Ehresmann connection on foliations generated by $\mathbb{R}^n$

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Abstract. We find sufficient conditions for the existence of the Ehresmann connection on manifold foliated by the locally free action of a commutative Lie group H in case codimension of the foliation equals 1. The connection constructed here is invariant with respect to the modified action of H.

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§ 1. Introduction.

The question of the existence of the Ehresmann connection on the foliated manifold was considered by number of authors . Usually it is assumed that this object already exists [1, 25, 18, 19, 20]. Nevertheless the existence problem was discussed before by number of authors. Note for example extensive work of G. Cairns on totally geodesic foliations [4] and monograph of P. Molino [13] which deals mainly with Riemannian foliations. There exists a number of criteria on existence of the Ehresmann connection. All of them naturally assume existence of some other structure on the foliated manifold — symplectic structure [22], vanishing cycles [24], absence of Reeb components (especially on manifolds of dimension 3) [16, 5], absence of so-called limit cycles [3], existence of the special Riemannian metric [12]. An abundant review of the existing results one can find also in [17].

Here we try to find conditions on the foliated manifold that somewhat generalise that explored before (absence of Reeb components or limit cycles).

Note that in this paper we consider foliations whose properties agree to the duality (totally geodesic foliations — Riemannian foliations) given in [22].

Let us define first Ehresmann connection on foliated manifold. Consider a manifold M with foliation F. Consider also the distribution $E \subset TM$ complement to F. We call a curve $\sigma : [0,1] \to M$ **horizontal** one if $d\sigma/dt \in E$. A curve $\gamma : [0,1] \to M$ we call **vertical** if $\gamma([0,1])$ belongs to one leaf. **Rectangle** is the mapping $\Pi : [0,1] \times [0,1] \to M$, such that for any $s \in [0,1]$ $\Pi(\cdot, s)$ is a horizontal curve, and for any $t \in [0,1]$ $\Pi(t, \cdot)$ is a vertical curve. The curves $\Pi(\cdot, 0)$, $\Pi(0, \cdot)$ are called **initial curves** of the rectangle Π.

Definition 1. [1]The distribution E is called **Ehresmann connection** if for any horizontal curve σ and vertical curve τ we can find necessarily unique rectangle for which σ and τ are the initial curves.

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Definition 2. Let us call a transversal P **H-complementary to the foliation** F if $\{hP\}_{h\in H}$ is the foliation on M complementary to F. Or, equivalently, if $hP\bigcap P\neq\varnothing$ implies $hP=P$.

Definition 3. We say that the transversal P is **homotopic to H-complementary** if we can define new group structure on H so that under it P becomes H-complementary.

Consider first a map $R: P\times H\to M$, $R(p,h)=ph$. It defines the set $B=R^{-1}(P)$.

Theorem 2. *Let (M,F) be 2-dimensional manifold foliated by the action of $H\cong\mathbb{R}$. Let there exist a total connected transversal P such that $\pi_0: B\to P$ $(\pi_0(p,h)=p)$ is a trivial covering.*

Then P is homotopic to H-complementary to the foliation F.

Description of the set B as a locally trivial fibre bundle is close to one of the graph of the foliation [23, 25].

Consider for each $x\in P$ the set $H_x=\{h\in H\mid hx\in P\}$. Note that $H_x\times\{x\}\subset B$. Now let us define a translation of $h\in H_x$ by $g\in H_x$ as follows: Let $(gx,h')\in B$ belong to the same connected component as $(x,h)\in B$. Since h' belongs to H_{gx} we have $gh'\in H_x$. Put $\Pi_g h=gh'$. In terms of functions of whose graphs consists B,

$$\Pi_g h=f_h(gx)+f_g(x). \quad (4)$$

Note that if foliation F on a manifold M does not have limit cycles [3] then translation of the element $h\in H_x$ by $g\in H_x$ is defined for any pair $h,g\in H_x$. The converse is evident for 2-dimensional manifolds.

Theorem 3. *If B is a trivial covering and for all $x\in P$ and $h\in H_x$ the translation Π_g is restriction of the shift in $\mathbb{R}^n$ then P is homotopic to H-complementary.*

Further on we consider partial case in which total transversal is diffeomorphic to $\mathbb{S}^1$.

Theorem 4. *If a transversal P is compact then $\pi_0: B\to P$ is a covering.*

It seems clear that in case we have a Reeb component on the foliated manifold M the transversal can not be compact; also if we have two of them then there is no connected total transversal.

Statement 4. *For any point $x\in P$ the set of translations $\Phi_x=\{\Pi_{\gamma,h}\}$, here $\Pi_{\gamma,h}$ are defined by the formula (4) is a transformation group of the set H_x.*

The group Φ_x acts transitively on H_x.

Statement 6. *If $\pi_0: B\to P$ is locally trivial bundle and $H\cong\mathbb{R}$ then $\pi_0: B\to P$ is globally trivial.*

Corollary 1. *Consider a 1-dimensional foliation F on a 2-dimensional manifold. If there exists a compact total connected transversal P then P is homotopic to H-complementary.*

Theorem 5. *Let a manifold M with foliation F meet the following conditions:*

1) $\mathrm{codim} F=1$.

2) *The foliation F is generated by locally free action of $\mathbb{R}^n$.*

3) *There exists a compact connected transversal on (M,F).*

Then there exist a transversal homotopic to H-complementary one and an Ehresmann connection on (M,F).

Theorem 6. *Let a manifold M with foliation F meet following conditions:*

1) $\mathrm{codim} F=1$.

2) *Foliation F is generated by action of* $\mathbb{R}^n$.

3) (M, F) *is transversally orientable.*

Then there exist a transversal homotopic to H-complimentary and an Ehresmann connection on (M, F).

§ 2. Basic existence theorem.

2.1 General assumptions

Consider a manifold M with foliation F generated by the action of locally free action of n-dimensional commutative Lie group H: $L : G \times M \to M$, $L : (g, p) \to L_g(p) = pg$.

Let $\rho : \widetilde{H} \to H$ be a universal covering. Action L induces locally free action $\widetilde{L}$ of a universal covering space $\widetilde{H}$: $\tilde{g}p = \rho(\tilde{g})p$ such that the orbits of actions coinside. Thus further on we assume that H is 1-connected, i.e. $H \cong \mathbb{R}^n$ [2].

Also we assume that

1) $\mathrm{codim} F = 1$.

2) There exists a closed transversal $P \subset M$ of the foliation F [13].

2.2 Stable subgroup of the transversal.

Since leaves of F are orbits of the action of a Lie group H each transversal P of the foliation F and each $h \in H$ gives us a submanifold hP transversal to F. Thus the action of H gives rise to the set of transversals $\{hP\}_{h \in H}$ to F.

Let P be some transversal of the foliation F (it exists by assumption 2) of a paragraph 1.1.) Put for each $x \in P$

$$H_x = \{h \in H \mid hx \in P\}.$$

Lemma 1. *The following statements are equivalent:*

i) *P is H-complementary to F transversal*;

ii) *there exists a subgroup H_0 of H such that $H_0 = H_x$ for all $x \in P$.*

Proof. i) $\Rightarrow$ ii) Let us show that for any $x, y \in P$ we have $H_x = H_y$. By definition of H-complementary transversal $h \in H_x$ implies $hP = P$, hence $h \in H_y$. Now put $H_0 = H_{x_0}$ for some $x_0 \in P$.

ii) $\Rightarrow$ i)

ii) implies that if h belongs to H_{x_0} for some $x_0 \in P$ then $hP \subset P$. Thus in this case by assumption h belongs to H_x for any $x \in P$.

Let $hP \cap P \neq \varnothing$. hen there exists x_0 such that $hx_0 = y_0 \in P$. Hence $hP \subset P$ and $h^{-1}P \subset P$ thus $hP = P$. This completes the proof. □

2.3 H-complementary transversal and Ehresmann connection.

Let a foliation F possess an H-complementary transversal P. Let us define a 1-dimensional distribution E on M as follows: First note that since the transversal P intersects all leaves of a foliation F (being orbits of action of H) we have for each $x \in M$ an element $p \in P$ and $h \in H$ such that $x = hp$. Put

$$E(x) = dh_p(T_pP).$$

Let us show that this definition is correct. Let $x = hp = h'p'$, here $h, h' \in H$, $p, p' \in P$. Then $p' = (h')^{-1}hp$, hence by definition of H-complementary transversal we get an equality $(h')^{-1}hP = P$. This implies a relation $d((h')^{-1}h)_p(T_pP) = T_{p'}P$, thus $dh'_{p'}(T_{p'}P) = dh_p(T_pP)$.

