Accordingly, the tensor product surface $f(t,s)$ is minimal in $\mathbb{R}^6_3$, if and only if

$$g(H, n_i) = 0, \quad i = 1, 2, 3, 4, 5.$$

On the other hand, by using the Beltrami's formula, a surface $f(t,s)$ is minimal in $\mathbb{R}^6_3$ if and only if

$$g(\Delta f, n_i) = 0, \quad i = 1, 2, 3, 4, 5. \tag{2.7}$$

Since the Laplacian of $f(t,s)$ is given by

$$\begin{aligned}\Delta f = \quad & g^{11}\frac{\partial^2 f}{\partial t^2} + 2g^{12}\frac{\partial^2 f}{\partial t\partial s} + g^{22}\frac{\partial^2 f}{\partial s^2} + \frac{1}{\sqrt{|g|}}\Big[(\frac{\partial}{\partial t}(\sqrt{|g|}g^{11}) \\ & + \frac{\partial}{\partial s}(\sqrt{|g|}g^{12}))\frac{\partial f}{\partial t} + (\frac{\partial}{\partial t}(\sqrt{|g|}g^{12}) + \frac{\partial}{\partial s}(\sqrt{|g|}g^{22}))\frac{\partial f}{\partial s}\Big],\end{aligned}$$

where $|g| = |det(g_{ij})|$, the minimality conditions (2.7) become

$$g(g_{11}\frac{\partial^2 f}{\partial s^2} - 2g_{12}\frac{\partial^2 f}{\partial t\partial s} + g_{22}\frac{\partial^2 f}{\partial t^2}, n_i) = 0, \quad i = 1,2,3,4,5. \tag{2.8}$$

## § 3. Tensor product surfaces of a Euclidean space curve and a Lorentzian plane curve

In this section, we classify all minimal and totally real tensor product surfaces in $\mathbb{R}^6_3$, and prove that there are no complex tensor product surfaces in the same space. Recall that a circle in $\mathbb{R}^2_1$ is defined in [3] as a curve with non-zero constant curvature.

**Theorem 3.1.** *The tensor product immersion $f(t,s) = \alpha(t)\otimes\beta(s)$ of a Euclidean space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{R}^2_1$, is a minimal surface in $\mathbb{R}^6_3$ if and only if :*

*(i) $\alpha$ is either the circle with the equation*

$$\alpha(t) = r_0(\cos(t), 0, \sin(t)), \quad r_0 \in R_0^+,$$

*or logaritmic spiral given by*

$$\alpha(t) = a_1 e^{a_2 t}(\cos(t), 0, \sin(t)), \quad a_1 \in R_0^+, \quad a_2 \in R_0,$$

*and $\beta$ is the curve with the equation*

$$\beta(s) = (a/\sqrt{\cosh(2s)})(\sinh(s), \cosh(s)), \quad a \in R_0,$$

*or the curve with the equation*

$$\beta(s) = (b/\sqrt{|\sinh(2s)|})(\sinh(s), \cosh(s)), \quad b \in R_0;$$

*(ii) $\alpha$ is orthogonal hyperbola given by*

$$\alpha(t) = (a/\sqrt{|\cos(2t)|})(\cos(t), 0, \sin(t)), \quad a \in R_0,$$

*and $\beta$ is the circle with the equation*

$$\beta(s) = \rho_0(\sinh(s), \cosh(s)), \quad \rho_0 \in R_0^+,$$

*or hyperbolic spiral given by*

$$\beta(s) = b_1 e^{b_2 s}(\sinh(s), \cosh(s)), \quad b_1 \in R_0^+, \quad b_2 \in R_0, \quad b_2 \neq \pm 1;$$

*(iii) $\beta$ is given by*

$$\beta(s) = \frac{a_2}{\sqrt{\cosh((1+k)s+a_1)}}(\sinh(s), \cosh(s)),$$

*or by*

$$\beta(s) = \frac{a_2}{\sqrt{|\sinh((1+k)s+a_1)|}}(\sinh(s), \cosh(s)),$$

*and $\alpha$ is given by*

$$\alpha(t) = \frac{b_2}{\sqrt{|\cos((1-k)t+b_1)|}}(\cos(t), 0, \sin(t)),$$

*where $a_1 \in R$, $a_2 \in R_0$, $b_1 \in R$, $b_2 \in R_0$, $k \in R_0$, $k \neq \pm 1$.*