Let us show that the distribution E is differentiable. Fist note that since action of the group H is locally free each point $p_0 \in P$ possess a neighbourhood $V(p_0)$ in P and a neighbourhood $W(0)$ in H such that the mapping $\alpha : V(p_0) \times W(0) \to U(p_0)$, $(p, h) \to hp$, here $U(p_0)$ is a neighbourhood of p_0 in M is diffeomorphism. It seems clear that α maps differentiable distribution $T_pV(p_0)$ on $V(p_0) \times W(0)$ into the restriction of the distribution E on $U(p_0)$. Thus a distribution E is differentiable i the neighbourhood of each point $p \in P$. Now note that by construction a distribution E is H-invariant. Thus E is differentiable in each point of M.

The distribution E is tangent to submanifolds hP by construction. Thus the set of submanifolds $\{hP\}$ gives rise to 1-dimensional foliation on M.

Theorem 1. *The distribution E defines an Ehresmann connection on M.*

Proof. The distribution E is complementary to the distribution TF by construction. Let us check that each horizontal curve σ and vertical curve τ define a rectangle for which σ and τ become initial curves (cf. definition [1]).

Let $\Pi : [0,1] \times [0,1] \to M$ be a rectangle. Then H-invariance of the distribution E implies that for any $h \in H$ the mapping $h\Pi : [0,1] \times [0,1] \to M$ also defines a rectangle. Note now that for any point $p \in M$ there exists $h \in H$ such that $h\,p$ belongs to P moreover the shift by h maps arbitrary horizontal (vertical) curve beginning in p into horizontal (vertical) curve starting at $h\,p$. Thus it suffices to prove that there exists a rectangle Π with initial curves σ and τ for any vertical curve σ and horizontal curve τ such that $\sigma(0) = \tau(0) = p_0 \in P$.

Since the mapping $\phi : H \to L_0$, $h \to hp_0$ is a universal covering (cp. [6]), we get that there exists a unique curve $h : [0,1] \to H$ such that $h(0) = 0 \in H$ and $\tau(t) = h(t)p_0$. Put

$$\Pi : [0,1] \times [0,1] \to M, \quad \Pi(s,t) = h(t)\sigma(s).$$

It seems clear that Π is rectangle. □

2.4 Conditions of the existence of the H-complementary transversal.

Statement 1. *The mapping $R : P \times H \to M$, $R(h,p) = ph$ is a local diffeomorphism.*

Proof. It suffices to show that for any $h \in H$, $p \in M$ the differential $dR : T_{(p,h)}(P \times H) \to T_{ph}M$ is isomorphism. Consider following diffeomorphisms:

$$R_h : M \to M, p \to ph$$
$$\widetilde{R}_h : P \times H \to P \times H, (p, h') \to (p, hh')$$

It seems clear that the followwing diagram is commutative:

$$\begin{tikzcd}
P \times H \arrow[r, "R"] \arrow[d, "\widetilde{R}_h"'] & M \arrow[d, "R_h"] \\
P \times H \arrow[r, "R"] & M
\end{tikzcd}$$

Hence it suffices to prove that

$$dR_{p,e} : T_{p,e}P \times H \to T_p M$$

is isomorphism. The definition of R implies that the following diagram is commutative:

$$\begin{tikzcd}
T_{(p,e)}(P \times H) \arrow[r, "dR_{p,e}"] \arrow[d] & T_p M \arrow[d] \\
T_p P \oplus T_e H \cong T_p P \oplus \mathfrak{g}(H) \arrow[r, "A"] & T_p P \oplus T_p L(p)
\end{tikzcd}$$

here $L(p)$ is the leaf of the foliation passing through p and vertical arrows are isomorphisms. Then the linear operator A has the following properties:

$$\forall V \in T_p P \quad A(V) = V, \qquad \forall a \in \mathfrak{g}(H) \quad A(a) = \sigma(a).$$

Since the action of H on M is locally free the mapping $\mathfrak{g}(H) \to T_p M$, $a \to \sigma(a)$ is monomorphism. Then the dimensional considerations give us that A is isomorphism. Hence $dR_{(p,e)}$ is also an isomorphism. □

Corollary 1.1. *The set $B = R^{-1}(P)$ is a submanifold of $P \times H$ of dimension equal to that of P.*

Proof. The mapping $R : P \times H \to M$ is local diffeomorphism. Hence its images are transversal to any submanifold of M. Hence the inverse image of the submanifold is a submanifold (cf. Theorem 3.2 of the Chapter 1 from [8]). □

2.4.1 Canonical covering.

The set B naturally possess two smooth maps: $\pi_0, \pi_1 : B \to P$, $\pi_0(p,h) = p$, $\pi_1(p,h) = ph$.

Statement 2. *The map $\pi_0 : B \to P$ is a local diffeomorphism.*

Proof. It suffices to show that

$$d\pi_0 : T_{(p,h)}B \to T_p P$$

is an isomorphism.

The local diffeomorphism R maps H ino the leaf transversal to P, hence B is transversal to H it the point $(p,h) \in B$. Thus

$$T_{(p,h)}B \subset T_{(p,h)}(P \times H) \cong T_p P \oplus T_h H$$

is transversal

$$T_{(p,h)}H \subset T_{(p,h)}(P \times H) \cong T_p P \oplus T_h H.$$

Hence, $\ker d\pi_0 = \{0\}$. The statement now follows from dimensional considerations. □

2.4.2 Conditions of existence of the H-complementary transversal on 2-dimensional manifold.

Our aim is to determine conditions under which the given transversal is homotopic to H-complementary.

Theorem 2. *Let (M, F) be a 2-dimensional manifold foliated by locally free action of a commutative Lie group $H \cong \mathbb{R}$. Let there exist a total connected closed transversal P on M for which $\pi_0 : B \to P$ is globally trivial.*

Then P is homotopic to H-complementary to the foliation F.

Proof. Fix a point $x \in P$. Let us show that $H_x \cong \mathbb{Z}$.

a) By assumption $\pi_0 : B \to B$ is a covering hence the leaf $\pi_0^{-1}(x) = \{(x, h) \mid h \in H_x\}$ is discrete in B. Thus this set is discrete also in $\{x\} \times H$. So, H_x is discrete in $H \cong \mathbb{R}$.

b) Let us prove that H_x is infinite. Consider $x \in P$ and $h \in H_x$. Let $\gamma : [0, 1] \to P$, $\gamma(0) = x$, $\gamma(1) = hx$. Let also $\widetilde{\gamma} : [0, 1] \to B$ be such that $\gamma(0) = (x, h)$, $\gamma(1) = (hx, h')$, here $h' \in H_{hx}$.

Since $\pi_0 : B \to P$ is globally trivial and $B_0 = \{(x, 0) \mid x \in P\}$ is a connected component of B the inequality $h' \neq 0$ holds true. Assume that $h' > 0$, hence $h_1 > h$.

Consider $h_1 = h + h'$, then $h_1 \in H_x$. Thus we construct $h'' \in H_{h_1x}$ and $h_2 = h_1 + h'' \in H_x$ by point hx and h_1. As (x, h), (hx, h'), (h_1x, h'') belong to the same connected component of B which does not coincide with B_0 $h'' > 0$ and $h_2 > h_1$. Continuing this process we obtain an increasing sequence of the elements of H_x. Hence, H_x is infinite.

c) Let us prove that the set H_x does not have either minimum or maximum.

To show this note that we have a natural order on $H_x \subset \mathbb{R}$. Assume that there exists a minimal element $h_{\min}$ for $x \in P$.

Let $h_{\min} = 0$. Consider $h > 0$ from H_x. Then $-h \in H_{hx}$. Now consider a curve $\gamma : [0, 1] \to P$ such that $\gamma(0) = hx$, $\gamma(1) = x$. Let $\widetilde{\gamma} : [0, 1] \to B$ be the lift of γ such that $\widetilde{\gamma}(0) = (hx, -h)$, $\widetilde{\gamma}(1) = (x, h')$. Then $h' < 0$ since the connected component of B which contains $\widetilde{\gamma}$ does not intersect B_0. Since by construction $h' \in H_x$ we arrive to the contradiction with the minimality of $h_{\min} = 0$.

Now let $h_{\min} < 0$. As in b) we construct the sequence $h_k \in H_x$ by point x and element $h_{\min}$. Then this sequence decreases. This again contradicts the assumption of minimality of $h_{\min}$.

d) Let us construct a diffeomorphism $\phi_0 : P \times \mathbb{R} \to P \times \mathbb{R}$, $\phi(B) = P \times \bigcup\limits_{i \in \mathbb{Z}} \{i\}$.

Paragraphs a)-c) imply that $B \subset P \times \mathbb{R}$ consists of disjoint union of the graphs of functions f_i, $i \in \mathbb{Z}$. Let us construct a mapping $\Phi_1 : P \times \mathbb{R} \to P \times \mathbb{R}$, $(x, y) \mapsto (x, \phi_1(x, y))$. The map $\phi_1(x, \cdot)$ for any fixed $x \in P$ is diffeomorphism of $\mathbb{R}$ which meets the following conditions:

$$\phi_1(x, y) = \begin{cases} 0, & \text{if } y = 0, \\ y/f_1(x), & \text{if } y \geq f_1(x), \\ y/|f_{-1}(x)|, & \text{if } y \leq f_{-1}(x) \end{cases}$$

Then $\Phi_1((x, f_1(x)) = (x, 1)$, $\Phi_1((x, f_{-1}(x)) = (x, -1)$. Consider now a diffeomorphism $\Phi_2 : P \times \mathbb{R} \to P \times \mathbb{R}$, $(x, y) \mapsto (x, \phi_2(x, y))$. Here ϕ_2 is again a diffeomorphism of $\mathbb{R}$ for each $x \in P$ which meets the following conditions:

$$\phi_2(x,y) = \begin{cases} y, & \text{if } -1 \leq y \leq 1, \\ 2f_1(x)y/f_2(x), & \text{if } y \geq f_2(x)/f_1(x), \\ 2|f_{-1}(x)|y/|f_{-2}(x)|, & \text{if } y \leq f_{-2}(x)/|f_{-1}(x)| \end{cases}$$

Note that $\Phi_2(\Phi_1)|_{\{(x,y)|f_{-1}(x)\leq y\leq f_1(x)\}} = \Phi_1|_{\{(x,y)|f_{-1}(x)\leq y\leq f_1(x)\}}$. Let us construct diffeomorphisms $\Phi_3, \Phi_4, \ldots$ in similar way. It follows from the construction that for each point (x,y) there exists a natural number n such that

$$\Phi_k(x,y)\cdots\Phi_{n+1}\Phi_n\cdots\Phi_1(x,y) = \Phi_n\cdots\Phi_1(x,y).$$

The desired diffeomorphism now is $\phi_0 = \Phi_1 \circ \Phi_2 \circ \cdots$.

e) Let us define now new group action. First put $h \star x = \pi_1(\phi_0^{-1}(x,h))$ for $h \in \mathbb{R}$ and $x \in P$.

Now consider any $y \in M$. Then there exist $x' \in P$ and $h' \in H$ such that $h'x' = y$. Put $h \star y = (h+h') \star x'$. Let us prove that this definition is correct, i.e. it does not depend on the choice of h' and x'. Assume that $y = h'x' = h''x''$. Then $h'' - h' \in H_{x''}$ and $x' = (h'' - h')x''$. Hence

$$(h+h'')x'' = ((h+h') + (h''-h'))x = (h+h')x'$$

so the action $\star$ is correctly defined.

Finally let us show that the transversal P is H-complementary with respect to the action $\star$.

Let $h \star P \cap P \neq \varnothing$. Then there exist $x \in P$ and $h \in H$ such that $h \star x \in P$. Let $\phi_0^{-1}(x,h) = (x,h')$ then

$$h \star x = \pi_1(\phi_0^{-1}(x,h)) = h'x \in P.$$

Hence $h'x \in P$, so $h' \in H_x$. Since $(x,h) = \phi_0(x,h')$ we get $h \in \mathbb{Z}$.

Now for any $x \in P$, $\phi_0^{-1}(x,h) = (x,h')$, here $h' \in H_x$. Then $h \star x = h'x \in P$. Thus $h \star P = P$. □

2.4.3 Sufficient condition of the existence of the H-complementary transversal in case $\dim H > 1$

Now we assume that $\dim H > 1$ and the bundle B is trivial. Then $B \subset P \times H$ consists of the set of graphs of the vector-functions $P \to H \cong \mathbb{R}^n$. Let us fix a point $x_0 \in P$. Let us denote by $f_h(x)$ a vector-function whose graph is a connection component of B passing through (x_0,h) for any $h \in H_{x_0}$. Then $f_h(x_0) = h$.

Now let us define a translation of $h \in H_x$ by $g \in H_x$ as follows: Let $(gx,h') \in B$ belong to the same connected component as $(x,h) \in B$. Since h' belongs to H_{gx} we have $gh' \in H_x$. Put $\Pi_g h = gh'$. In terms of functions of whose graphs consists B,

$$\Pi_g h = f_h(gx) + f_g(x).$$

Theorem 3. *If B is a trivial covering and for all $x \in P$ and $h \in H_x$ the translation Π_g is the restriction of a shift in $\mathbb{R}^n$ then P is homotopic to H-complementary.*

Proof. a) Fix $x \in P$. Since Π_g is a shift for each $g \in H_x$, we get $\Pi_g(h) = h + q(g, x)$. Let us put in the last equation $h = 0$. Then

$$q(g, x) = \Pi_g(0) = f_0(gx) + f_g(x) = f_g(x),$$

since $f_0(x) = 0$ for any x. Henceforth

$$f_h(gx) + f_g(x) = \Pi_g(h) = h + f_g(x), \tag{2.1}$$

thus

$$f_h(gx) = h = f_h(x)$$

for any $g \in H_x$.

b) Condition (2.1) implies that

$$f_{g+h}(x) = f_g(x) + f_h(x), \tag{2.2}$$

since $h = f_h(x)$.

Let us choose a base h_1, h_2, ... , h_q of the group H_{x_0}. Then (2.2) implies that for any $x \in P$ $f_{h_1}(x)$, $f_{h_2}(x)$, ... , $f_{h_q}(x)$ is a base of H_x. Note that the system $\{f_i(x)\}$ is linearly independent over $\mathbb{R}$ for any x. Now assume that $f_{q+1}(x)$, ... , $f_n(x)$ is the set of vector-functions such that

$$f_{h_1}(x), f_{h_2}(x), \dots, f_{h_q}(x), f_{q+1}(x), \dots, f_n(x) \tag{2.3}$$

is a base of H for any x.

c) Now we construct a diffeomorphism $\phi_0 : P \times H \to P \times H$, $\phi_0(B) = P \times \mathbb{Z}^q$ using base (2.3) similarly to the paragraph d) of the proof of theorem 2. Put

$$h \star x = \pi_1(\phi_0^{-1}(x, h)), \quad x \in P, h \in H$$

Now take $y \in M$. Then there exist $x' \in P$ and $h' \in H$ such that $h'x' = y$. Put $h \star y = (h + h') \star x'$. The correctness of the definition is proved as theorem 3. □

Note 1. We may not use the translation to be a shift. We may use only triviality of B and commutativity of the set of translations Π_g. If $H \simeq \mathbb{R}^2$ then we are able to construct a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ passing throughh_i, $i \in \mathbb{Z}$ without selfintersections. The construction of such a curve is inductive. We note first that after the addition of the n-s point to the curve γ the set $\mathbb{R}^2 \setminus \gamma_n$ remains 1-connected. Thus we can add the next point so that the curve γ keeps desired properties. Hence, discreteness of $L \bigcap P$ and closedness of P imply that the curve γ passes through $\{\infty\}$ in the extended plane $\mathbb{R} \cup \{\infty\} \cong \overline{\mathbb{C}}$.

Jordan theorem then says that this curve divides $\mathbb{R}^2 \bigcup \{\infty\}$ into two different 1-connected sets. Hence, Riemann theorem gives us the transformation $f : \mathbb{C} \to \mathbb{C}$ such that these sets map into halfplanes and a curve γ — to the line separating them.

Since different translations commute curves corresponding to different h_i intersect only by 0.

2.4.4 Condition of triviality of the covering $\pi_0 : B \to P$

Let $\pi_0 : B \to P$ be a covering. Then the curve $\gamma : [0,1] \to P$ and $b \in \pi_0^{-1}(\gamma(0))$ possess the lift $\widetilde{\gamma} : [0,1] \to P$, $\widetilde{\gamma}(0) = b$. Hence we get a parallel translation of the points of the leaf along path in P.

Statement 3. *Assume that for any elements $x \in P$, $g \in H_x$ and the path $\gamma : [0,1] \to P$ such that $\gamma(0) = x$, $\gamma(1) = gx$, the parallel translation along γ is a shift. Then $\pi_0 : B \to P$ is a trivial covering.*

Proof. If $P \cong \mathbb{R}$ then any covering is trivial.

Assume now that $P \cong \mathbb{S}^1$. Since B_0 maps into itself under parallel translation the shift is necessarily trivial. Hence holonomy group is trivial as is the bundle. □

2.4.5 Examples

Let a covering $\pi_0 : B \to P$ be not necessarily trivial. Let us define a translation of the points of the leaf $H_x = \pi_0^{-1}(x)$ along paths γ connecting x with gx, here $g \in H_x$. In this case we have a unique path $\widetilde{\gamma}$ in B starting into $h \in H_x$ and ending into $h' \in H_{gx}$. Put $\Pi_{\gamma,g}(h) = h'g \in H_x$. Thus $\Pi_{\gamma,g} : H_x \to H_x$.