*Proof.* Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a Euclidean space curve and $\beta(s) = (\beta_1(s), \beta_2(s))$ be a Lorentzian plane curve. Then their tensor product $f(t,s) = \alpha(t) \otimes \beta(s)$ is given by (2.2). Let us first suppose that $f(t,s)$ is a minimal surface in $\mathbb{R}^6_3$. By using (2.2), we easily find

$$\text{(3.1)} \quad \frac{\partial^2 f}{\partial t^2} = \alpha''(t) \otimes \beta(s), \quad \frac{\partial^2 f}{\partial s^2} = \alpha(t) \otimes \beta''(s), \quad \frac{\partial^2 f}{\partial t \partial s} = \alpha'(t) \otimes \beta'(s).$$

The normal space of $f(t,s)$ is spanned by vectors $\{n_1, n_2, n_3, n_4\}$, given by (2.5). Moreover, for the vector $n_5$ given by (2.6), we have $n_5 \in \text{span}\,\{n_1, n_2, n_3, n_4\}$. Consequently, relations (2.1), (2.5), (2.6) and (3.1) yield

$$\text{(3.2)} \quad g(\frac{\partial^2 f}{\partial t^2}, n_i) = g(\frac{\partial^2 f}{\partial s^2}, n_i) = g(\frac{\partial^2 f}{\partial t \partial s}, n_j) = 0, \quad i = 1,2, \quad j = 3,4,5.$$

On the other hand, relations (2.1), (2.5), (2.6) and (3.1) also imply

$$\text{(3.3)} \quad \begin{aligned} g(\frac{\partial^2 f}{\partial t \partial s}, n_i) &= g_1(\alpha', J_i(\alpha))g_2(\beta', J(\beta)), \quad i = 1,2, \\ g(\frac{\partial^2 f}{\partial t^2}, n_j) &= g_1(\alpha'', J_{j-2}(\alpha'))g_2(\beta, J(\beta')), \quad j = 3,4,5, \\ g(\frac{\partial^2 f}{\partial s^2}, n_j) &= g_1(\alpha, J_{j-2}(\alpha'))g_2(\beta'', J(\beta')), \quad j = 3,4,5. \end{aligned}$$

Combining (2.8) and (3.2), we obtain that the minimality conditions are given by

$$\text{(3.4)} \quad g(\frac{\partial^2 f}{\partial t \partial s}, n_i) = 0, \quad g(g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_j) = 0,$$

where $i = 1,2$ and $j = 3,4,5$, or else by

$$\text{(3.5)} \quad g_{12} = 0, \quad g(g_{22}\frac{\partial^2 f}{\partial t^2} + g_{11}\frac{\partial^2 f}{\partial s^2}, n_j) = 0, \quad j = 3,4,5.$$

Therefore, we may distinguish the following two cases: the case when the conditions (3.4) hold and the case when the conditions (3.5) hold.

**Case I.** Assume that conditions (3.4) hold. Then relations (3.3) and (3.4) imply

$$(3.6)\qquad \begin{aligned} &g_1(\alpha', J_i(\alpha))g_2(\beta', J(\beta)) = 0,\\ &g_{22}g_1(\alpha'', J_{j-2}(\alpha'))g_2(\beta, J(\beta'))\\ &+g_{11}g_1(\alpha, J_{j-2}(\alpha'))g_2(\beta'', J(\beta')) = 0, \end{aligned}$$

for $i = 1, 2$ and $j = 3, 4, 5$. Next we consider two subcases.

**Case I.1.** If $g_2(\beta', J(\beta)) = 0$, then $-\beta_1'\beta_2 + \beta_1\beta_2' = 0$, so $\beta$ is a straight line passing through the origin. Then we have $g_{11}g_{22} - g_{12}^2 = 0$, which means that the surface $f(t, s)$ is not regular. Hence we obtain a contradiction.