Let a leaf L of the foliation F be diffeomorphic to $\mathbb{R}^n$, and intersect P only in finite set of points. Then for any $x \in L \cap P$ neither of $\Pi_{\gamma,g}$ is a shift. Assume on the contrary that the set $\{(\Pi_{\gamma,g})^k\}_{k \in N}$ is infinite then the set $L \cap P$ is also infinite.

If a transversal P is H-complementary and a leaf L of the foliation F is diffeomorphic to $\mathbb{R}^n$ then $L \cap P$ either consists only of one point or infinite. Assume that there exists $x \in P$ such that for some $h \in H$ $hx \in P$ then h belongs to H_x. But $H_x = H_P$ by lemma 1. Hence $h^k x$ belongs to $L \cap P$ for any $k \in \mathbb{N}$. Since $L \cong \mathbb{R}^n$ the points $h^k x$ are different.

Example 1. Consider a foliation on $M = \mathbb{R}^2 \times \mathbb{S}^1$ as the natural foliation with the leaf $\mathbb{R}^2$. Consider a trajectory of a rational flow on $\mathbb{S}^1 \times \mathbb{S}^1$ as transversal P. Since the set of intersection points of the leaf with P is finite, the transversal P does not give rise to the Ehresmann connection invariant under the action of $\mathbb{R}^2$.

Note 2. If we do not want to deform action of a group then we may consider a set of transversals which may converge (pointwise) to the transversal which defines Ehresmann connection. Assume as in the proof of the previous statement that $x \in P$, $h_1 \in H_x$. Put $P_1 = h_1 P$. Then $P_2 = (h_1 + 1/2h(t))P_1$, here $h(0) = h_{12}(0)$, $h_{12}^{-1}(x)x \in h_1 P$ for $x \in P$ and $h(t_0) = e$ for $P(t_0) = h_1 x$; we assume also that $[0,1]h_{12}^{-1} \bigcap P_2 = \varnothing$. And so on. This process converges if and only if the series $\sum\limits_{i \in \mathbb{N}} a_i$ with general summand $a_i = h_{i,i+1}(x)$ converges for any leaf of the foliation F.

Example 2. Consider a foliation on $\mathbb{S}^1 \times \mathbb{R}$, generated by images of curves l_b: $y = ax + b$, (the constant $a \neq 0$), $b \in \mathbb{R}$ under natural mapping $\mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}$, $(x,y) \mapsto ([x], y)$, here $[x] \in \mathbb{S}^1$ is the equivalence class of $x \in \mathbb{R}$ with respect to the relation $x \sim x + k$, $k \in \mathbb{Z}$. Consider the transversal P given by the following equation in the coordinate chart $U \cong (-1/2, 1/2) \times \mathbb{R}$, $x \in \mathbb{R}$, here $y \in (-1/2, 1/2)$, $(-1/2, 1/2)$ is the coordinate chart on $\mathbb{S}^1$:

$$y = \begin{cases} 0 & x \leq 0; \\ x & x \in (0, 1/4]; \\ -x + 1/2 & x \in (1/4, 1/2]; \\ 0 & x > 1/2 \end{cases}$$

This subset of M is not a smooth manifold (though it can be approximated by a smooth manifold). Nevertheless it suffices to consider this set for the illustration of the method. This transversal converges to the curve given by $y = 0$.

Example 3. Consider a foliation on $\mathbb{R}^2 \setminus \{0\}$ generated by circles with common center in 0. Let us define action of the group $\mathbb{R}$ as follows: $R : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$: $R_h x = e^{ih/|x|}x$. Then there is no connection invariant under the action of the group H. Note that at the same time we are able to deform action of $\mathbb{R}$ (following theorem 2) to the following: $R^{'}_h x = e^{ih}x$. Let us show that there is no invariant Ehresmann connection. Assume that it exists. Then the tangent vector at the point $x \in \mathbb{R}^2$ to the transversal under the action of an element from isotropy group $H_x \cong \mathbb{Z}$ of the leaf L_x, comprising x maps to the collinear vector. But this is impossible since $\frac{d}{dr}e^{ih/r} = -\frac{1}{r^2}e^{ih/r} \neq 0 \mathrm{mod} \pi$. Thus this vector turns by angle which is not a multiple of π.

§ 3. Existence of H-complementary transversal if there exists a compact transversal on M.

Let (M, F) be a foliation of codimension 1 generated by locally free action of the commutative group H. Let P be a transversal of the foliation F and $\pi_0 : B \to P$ be a map defined in paragraph 1.4.1.

Theorem 4. *If the transversal P is compact then $\pi_0 : B \to P$ is covering.*

Proof. a) Assume that H is compact then leaves of the foliation F are also compact. Then for any point $x \in P$ the set H_x is finite.

Let us recall that the map $R : P \times H \to M$, $(x, h) \to hx$ is local diffeomorphism. Now leaves of the trivial bundle $\pi : P \times H \to P$ pass into leaves of the foliation F under the map R and P is transverse to the foliation F. Consider $(x, h) \in B = R^{-1}(P)$. Then $B \subset P \times H$ is transverse to the leaves of π. Hence there exist a neighbourhood $V_h(x)$ and a mapping $s_h : V_h(x) \to B$ such that $\pi_0 s_h = \mathrm{Id}_{V(x)}$ and $s_h(x) = (x, h)$.

Consider $V(x) = \bigcap\limits_{h \in H_x} V_h(x)$. Then for any two points $x, y \in P$ consider the path $\gamma : [0, 1] \to P$ such that $\gamma(0) = x$, $\gamma(1) = y$. There exists a division $t_0 = 0 < t_1 < \ldots t_N = 1$ such that $\gamma([t_i, t_{i+1}]) \subset V(\gamma(t_i))$. Then for any $h \in H_x$ we have $s_h(\gamma(t_1) = (\gamma(t_1), h_1)$ and

$$s_h(V(\gamma_{t_0}) \cap s_{h_1}(V(\gamma(t_1)) \neq \varnothing .$$

Thus there exists a path (necessarily unique) $\widetilde{\gamma} : [t_0 = 0, t_1] \to B$ such that $\widetilde{\gamma}(0) = (x, h)$ $\pi_0 \widetilde{\gamma}(t) = \gamma(t)$, $t \in [0, t_1]$.

Similarly we construct a unique path $\widetilde{\gamma} : [0, 1] \to B$ such that $\widetilde{\gamma}(0) = (x, h)$ $\pi_0 \widetilde{\gamma}(t) = \gamma(t)$, $t \in [0, 1]$. Thus $\pi_0 : B \to P$ is a covering.

b) Assume that H is not compact.

Let $V(x_0)$ be a neighbourhood of he point x_0 in P. Let us prove that for any $h_0 \in H_{x_0}$ there exists a map $s_{h_0} : V(x_0) \to B$ such that $\pi_0 s_{h_0} = \mathrm{Id}_{V(x_0)}$ and $s_{h_0}(x_0) = (x_0, h_0)$.

Note that repeating considerations of a) we show that for $h_0 \in H_{x_0}$ there exists s_{h_0} defined in a neighbourhood $U(x_0) \subset V(x_0)$ depending on the choice of h_0. Consider $U(x) = (x_1, x_2)$. Let us show that there exists $\lim\limits_{x \to x_1} s_{h_0}(x)$.

Let $s_{h_0}(x) = (x, h(x))$. By compactness of P the angle of the inclination of the leaves of F to P is bounded from above. Hence $|h'(x)| \leq K$, $x \in U(x_0)$. Then $|h(x)| \leq |h(x_0)| + K|x - x_0|$. Since $|h'|$ is bounded on $U(x)$ h is a function with finite change. Then by Corollary 2 of the theorem 6 from the fifth chapter of [14] there exist a limit $\lim\limits_{x \to x_1} h(x)$.

Let $\lim\limits_{x \to x_1} s_{h_0}(x) = (x_1, h_1)$. Then there exists $s_{h_1}(x)$ defined in a neighbourhood $U(x_1)$ such that $\pi_0 s_{h_1} = \mathrm{Id}_{U(x_1)}$ and $s_{h_1}(x_1) = (x_1, h_1)$. It seems clear that for any $x \in U(x_0) \cap U(x_1)$ $s_{h_1}(x) = s_{h_0}(x)$. Then we proceed as in the case of the compact group H. □

3.1 Properties of the set of translations.

Let $\pi_0 : B \to P$ be a covering. Consider $x \in P$, $g \in H_x$, $\gamma : [0, 1] \to P$, $\gamma(0) = x$, $\gamma(1) = gx$. Then for any $(x, h) \in \pi_0^{-1}(x)$ there exists a unique path $\widetilde{\gamma} : [0, 1] \to B$, $\widetilde{\gamma}(0) = (x, h)$, $\widetilde{\gamma}(1) = (gx, h')$ covering γ. Let us define a mapping

$$\Pi_{\gamma,g} : H_x \to H_x, \quad \Pi_{\gamma,g}(h) = h'g. \tag{3.4}$$

Let us call $\Pi_{\gamma,g}$ a translation of the leaf $H_x = \pi_0^{-1}(x)$ along γ connecting x to $gx \in P$.

Statement 4. *For any point $x \in P$ the set of translations $\Phi_x = \{\Pi_{\gamma,h}\}$, here $\Pi_{\gamma,h}$ are defined by the formula (3.4) is a group of transformations of the set H_x.*

This group Φ_x acts transitively on H_x.

Proof. Let us prove first that the set Φ_x is closed under composition.

Consider $\gamma_1 : [0, 1] \to P$, $\gamma_2 : [0, 1] \to P$, $\gamma_1(0) = \gamma_2(0) = x$, and $g_1, g_2 \in H_x$ such that $g_1 x = \gamma_1(1)$, $g_2 x = \gamma_2(1)$. Let us show that there exist $\gamma_0 : [0, 1] \to P$, $\gamma_0(0) = x$, and $g_0 \in H_x$ (here $g_0 x = \gamma_0(1)$) such that

$$\Pi_{\gamma_2,g_2}(\Pi_{\gamma_1,g_1}(h)) = \Pi_{\gamma_0,g_0}(h), \forall h \in H_x. \tag{3.5}$$

Let $\widetilde{\gamma}_1(t) = (\gamma_1(t), h_1(t))$ be the lift of the path γ_1, $\widetilde{\gamma}_1(0) = (x, h)$ and $\widetilde{\gamma}_2'(t) = (\gamma_2(t), h_2(t))$ be the lift of γ_2, $\widetilde{\gamma}_2'(0) = (x, g_1 + h_1(1))$, then $\widetilde{\gamma}_2''(t) = (\gamma_2(t), g(t))$ is the lift of the curve γ_2, $\widetilde{\gamma}_2''(0) = (x, g_1)$. Consider

$$\delta : [0, 1] \to P, \delta(t) = \pi_1(\widetilde{\gamma}_2(t)) = g(t)\gamma_2(t).$$

Now put $\gamma_0 = \delta \cdot \gamma_1$, here $\cdot$ is the composition of the paths. Put also $g_0 = g(1)g_2$. Let us show that then (3.5) holds true.

Let $\widetilde{\delta} : [0, 1] \to B$, $\widetilde{\delta}(t) = (\delta(t), h_0(t))$, $\widetilde{\delta}(0) = (g_1 x, h_1(1))$ be the lift of the curve δ.

Now in the equality (3.5) we have $\Pi_{\gamma_2,g_2}(\Pi_{\gamma_1,g_1}(h)) = g_2 + h_2(1)$ on one side; and $\Pi_{\gamma_0,g_0}(h) = h_0(1) + g(1) + g_2$ on the other. Let us show that these elements coincide. It suffices to prove that $h_0(1) + g(1) = h_2(1)$. Note first that $h_0(t) + g(t) \in H_{\gamma_2(t)}$ since $g(t) \in H_{\gamma_2(t)}$, moreover $g(t)\gamma_2(t) = \delta(t) \quad h_0(t) \in H_{\delta(t)}$. Hence the path $\Gamma(t) = (\gamma_2(t), h_0(t) + g(t))$ belongs to B and $\Gamma(0) = (x, g_1 + h_1(1))$, i.e. $\Gamma = \widetilde{\gamma}_2$. So, by uniqueness of the covering path we get $h_0(1) + g(1) = \Gamma(1) = \widetilde{\gamma}_2(1) = h_2(1)$.

Let us now show that there exists an inverse element for any $\Pi_{\gamma,h} \in \Phi_x \quad \Phi_x$. To prove this consider a path $\gamma^{-1} : [0,1] \to P$, $\gamma^{-1}(0) = gx$, $\gamma^{-1}(1) = x$. Let $\widetilde{\gamma}_1(t) = (\gamma^{-1}(t), h_1(t))$ be the lift of the curve γ^{-1}, $\widetilde{\gamma}_1(0) = (hx, -h)$. Now consider $\gamma'(t) = h_1(t)\gamma^{-1}(t)$ and $h' = h_1(1)$.

Let us show that $\Pi_{\gamma',h'}$ is inverse to $\Pi_{\gamma,h}$. The identity (3.5) implies that

$$\Pi_{\gamma,h}\Pi_{\gamma',h'} = \Pi_{\gamma\cdot\gamma^{-1},h+h_1(0)} = \Pi_{\gamma\cdot\gamma^{-1},0}.$$

Since the loop $\gamma'' = \gamma\cdot\gamma^{-1}$ is contractible the lift $\widetilde{\gamma}''$ of the loop γ'', starting into (x,g) also ends in (x,g) for each $g \in H_x$. So by the definition of $\Pi_{\gamma,h}$ we get $\Pi_{\gamma'',0} = \mathrm{Id}$. Thus the set Φ_x is transformation group of H_x.

Note now that since P is connected $0 \in H_x$ can be mapped into any $h \in H_x$ by element from Φ_x. Then action of Φ_x on H_x is transitive. □

Statement 5. *If* $\Pi_{\gamma,g} = \mathrm{Id}$ *then* $g = 0$ *and* γ *is the loop in* P.

Proof. Since there exists a zero section $B_0 \subset B$ $\Pi_{\gamma,g}(0) = g + 0 = 0$, i.e. $g = 0$. □

Statement 6. *If* $\pi_0 : B \to P$ *is locally trivial bundle and* $H \cong \mathbb{R}$ *then* $\pi_0 : B \to P$ *is trivial.*

Proof. If P is diffeomorphic to $\mathbb{R}$ then the covering $\pi_0 : B \to P$ is naturally trivial.

Now let P be diffeomorphic to $\mathbb{S}^1$. Recall that the bundle $\pi_0 : B \to P$ possess global section $\sigma : P \to B$, $\sigma(x) = (x,0)$. Now consider a loop $\delta : [0,1] \to P$, $\delta(0) = \delta(1) = x$. Let $(\delta(t), h(t))$ be the lift of the path δ into B such that $(\delta(0), h(0)) = (x,h)$. Since the group $\mathbb{R}$ is orientable $h_1 = h(1)$ has the same sign as h. Assume that $h > 0$. If $h_1 < h$ we consider the loop $\delta\cdot\delta$ and h_2 which is generated similarly to h_1. Again we have $h_2 < h_1$. Let us continue the construction of h_n. Since the sequence $(h_n)_{n\in\mathbb{N}}$ decreases and is bounded from below ($\forall n \in \mathbb{N}$, $0 < h_n$ since B_0 is connected component; then if the lift of the path starts into the point which does not belong to B_0 this lift does not intersect B_0), there exists an element $h_0 \in H$ such that $h_n \to h_0$, $n \to \infty$. Since action of H is continuous $h_0 \in H_x$. But H_x is discrete. This is a contradiction. Assume that $h_1 > h$. Consider a set h_1, δ^{-1}, h instead of the set h, δ, h_1 of the previous considerations. Then we again arrive to the contradiction. Hence for any loop $\delta \subset P$ and $h \in H_x$ and the lift $(\delta(t), h(t))$ into B of the curve δ such that $(\delta(0), h(0)) = (x,h)$ we get $h(1) = h$.

□

Corollary 1. *Let a foliation* F *on* 2*-dimensional manifold be generated by* $\mathbb{R}$. *Let* P *be a total connected compact transversal to* F. *Then* P *is homotopic to* H*-complementary.*

Proof. By statement 6 $\pi_0 : B \to P$ is globally trivial. Now apply theorem 2. □

Lemma 2. *Assume that for $x \in P$, $h \in H_x$ the paths $\gamma, \gamma' \subset P$, $(\gamma(0) = \gamma'(0) = x$, $\gamma(1) = \gamma'(1) = hx)$ are homotopic to each other then $\Pi_{\gamma,h} = \Pi_{\gamma',h}$.*

Proof. If paths are homotopic then their lifts into B are also homotopic to each other and end at the same point. □

Corollary 2. *The set Φ_x is countable for any $x \in P$.*

Proof. There are no more than countable number of not homotopic to each other paths between any pair of points of $\mathbb{S}^1$. The set of intersection points $L \bigcap P$ is also at most countable for any $L \in F$. Hence the set Φ_x is countable. □

Statement 7. *For any pair of points $x, y \in P$, $\Phi_x \cong \Phi_y$.*

Proof. Let a pair $x, y \in P$ be fixed. Let us construct a map $\psi : \Phi_x \to \Phi_y$. Let us fix a leaf $\delta : [0,1] \to P$ such that $\delta(0) = x$, $\delta(1) = y$.