**Case I.2.** If $g_1(\alpha', J_i(\alpha)) = 0$, for $i = 1, 2$, the system of equations (3.6) becomes

$$(3.7)\qquad \begin{aligned} &-\alpha_1'\alpha_2 + \alpha_1\alpha_2' = 0, \quad \alpha_2'\alpha_3 - \alpha_2\alpha_3' = 0,\\ &g_{22}(-\alpha_1''\alpha_2' + \alpha_1'\alpha_2'')g_2(\beta, J(\beta'))\\ &+g_{11}(-\alpha_1\alpha_2' + \alpha_1'\alpha_2)g_2(\beta'', J(\beta')) = 0,\\ &g_{22}(-\alpha_2'\alpha_3'' + \alpha_2''\alpha_3')g_2(\beta, J(\beta'))\\ &+g_{11}(-\alpha_2'\alpha_3 + \alpha_2\alpha_3')g_2(\beta'', J(\beta')) = 0,\\ &g_{22}(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3'')g_2(\beta, J(\beta'))\\ &+g_{11}(-\alpha_1\alpha_3' + \alpha_1'\alpha_3)g_2(\beta'', J(\beta')) = 0. \end{aligned}$$

Now we distinguish the following possibilities. If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, from the first two equations of (3.7) we get that $\alpha$ is a straight line passing through the origin. Then the surface $f(t, s)$ is not regular, which is a contradiction. Next, if $\alpha_1 = 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, or $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 = 0$, in a similar way from (3.7) we obtain that $\alpha$ is a straight line passing through the origin, which implies a contradiction. Finally, if $\alpha_1 \neq 0$, $\alpha_2 = 0$ and $\alpha_3 \neq 0$, the system of equations (3.7) reduces to the equation

$$g_{22}(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3'')g_2(\beta, J(\beta')) + g_{11}(-\alpha_1\alpha_3' + \alpha_1'\alpha_3)g_2(\beta'', J(\beta')) = 0.$$

Therefore, the last equation yields

$$(3.8)\qquad \frac{g_1(\alpha, \alpha)(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3'')}{g_1(\alpha', \alpha')(\alpha_1\alpha_3' - \alpha_1'\alpha_3)} = \frac{g_2(\beta, \beta)g_2(\beta'', J(\beta'))}{g_2(\beta', \beta')g_2(\beta, J(\beta'))} = k, \qquad k \in R_0.$$

Since $\alpha$ is arbitrary curve in $\mathbb{R}^3$, lying in the plane $x_2 = 0$, we may assume that $\alpha$, in polar coordinates, has the equation

$$(3.9)\qquad \alpha(t) = r(t)(\cos(t), 0, \sin(t)).$$

Similarly, we may assume that the equation of a regular curve $\beta$ in $\mathbb{R}^2_1$ (which is not a straight line containing the origin), reads $\beta(s) = \rho(s)(\sinh(s), \cosh(s))$ or $\beta(s) = \rho(s)(\cosh(s), \sinh(s))$, where $\rho(s)$ is arbitrary function. Without loss of generality, assume that $\beta$ has the equation

$$(3.10)\qquad \beta(s) = \rho(s)(\sinh(s), \cosh(s)).$$

Equations (3.8), (3.9) and (3.10) imply

$$(3.11)\qquad \frac{\rho\rho'' - 2\rho'^2 + \rho^2}{\rho'^2 - \rho^2} = \frac{-rr'' + 2r'^2 + r^2}{r^2 + r'^2} = k, \qquad k \in R_0.$$

If $k = 1$, from (3.11) we get two differential equations $\rho\rho'' - 3\rho'^2 + 2\rho^2 = 0$ and $rr'' - r'^2 = 0$. The solutions of the previous equations are of the form $\rho(s) = b_1/\sqrt{\cosh(2s+b_2)}$, $b_1 \in R_0$, $b_2 \in R$, if $|\rho'| < |\rho|$, or $\rho(s) = b_3/\sqrt{|\sinh(2s+b_4)|}$, $b_3 \in R_0$, $b_4 \in R$, if $|\rho'| > |\rho|$, and $r(t) = r_0 \in R_0^+$, or $r(t) = a_1 e^{a_2 t}$, $a_1 \in R_0^+$, $a_2 \in R_0$. Taking $b_2 = b_4 = 0$, we obtain that $\beta$ is given by

$$\beta(s) = \frac{b_1}{\sqrt{\cosh(2s)}}(\sinh(s), \cosh(s)),$$

or by

$$\beta(s) = \frac{b_3}{\sqrt{|\sinh(2s)|}}(\sinh(s), \cosh(s)).$$

On the other hand, $\alpha$ is the circle or logaritmic spiral, respectively given by

$$\alpha(t) = r_0(\cos(t), 0, \sin(t)),$$

and

$$\alpha(t) = a_1 e^{a_2 t}(\cos(t), 0, \sin(t)),$$

which proves statement (i).

If $k = -1$, in a similar way from equations (3.11) we get two differential equations $-rr'' + 3r'^2 + 2r^2 = 0$ and $\rho\rho'' - \rho'^2 = 0$. It follows that $r(t) = a_3/\sqrt{|\cos(2t+a_4)|}$, $a_3 \in R_0$, $a_4 \in R$ and $\rho(s) = \rho_0 \in R_0^+$ or $\rho(s) = b_3 e^{b_4 s}$, $b_3 \in R_0^+$, $b_4 \in R_0$, $b_4 \neq \pm 1$. If $b_4 = \pm 1$, then $\rho(s) = b_3 e^{\pm s}$, so equation (3.10) implies that $\beta(s)$ is a null curve, which is a contradiction. Taking $a_4 = 0$, we find that $\alpha$ is orthogonal hyperbola with the equation

$$\alpha(t) = \frac{a_3}{\sqrt{|\cos(2t)|}}(\cos(t), 0, \sin(t)),$$

and $\beta$ is the circle or hyperbolic spiral, respectively given by

$$\beta(s) = \rho_0(\sinh(s), \cosh(s)),$$

and

$$\beta(s) = b_3 e^{b_4 s}(\sinh(s), \cosh(s)).$$

Hence statement (ii) is proved.

Finally, if $k \neq \pm 1$, putting $v = \rho'/\rho$ and $w = r'/r$, from (3.11) we find $v' = (1+k)(v^2-1)$ and $w' = (1-k)(w^2+1)$. After integration, we get $\rho(s) = a_3/\sqrt{\cosh((1+k)s+a_4)}$, $a_3 \in R_0$, $a_4 \in R$ if $|v| < 1$, or else $\rho(s) = a_3/\sqrt{|\sinh((1+k)s+a_4)|}$, $a_3 \in R_0$, $a_4 \in R$ if $|v| > 1$. We also find $r(t) = b_1/\sqrt{|\cos((1-k)t+b_2)}$, $b_1 \in R_0$, $b_2 \in R$. Consequently, the curve $\alpha$ is given by

$$\alpha(t) = \frac{b_1}{\sqrt{|\cos((1-k)t+b_2)|}}(\cos(t), 0, \sin(t)),$$

and $\beta$ is given by

$$\beta(s) = \frac{a_3}{\sqrt{\cosh((1+k)s+a_4)}}(\sinh(s), \cosh(s)),$$

or by

$$\beta(s)=\frac{a_3}{\sqrt{|\sinh((1+k)s+a_4)|}}(\sinh(s),\cosh(s)).$$

In this way, statement (iii) is proved.

**Case II.** Assume that conditions (3.5) hold. Since $g_{12}=g_1(\alpha,\alpha')g_2(\beta,\beta')=0$, we consider two subcases: $g_1(\alpha,\alpha')=0$ and $g_2(\beta,\beta')=0$.

**Case II.1.** If $g_1(\alpha,\alpha')=0$, then $g_1(\alpha,\alpha)=c^2$, $c\in\mathbb{R}_0^+$, which means that $\alpha$ lies in sphere $\mathbb{S}^2(c)$ centered at the origin and with radius $c$. By reparameterizing $\alpha$, we may assume that $g_1(\alpha',\alpha')=1$. Next, equations (3.3) and (3.5) imply

$$g_{22}g_1(\alpha'',J_{j-2}(\alpha'))g_2(\beta,J(\beta'))+g_{11}g_1(\alpha,J_{j-2}(\alpha'))g_2(\beta'',J(\beta'))=0, \tag{3.12}$$

for $j=3,4,5$, and consequently

$$\begin{aligned}\frac{c^2(-\alpha_1{}''\alpha_2{}'+\alpha_1{}'\alpha_2{}'')}{\alpha_1\alpha_2{}'-\alpha_1{}'\alpha_2}&=\frac{c^2(\alpha_2{}''\alpha_3{}'-\alpha_2{}'\alpha_3{}'')}{-\alpha_2\alpha_3{}'+\alpha_2{}'\alpha_3}=\frac{c^2(-\alpha_1{}''\alpha_3{}'+\alpha_1{}'\alpha_3{}'')}{\alpha_1\alpha_3{}'-\alpha_1{}'\alpha_3}\\&=\frac{g_2(\beta,\beta)g_2(\beta'',J(\beta'))}{g_2(\beta',\beta')g_2(\beta,J(\beta'))}=k,\quad k\in R_0.\end{aligned} \tag{3.13}$$