Consider $\Pi_{\gamma,g} \in \Phi_x$. There exists the lift δ_g of the path δ $\delta_g = (\delta(t), g(t))$ into B starting at (x, g). Consider a path $\delta_1 : [0,1] \to P$, $\delta_1(t) = \delta(t)g(t)$. Put $\psi(\Pi_{\gamma,g}) = \Pi_{\delta^{-1}\cdot\gamma\cdot\delta_1, g(1)}$.

Let us show that $\psi(\Pi_{\gamma_1,g_1} \circ \Pi_{\gamma_2,g_2}) = \psi(\Pi_{\gamma_1,g_1}) \circ \psi(\Pi_{\gamma_2,g_2})$. Let $(\gamma_2(t), g_1(t))$ be the lift of the path γ_2 into B, $(\gamma_2(0), g_1(0)) = (x, g_1)$. Consider $\gamma_2^{'} : [0,1] \to P$, $\gamma_2^{'}(t) = g_1(t)\gamma_2(t)$. Let $(\delta(t), g'(t))$ be the lift of δ into B such that $(\delta(0), g'(0)) = (x, g_1(1) + g_2)$. Consider $\delta' : [0,1] \to P$, $\delta'(t) = \delta(t)g'(t)$. Then $\psi(\Pi_{\gamma_1,g_1} \circ \Pi_{\gamma_2,g_2}) = \psi(\Pi_{\gamma_1\cdot\gamma_2^{'}, g_1(1)+g_2}) = \Pi_{\delta^{-1}\cdot\gamma_1\cdot\gamma_2^{'}\cdot\delta', g'(1)}$.

On the other hand $\psi(\Pi_{\gamma_1,g_1}) = \Pi_{\delta^{-1}\cdot\gamma_1\cdot\delta_1^{'}, g_1^{'}(1)}$, $\psi(\Pi_{\gamma_2,g_2}) = \Pi_{\delta^{-1}\cdot\gamma_2\cdot\delta_2^{'}, g_2^{'}(1)}$. Here $(y, g_i^{'}(1))$ is the endpoint of the lift $(\delta(t), g_i^{'}(t))$ of the path δ starting at (x, g_i), $i = 1, 2$. $\delta_i^{'}(t) = \delta(t)g_i^{'}(t)$. Then $\psi(\Pi_{\gamma_1,g_1}) \circ \psi(\Pi_{\gamma_2,g_2}) = \Pi_{\delta^{-1}\cdot\gamma_1\cdot\delta_1^{'}, g_1^{'}(1)} \circ \Pi_{\delta^{-1}\cdot\gamma_2\cdot\delta_2^{'}, g_2^{'}(1)} = \Pi_{\delta^{-1}\cdot\gamma_1\cdot\gamma_2^{'}\cdot\delta', g'(1)}$ since the lift $\delta^{-1}\cdot\gamma_2\cdot\delta_2^{'}$ into B starting at $(y, g_1^{'}(1))$ is $(\delta(t), g_1^{'}(t))^{-1} \cdot (\gamma_2(t), g_1(t)) \cdot (\delta(t), g^{'}(t))$.

To prove that ψ is isomorphism we first show that $\mathrm{Ker}(\psi) = \{e\}$. Let $\psi(\Pi_{\gamma,g}) = \Pi_{\delta^{-1}\cdot\gamma\cdot\delta', g'} = \mathrm{Id}$. Hence $g' = 0$ and by construction of ψ we get $g(t) = 0$. Hence γ is a loop and $\delta' = \delta^{-1}$. But $\delta' \cdot \gamma\delta^{-1}$ is contractible. Hence the loop γ is also contractible. Then $\Pi_{\gamma,g} = \Pi_{const_x,0} = Id$. Thus ψ is a monomorphism.

Let us show first that there exists an element $\Pi_{\gamma,g} \in \Phi_x$ such that $\Pi_{\gamma',g'} = \psi(\Pi_{\gamma,g})$ for arbitrary $\Pi_{\gamma',g'} \in \Phi_y$. To do this consider the lift $((\delta)^{-1}(t), g(t))$ of the path $(\delta)^{-1}$ into B such that $((\delta)^{-1}(0), g(0)) = (y, g')$. Assume that $\delta' : [0,1] \to P$ is defined by the equation $\delta'(t) = (\delta)^{-1}(t)g(t)$. Put $\gamma = \delta \cdot \gamma \cdot \delta'$ and $g = g(1)$. Then $\psi(\Pi_{\gamma,g}) = \Pi_{\delta^{-1}\cdot\delta\cdot\gamma\cdot\delta'\cdot\delta'^{-1}, g'}$ since the lift $(\delta(t), g'(t))$ of the curve δ into B starting at $(x, g(1))$ is uniquely defined. At the same time $(\delta(0), g'(0)) = ((\delta)^{-1}(1), g(1))$. Note now that $\delta^{-1} \cdot \delta \cdot \gamma \cdot \delta' \cdot \delta^{'-1} = \gamma$. Then ψ is an epimorphism. □

Statement 8. *If the action of the group Φ_x on H_x is free then the bundle $\pi_0 : B \to P$ is trivial. If $\pi_0 : B \to P$ is trivial and for any $x, y \in P$, $H_x = H_y$ then the action of Φ_x on H_x is free.*

Proof. Since action of Φ_x is free then $\Pi_{\gamma,0}(0) = 0$ implies that $\forall h \in H_x$ $\Pi_{\gamma,0}(h) = h$. Thus lift of any loop is again a loop. Hence, $\pi_0 : B \to P$ is trivial.

Assume now that for some $h \in H_x$ $\Pi_{\gamma,g}(h) = h$. By definition $\Pi_{\gamma,g}(h) = g + h(1)$, here $h(1)$ is the endpoint of the lift $(\gamma(t), h(t))$ of the path γ into B starting at (x, h). Since the bundle $\pi_0 : B \to P$ is trivial and $H_x = H_{\gamma(t)}$ for any t, we get $h(1) = h$. Henceforth, $g = 0$, i.e. γ is a loop on P. Then the triviality of a bundle implies that the lift of any loop from B is again a loop, i.e. for any $h' \in H_x$ the equality $\Pi_{\gamma,g}(h') = h'$ holds true. □

3.2 Existence of the invariant transversal for crystallographic group of translations.

Assume that the action of Φ_x on H_x is free for any $x \in P$. Hence, statement 8 implies triviality of the covering $\pi_0 : B \to P$.

Assume also the following:

A) There exist a diffeomorphism $\alpha : P \times H \to P \times \mathbb{R}^n$ and a subset $\Delta \subset \mathbb{R}^n$ such that the diagram (vertical arrows are the projections on the first factor)

$$\begin{tikzcd}
P \times H \arrow[r, "\alpha"] \arrow[d] & P \times \mathbb{R}^n \arrow[d] \\
P \arrow[r, "\mathrm{Id}"] & P
\end{tikzcd}$$

is commutative and $\alpha(B) = P \times \Delta$. then $\alpha(x, h) = (x, \alpha_x(h))$ and $\alpha_x(H_x) = \Delta$.

B) There exists a subgroup Φ of the isometry group of Euclidean space $\mathbb{R}^n$ such that

i) Φ is a semi-direct product of finite subgroup G of the group $O(\mathbb{R}^n)$ and finitely-generated subgroup L of the group of shifts of $\mathbb{R}^n$ (in this case Φ is a simmorphic crystallographic group [21]);

ii) Δ is the orbit of the action of Φ on $\mathbb{R}^n$;

iii) for any $x \in P$

$$\Phi_x = \{\alpha_x^{-1} \phi \alpha_x \mid \phi \in \Phi\}.$$

Conditions A) and B) imply that the action of Φ_x on H_x can be extended from the set $H_x \subset H \cong \mathbb{R}^n$ onto the whole H. Let $\Pi_g \in \Phi_x$. Let $h, g \in H$ and $\alpha(x, h) = (x, h_0)$, $\alpha(x, g) = (x, g_0)$ and for $g \in H_x$ $\alpha(\Pi_g) = \Pi'_{g_0}|_\Delta$. By definition of α we have $\alpha_x^{-1}(x, \Pi'_{g_0}(h_0)) = (x, \Pi_g(h))$ for $h \in H_x$. Put $\Pi_g(h') = h''$ for $h' \in \mathbb{R}^n$, here $(x, h'') = \alpha_x^{-1}(x, \Pi_{g_0}(h'_0))$ and $\alpha(x, h') = (x, h'_0)$.