¿From equations (3.13), we obtain the system of equations

$$\begin{aligned}c^2(-\alpha_1{}''\alpha_2{}'+\alpha_1{}'\alpha_2{}'')-k(\alpha_1\alpha_2{}'-\alpha_1{}'\alpha_2)&=0,\\c^2(\alpha_2{}''\alpha_3{}'-\alpha_2{}'\alpha_3{}'')-k(-\alpha_2\alpha_3{}'+\alpha_2{}'\alpha_3)&=0,\\c^2(-\alpha_1{}''\alpha_3{}'+\alpha_1{}'\alpha_3{}'')-k(\alpha_1\alpha_3{}'-\alpha_1{}'\alpha_3)&=0.\end{aligned}$$

The above equations can be easily rewritten as

$$\begin{aligned}(c^2\alpha_2{}''+k\alpha_2)\alpha_1{}'-\alpha_2{}'(c^2\alpha_1{}''+k\alpha_1)&=0,\\(c^2\alpha_2{}''+k\alpha_2)\alpha_3{}'-\alpha_2{}'(c^2\alpha_3{}''+k\alpha_3)&=0,\\(c^2\alpha_3{}''+k\alpha_3)\alpha_1{}'-\alpha_3{}'(c^2\alpha_1{}''+k\alpha_1)&=0.\end{aligned} \tag{3.14}$$

Note that the equations (3.14) can be interpreted as the necessary and the sufficient conditions for vectors $c^2\alpha''+k\alpha$ and $\alpha'$ to be linearly independent. Hence there exist a function $\lambda(t)$ such that

$$c^2\alpha''+k\alpha=\lambda\alpha'.$$

Taking the inner product on both sides of the previous equation with $\alpha'$ gives $\lambda(t)=0$. Therefore, $c^2\alpha''=-k\alpha$, which in particular implies that $\alpha$ lies in a plane $\pi$ passing through the origin. Up to isometries of $\mathbb{R}^3$, we may assume that the normal to the plane $\pi$ is $(1,0,0)$. Then $\alpha_2^2+\alpha_3^2=c^2$, $c\in R_0$, which means that $\alpha$ is the circle given by

$$\alpha(t)=c(0,\cos(t/c),\sin(t/c)). \tag{3.15}$$

Since $\beta$ is a regular non-null curve in $\mathbb{R}_1^2$ (which is not a straight line containing the origin), the equation of $\beta$ is given by (3.10). By using the equations (3.13) and (3.15), we easily obtain $k=1$. Then equations (3.10) and (3.13) imply differential equation $\rho\rho''-3\rho'^2+2\rho^2=0$. The solution of the previous equation is given by $\rho(s)=$

$c_2/\sqrt{\cosh(2s+c_1)}$, $c_1 \in R$, $c_2 \in R_0$, if $|\rho'| < |\rho|$, or by $\rho(s) = c_4/\sqrt{|\sinh(2s+c_3)|}$, $c_3 \in R$, $c_4 \in R_0$, if $|\rho'| > |\rho|$. We may take $c_1 = c_3 = 0$, so $\beta$ has the equation

$$\beta(s) = \frac{c_2}{\sqrt{\cosh(2s)}}(\sinh(s), \cosh(s)),$$

or

$$\beta(s) = \frac{c_4}{\sqrt{|\sinh(2s)|}}(\sinh(s), \cosh(s)).$$

Therefore, in this case, we obtain the equations of curves $\alpha$ and $\beta$ as in the previous case I.2.

**Case II.2.** If $g_2(\beta, \beta') = 0$, then $g_2(\beta, \beta) = \pm c^2$, $c \in R_0^+$, which means that $\beta$ is the spacelike or the timelike circle, respectively given by $\beta(s) = c(\cosh(s), \sinh(s))$ and $\beta(s) = c(\sinh(s), \cosh(s))$. In both cases, by using equations (3.12), we obtain

$$\begin{aligned} \frac{g_1(\alpha,\alpha)(-\alpha_1''\alpha_2' + \alpha_1'\alpha_2'')}{g_1(\alpha',\alpha')(\alpha_1\alpha_2' - \alpha_1'\alpha_2)} &= \frac{g_1(\alpha,\alpha)(\alpha_2''\alpha_3' - \alpha_2'\alpha_3'')}{g_1(\alpha',\alpha')(-\alpha_2\alpha_3' + \alpha_2'\alpha_3)} \\ &= \frac{g_1(\alpha,\alpha)(-\alpha_1''\alpha_3' + \alpha_1'\alpha_3'')}{g_1(\alpha',\alpha')(\alpha_1\alpha_3' - \alpha_1'\alpha_3)} = -1. \end{aligned} \tag{3.16}$$

Putting $g_1(\alpha', \alpha')/g_1(\alpha, \alpha) = \lambda(t)$ in (3.16), we easily get the system of equations

$$\begin{aligned} &(-\alpha_1'' + \lambda\alpha_1)\alpha_2' + (\alpha_2'' - \lambda\alpha_2)\alpha_1' = 0, \\ &(\alpha_2'' - \lambda\alpha_2)\alpha_3' + (-\alpha_3'' + \lambda\alpha_3)\alpha_2' = 0, \\ &(-\alpha_1'' + \lambda\alpha_1)\alpha_3' + (\alpha_3'' - \lambda\alpha_3)\alpha_1' = 0. \end{aligned}$$

Consequently, there exist a function $m(t)$ such that $\alpha'' - \lambda\alpha = m\alpha'$. It follows that $\alpha$ lies in the plane $\pi$ passing through the origin. Up to isometries of $\mathbb{R}^3$, we may assume that the normal to the plane $\pi$ is $(0, 1, 0)$. Thus $\alpha$ is given by

$$\alpha(t) = r(t)(\cos(t), 0, \sin(t)). \tag{3.17}$$

Next, equations (3.16) and (3.17) imply differential equation $rr'' - 3r'^2 - 2r^2 = 0$, with the solution $r(t) = b_2/\sqrt{|\cos(2t+b_1)|}$, $b_1 \in R$, $b_2 \in R_0$. Taking $b_1 = 0$, we get that $\alpha$ is orthogonal hyperbola with the equation

$$\alpha(t) = \frac{b_2}{\sqrt{|\cos(2t)|}}(\cos(t), 0, \sin(t)).$$

In this case, we obtain the equations of curves $\alpha$ and $\beta$ as in the previous case I.2.

Conversely, if one of statements (i), (ii) or (iii) holds, a straightforward calculation shows that $f(t, s)$ is a minimal surface in $\mathbb{R}^6_3$. □

In the next two theorems, we classify totally real tensor product surfaces in the semi-Euclidean space $\mathbb{R}^6_3$ and prove that there are no complex tensor product surfaces in the same space.

**Theorem 3.2.** *The tensor product immersion $f(t, s) = \alpha(t) \otimes \beta(s)$ of a Euclidean space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{R}^2_1$, is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure $J_0$, given by $J_0(p, q, u, v, z, w) = (-q, p, -v, u, -w, z)$ on $\mathbb{R}^6_3$, if and only if $\alpha$ lies in a sphere $\mathbb{S}^2$, centered at the origin, or $\beta$ is the curve given by*

$$\beta(s) = \frac{a}{\sqrt{|\sinh(2s)|}}(\sinh(s), \cosh(s)), \quad a \in R_0. \tag{3.18}$$

*Proof.* Let $\alpha(t)$ be a Euclidean space curve and $\beta(s)$ be a Lorentzian plane curve. Assume that the tensor product $f(t,s) = \alpha(t) \otimes \beta(s)$ is a totally real Lorentzian immersion in $\mathbb{R}^6_3$. By definition, there hold the conditions

$$g(J_0(\frac{\partial f}{\partial t}), \frac{\partial f}{\partial s}) = 0, \quad g(J_0(\frac{\partial f}{\partial s}), \frac{\partial f}{\partial t}) = 0. \tag{3.19}$$

Let $J^* : \mathbb{R}^2_1 \to \mathbb{R}^2_1$ be the map defined by

$$J^*(x,y) = (y, -x).$$

By using equations (2.2) and (2.3), we easily find

$$\begin{aligned} J_0(\frac{\partial f}{\partial t}) &= J_0(\alpha' \otimes \beta) = \alpha' \otimes J^*(-\beta), \\ J_0(\frac{\partial f}{\partial s}) &= J_0(\alpha \otimes \beta') = \alpha \otimes J^*(-\beta'). \end{aligned} \tag{3.20}$$