Note 3. Consider lift of the path $\gamma : [0, 1] \to P$ into $P \times \mathbb{R}^n$ starting at (x, h_0), $h_0 \in \mathbb{R}^n$. By triviality of the bundle B and independence of $\Pi_{g_0}(h_0)$ of $x \in P$ we get $\gamma(t) = (x(t), h_0)$. Then $\gamma' = \alpha^{-1}(\gamma)$ is the lift of the path $\gamma : [0, 1] \to P$ into $P \times H$, $\gamma'(t) = \alpha_{\gamma(t)}^{-1}(\gamma(t), h_0)$. Note that action of the group does not preserve under the trivialisation α, i.e. $\Pi_{g_0}(h_0)x$ does not equal $(h_0 + g_0)x$. But under the action of α lifts $(x(t), \Pi_{g(t)}h(t))$ of the path γ starting at $(x, g + h')$ map to $(x(t), \Pi_{g_0}(h_0))$.

Statement 9. *Let there exist a closed compact transversal on the orientable manifold M with foliation F of codimension 1, generated by the locally free action of a commutative Lie group H. Assume also that the conditions A and B are true for (M, F). Then there exists a transversal P' on (M, F) such that for any $x \in P'$ the group Φ_x is a free Abel group.*

Proof. Let $y_0 \in \mathbb{R}^n$ be a fixed point of action of the elements $g \in G$ (definition of G is given in the condition B i)). Then since by assumption the action of Φ_x on H_x is free the action of the group Φ on Δ is also free. Hence, y_0 does not belong to Δ.

Let $(x, h(x)) = \alpha^{-1}(x, y_0)$. Consider a map $\rho : P \to M$, $\rho(x) = h(x)x$. Let $P' = \rho(P)$.

Let us prove first that $P' = \rho(P)$ is a manifold. Note first that P' is factor of P by the free action of finite group. Let $h(x)x = h(y)y$ and $\gamma : [0,1] \to P$, $\gamma(0) = x$, $\gamma(1) = y$. Since (y, z) is the fixed point of the transformation group$G$ and the mapping α is isomorphism the lift γ_0 of γ $\gamma_0(t) = \alpha^{-1}_{\gamma(t)}(\gamma(t), z)$ starting at $(x, h(x))$ ends in $(y, h(y))$. Then $\Pi_{h(x)-h(y)}(h(x)) = h(y) + h(x) - h(y) = h(x)$. Hence, $\Pi_{h(x)-h(y)} \in G$. Note now that by triviality of $\Pi_0 : B \to P$ each map $\Pi_g \in \Phi_x$ defines homeomorphism $\Pi'_g : P \to P$, $x(t) \mapsto g(x(t))x(t)$. That is, $h(x)x = h(y)y$ implies $y = u'(x)$, $u \in G$.

Assume that $y = u'(x)$, $u \in G$. Then since there exists only one fixed point of the set of transformations G $h(x)x = h(y)y$. Hence, $xh(x) = yh(y)$ if and only if $y = u'(x)$, here $u \in G$.

Consider now the following group of diffeomorphisms of the manifold P $G' = \{\Pi'_g | \Pi_g \in G\}$. Since $xh(x) = yh(y)$ if and only if $y = u'(x)$, $u \in G$ we obtain $P' \cong P/G'$. As M is an orientable manifold homeomorphisms $u' \in G'$ preserve orientation on P. Now the finiteness of G' for any $u' \in G'$ implies the existence of a number $n \in \mathbb{N}$ such that ${u'}^n = \mathrm{Id}$. Hence, there exists a diffeomorphism $\psi_{u'} : P \to P$ such that $\psi \circ u' \circ \psi^{-1}(x)$ is a turn onto angle $2\pi/n$. Then there exists a neighbourhood $V_{u'}(x)$ such that for each pair $x_1, x_2 \in V_{u'}(x)$ and any $n \in \mathbb{Z}$, $x_1 \neq {u'}^n(x_2)$ for any $x \in P$. Consider $V(x) = \bigcap\limits_{u' \in G'} V_{u'}(x)$. Hence, P/G' is a manifold.

Consider a natural projection $\pi : P \to P/G'$. Next consider the mapping $\beta : P/G' \to P'$ such that $\beta(t) \circ \pi = \rho$. This mapping β is correctly defined since if $\rho(x) = \rho(y)$ then $x = u'(y)$, $u' \in G'$, so $\pi(x) = \pi(y)$. The mapping β is a local homeomorphism since for any $t \in P/G'$ there exist a neighbourhood $U(t) \subset \pi(V(\pi^{-1}(t)))$ and a homeomorphism ρ of $U(t)$ onto $\pi^{-1}(U(t))$. The set P' is transversal to the foliation F since $P' = \pi_0(\alpha^{-1}(P \times \{y_0\}))$, α is diffeomorphism that maps $x \times H$ in $x \times \mathbb{R}^n$ (i.e. it conserves leaves and maps transversal to transversal) and π_0 is a local diffeomorphism. The equality $\rho(x) = \pi_0(\alpha^{-1}((x, z)))$ implies that $\dfrac{d\rho}{dx} \neq 0$, hence $d\rho$ is an isomorphism of T_xP onto $T_{\rho(x)}P'$.

Let $\pi'_0 : B' \subset P' \times H \to P'$ be the canonical covering for the transversal P' (cf 1.4.1.). Consider now the mapping $\rho^* : B \to B'$, $\rho^*(x, g) = (\rho(x), h(gx) + g - h(x))$. This mapping maps each path $\gamma : [0,1] \to B$ into the path $\rho^*(\gamma) \subset B'$. Let us show that ρ^* is a covering. Let $\gamma' : [0,1] \to B'$ be th path in B' starting in $(h(x_0)x_0)$. Here x_0 is defined up to some element of G'. Let $\gamma'(t) = (x'(t), g'(t))$. Consider a neighbourhood $V(x_0) \subset P$ such that the mapping $h(x)$, $h(x)x \in P'$ is uniquely defined in it. Hence we get $h(x(t))$ (we extend this function considering it in the neighbourhoods $V(y)$ and $V(z)$ in $V(x_0)$, here $V(x) = (y, z)$). Note now that $g'(0)x'(0) = h(x_1)x_1$, here x_1 is again defined up to element of G'. Then there exists the element $g_1 \in H_x$ such that $x_1 = g_1x_0$. Here g_1 is defined up to addition with the element of the isotropy group of the leaf passing through x_0, x_1, but since $L = H/H_L$ we can assume that g_1 is uniquely defined. Then in the neighbourhood $V(x_1)$ we get the mapping $g_1(t)$ such that $g_1(t)x(t) = x_1(t)$. Hence $g'(t)x'(t) = h(x_1(t))x_1(t) = (g_1(t) + h(g_1(t)x(t)))x(t) = (g_1(t) + h(g_1(t)x(t)) - h(t))x'(t)$, so $g'(t) =$

$g_1(t)+h(g_1(t)x(t))-h(t)$. Thus $(x'(t),g'(t))=\rho^*(x(t),g_1(t))$. Note that the right-hand side of the last equality does not depend on $u'\in G'$. Note also that if there exists a point $x\in P$ such that $u'(x)=x$ then for any other point $x'\in P$, $u'(x')=x'$ (otherwise there does not exist $n\in\mathbb{N}$ such that $u'^{n}=\mathrm{Id}$). Hence power of the set $\rho^{*-1}((x',g'))$ equals that of G.

Note now that the diagram

$$\begin{tikzcd}
B \arrow[r, "\rho^*"] \arrow[d, "\pi_0"'] & B' \arrow[d, "\pi_0'"] \\
P \arrow[r, "\rho"] & P'
\end{tikzcd}$$

is commutative, here $\pi_0'(x',g')=x'$.

Let us show that $\pi_0':B'\to P'$ is a trivial covering. Consider a loop $\gamma:[0,1]\to P'$, $\gamma(0)=\gamma(1)=x'$, a point $(x_0',g')\in B'$ and a lift $\gamma':[0,1]\to B'$, $\gamma'(0)=(x_0',g')$. Then the curve $\gamma_0\in\rho^{-1}(\gamma)$, starting at x_0, here $h(x_0)x_0=x_0'$, finishes at the point $u(x_0)$. Then we have the uniquely defined (since the diagram given above is commutative) curve $\gamma_1\subset B$, such that $\pi_0(\gamma_1)=\gamma_0$ and $\rho^*(\gamma_1)=\gamma'$. At the same time $\gamma_1(1)=(u(x_0),g_{u(x_0)})$. Then $\rho^*(\gamma_1(1))=(\rho(u(x_0)),h(g_{u(x_0)u(x_0)})+g_u(x_0)-h(u(x_0)))=(\rho(x_0),h(gx_0)+g-h(x_0))=\gamma'(0)$. This means that γ' is a loop.

The mapping ρ^* naturally defines the mapping $\rho':\Phi_x\to\Phi_{\rho(x)}$. Let $\Pi_g\in\Phi_x$. Then $\Pi_{g_1}(g_2)=g_2(1)+g_1$, here $\gamma_1(t)=(x(t),g_2(t))$ is a lift of the path γ_0, connecting x with g_1x. Consider $\rho^*(x,g_2)=(\rho(x),g_2')$ and $\rho^*(\gamma_1(1))=(g_1'\rho(x),g_2'(1))$. Then $g_i'\rho(x)=\rho(g_ix)$. Put $\rho'(\Pi_{g_1})(g_2')=g_2'(1)+g_1'=\Pi_{g_1'}(g_2')$ or $\rho'(\Pi_{g_1})=\Pi_{g_1'}$.