Moreover, by using (2.1) and (3.20), the conditions (3.19) become

$$\begin{aligned} g(\alpha' \otimes J^*(-\beta), \alpha \otimes \beta') = g_1(\alpha, \alpha')g_2(J^*(-\beta), \beta') = 0, \\ g(\alpha \otimes J^*(-\beta'), \alpha' \otimes \beta) = g_1(\alpha, \alpha')g_2(J^*(-\beta'), \beta) = 0. \end{aligned} \tag{3.21}$$

If $g_1(\alpha, \alpha') = 0$, then $g_1(\alpha, \alpha) = c^2$, $c \in R_0^+$, which means that $\alpha$ lies in a sphere $\mathbb{S}^2$, centered at the origin and with radius $c$. Next, if $g_2(J^*(-\beta'), \beta) = g_2(J^*(-\beta), \beta') = 0$, then $\beta_1{}'(s)\beta_2(s) + \beta_1(s)\beta_2{}'(s) = 0$ and thus $\beta_1(s)\beta_2(s) = c$, $c \in \mathbb{R}_0$. The last equation and (3.10) imply $\rho^2(s) = 2c/\sinh(2s)$. Consequently, $\beta$ is the curve given by (3.18).

Conversely, if $\alpha$ lies in a sphere centered at the origin in $\mathbb{R}^3$, or if $\beta$ is the curve given by (3.18), then equations (3.21) are satisfied, which implies that the conditions (3.19) hold. Therefore, $f(t,s)$ is a totally real surface, which proves the theorem. □

**Theorem 3.3.** *There are no complex tensor product surfaces $f(t,s) = \alpha(t) \otimes \beta(s)$ of a Euclidean space curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ and a Lorentzian plane curve $\beta : \mathbb{R} \to \mathbb{R}^2_1$, with respect to the pseudo-Hermitian structure $J_0$, given by $J_0(p,q,u,v,z,w) = (-q,p,-v,u,-w,z)$ on $\mathbb{R}^6_3$.*

*Proof.* Let us suppose that the tensor product $f(t,s)$ of a Euclidean space curve $\alpha(t)$ and a Lorentzian plane curve $\beta(s)$ is a complex Lorentzian immersion. By definition, the following equations are satisfied

$$g(J_0(\frac{\partial f}{\partial t}), n_i) = 0, \quad g(J_0(\frac{\partial f}{\partial s}), n_i) = 0, \quad i = 1,2,3,4,5, \tag{3.22}$$

where the vectors $n_1$, $n_2$, $n_3$, $n_4$, $n_5$ are given by (2.5) and (2.6). Moreover, $\{n_1, n_2, n_3, n_4\}$ is a basis of the normal space and $n_5 \in \text{span}\{n_1, n_2, n_3, n_4\}$. Next, equations (2.1), (2.5), (2.6), (3.20) and (3.22) imply the following equations

$$g(J_0(\frac{\partial f}{\partial s}), n_i) = 0, \quad g(J_0(\frac{\partial f}{\partial t}), n_j) = 0,$$

(3.23) $$g(J_0(\frac{\partial f}{\partial s}), n_j) = -g_1(\alpha, J_{j-2}(\alpha'))g_2(J^*(\beta'), J(\beta')) = 0,$$

$$g(J_0(\frac{\partial f}{\partial t}), n_i) = -g_1(\alpha', J_i(\alpha))g_2(J^*(\beta), J(\beta)) = 0,$$

where $i = 1, 2$ and $j = 3, 4, 5$. Since $g_1(\alpha, J_i(\alpha')) = -g_1(\alpha', J_i(\alpha))$ for $i = 1, 2, 3$ and $g_2(J^*(\beta), J(\beta)) \neq 0$, $g_2(J^*(\beta'), J(\beta')) \neq 0$, from equations (3.23) it follows that

$$g_1(\alpha, J_{j-2}(\alpha')) = 0, \quad j = 3, 4, 5.$$

The last system of equations is equivalent with the system of equations

$$\alpha_1{}'\alpha_2 - \alpha_1\alpha_2{}' = 0, \quad \alpha_2{}'\alpha_3 - \alpha_2\alpha_3{}' = 0, \quad \alpha_1{}'\alpha_3 - \alpha_1\alpha_3{}' = 0.$$

Therefore, $\alpha$ is a straight line passing through the origin. Then the tensor product surface $f(t, s)$ is not regular, which gives a contradiction. Hence the theorem is proved. □

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