Let us show that if $\Pi_g\in G$ then $\rho'(\Pi_g)=\mathrm{Id}$. Indeed the former means that $\rho(gx)=\rho(x)$, thus $\rho^*(\gamma_0)$ is a loop. Now triviality of B' implies triviality of $\Pi_{\rho^*(\gamma_0),0}\in\Phi'_{\rho(x)}$. Assume that $\Pi_g\notin G$, thus $gx\neq ux$, $u\in G$. Hence $\rho(gx)\neq\rho(x)$, this implies that $g'x'=\rho(gx)\neq\rho(x)=x'$. Hence $\Pi_{g'}(0)=g'\neq 0$ and $\Pi_{g'}\neq\mathrm{Id}$. Thus $\mathrm{Ker}(\rho')=G$.

Let us show now that ρ' is a group homomorphism. To do this first note that $\rho'(\Pi_{g_1}\circ\Pi_{g_2})=\rho'(\Pi_{g_1})\circ\rho'(\Pi_{g_2})$. Then by definition of the group operation on Φ_x we have $\Pi_{g_1}\circ\Pi_{g_2}=\Pi_{g_1+g_2(1)}$, here $(x(t),g_2(t))$ is the lift of the path connecting x with g_2x to B starting at the point g_1. Thus it suffices to show that $\Pi_{h((g_1+g_2(1))x)+g_1+g_2(1)-h(x)}=\Pi_{h(g_1x)+g_1-h(x)}\circ\Pi_{h(g_2x)+g_2-h(x)}$. This follows from commutativity of the last diagram.

Since ρ^* is a covering ρ' is an epimorphism.

Thus we get an epimorphism $\rho':\Phi_x\to\Phi_{\rho(x)}$, such that $\mathrm{Ker}(\rho')=G$. Hence $\Phi_{\rho(x)}=\Phi_x/\mathrm{Ker}(\rho')$. □

Now we combine the last two statements with theorem 3 to get the

Corollary 3. *Let a foliation F of codimension 1 on the manifold M be generated by locally free action of a commutative Lie group H. Let there exist a compact connected transversal P on (M,F). Assume that the triple (M,F,P) meet conditions A and B. Then there exists an H-invariant Ehresmann connection on (M,F).*

Let us give examples of discrete subsets of $\mathbb{R}^n$, which possess a free and transitive action of the discrete group Φ though there is no diffeomorphism $\alpha:\mathbb{R}^n\to\mathbb{R}^n$ such that $\alpha\circ\Phi\circ\alpha^{-1}$ becomes a crystallographic group.

Example 4. Consider $Z = \bigcup\limits_{k\in\mathbb{Z}} (\{(k-1,k)\}\bigcup\{(k,k)\}\bigcup\{(k+1,k)\}) \subset \mathbb{R}^2$.

Consider then the transformations $\phi_1, \phi_2 : Z \to Z$,

$$\phi_1(x,y) = (x+1, y+1);$$

$$\phi_2 = \begin{cases} (x+1,y) & x = y = 2k; \\ (x+1,y+1) & x = 2k+1, y = 2k; \\ (x,y+1) & x = 2k, y = 2k-1; \\ (x+1,y+1) & x = 2k-1, y = 2k; \\ (x+1,y) & x = 2k, y = 2k+1; \\ (x,y+1) & x = y = 2k+1 \end{cases}$$

Here $k \in \mathbb{Z}$. Then $\phi_1^2 = \phi_2^3$. At the same time $\phi_1 \circ \phi_2 \neq \phi_2 \circ \phi_1$.

Let there exist a diffeomorphism $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ such that $\alpha \circ \phi_i \circ \alpha^{-1}$ turn to be elements of a crystallographic group. Then $\alpha\circ\phi_1\circ\alpha^{-1} = e^{i\alpha}z+\beta$, $\alpha\circ\phi_2\circ\alpha^{-1} = e^{i\gamma}z+\delta$. Since $\phi_1^2 = \phi_2^3$ we have an equality $e^{3\gamma i}z + (e^{2\gamma i} + e^{\gamma i} + 1)\delta = e^{2\alpha i}z + (e^{\alpha i} + 1)\beta$ for any $z \in \mathbb{C} \cong \mathbb{R}^2$. Hence $3\gamma = 2\alpha$. Since by assumption ϕ_i belongs to crystallographic group, there are four different cases:

1) $\alpha = \pi$, $\gamma = \pm\frac{2\pi}{3}$;
2) $\alpha = 0$, $\gamma = \pm\frac{2\pi}{3}$;
3) $\alpha = \pi$, $\gamma = 0$;
4) $\alpha = \gamma = 0$.

In 1) we have $(e^{4/3\pi i} + e^{2/3\pi i} + 1)\delta = (e^{i\pi} + 1)\beta$, or $0 = 0$, hence $\alpha \circ \phi_1 \circ \alpha^{-1}(z) = e^{i\pi}z + \beta$. But $e^{i\pi}(e^{i\pi}z + \beta) + \beta = z$. Then the orbit of the action of $\alpha \circ \phi_1 \circ \alpha^{-1}$ is finite. This is impossible since orbit of any point $x \in Z$ with respect to the action of ϕ_1 is infinite by construction.

Consider the second case. The equality $e^{3\gamma i}z+(e^{2\gamma i}+e^{\gamma i}+1)\delta = e^{2\alpha i}z+(e^{\alpha i}+1)\beta$ implies $\beta = 0$. Thus again the orbit of the action of the group generated by $\alpha\circ\phi_1\circ\alpha^{-1}$ is finite.

The third case provides us with the equality $\delta = 0$. Hence again the orbit $\alpha \circ \phi_2 \circ \alpha^{-1}$ is finite.

The fourth case is impossible since $\phi_1 \circ \phi_2 \neq \phi_2 \circ \phi_1$.

Statement 10. *Let a foliation F on two-dimensional manifold M be generated by locally free action of $\mathbb{R}$, orientable [15] and possess complete transversal P. Then there exists a transversal homotopic to H-complementary on (M, F).*

Proof. We consider several cases.

Let there exist a compact leaf $L \in F$ such that the set $P \bigcap L$ consits of at least two points. Then we can apply construction from the proof of lemma 1 of [15] and get a compact transversal P'. Now apply to P' corollary 1 of the section 2.1.

Now assume that all leaves of F are non-compact. Let there exist a leaf $L \in F$, such that $L \bigcap P = \{x_1, x_2, \dots\}$. Consider a Poincare map $\phi : U(x_1) = (x_0, x_0^{'}) \subset P \to P$, $\phi(x_1) = x_2$. Then construct a continuous mapping $f : U(x_1) \to \mathbb{R}$, $f(t) = h_t$, here $h_t t \in P$, $t \in U(x_1)$ and $f(x_1)x_1 = x_2$. Let us investigate the behaviour of f at the ends of the interval $(y_0, y_0^{'})$. If there exists a finite limit $\lim\limits_{x\to y_0} f(x)$ then the mapping ϕ can be extended continuously into y_0 and apply similar procedure in the neighbourhood $U(y_0) = (y_1, y_1^{'})$. Now let $\lim\limits_{x\to y_0} f(x) = \infty$. Consider $z = \lim\limits_{x\to y_0} \phi(x)$. If z and y_0 belong to the same leaf consider new transversal $P' = (P \setminus (\phi(x'), \infty)) \bigcup f(x')(x', \infty)$, here $(x', \infty) \subset P$. Thus let y_0, z belong to different leaves. Then the set $U(x) = \bigcup\limits_{i=0}^{\infty} (y_i, y_i^{'})$ possess two maximal elements with respect to the order of [15].

Now let P intersect each leaf in one point. Then for any $x \in P$ the set H_x is isotropy group of the leaf L_x. Now constructions similar to that of statement 7 tell us that these groups are isomorphic to each other for different $x, x' \in P$. Let there exist $x \in P$, $H_x \cong \mathbb{Z}$, otherwise the statement is obvious. Note first that if $H_x \cong \mathbb{Z}$ then there exists a neighbourhood $U(X) \subset P$, such that $\forall y \in U(X)$, $H_y \cong H_x \cong \mathbb{Z}$. Assume that $S = \bigcup\limits_{x \in P, H_x \cong \mathbb{Z}} \neq P$, i.e. there exists a limit point $z \in P$ of the open set S. Let the set $R = \{h_x | x \in S \bigcap U(z), < h_x >= H_x\}$ be bounded. Then there exists a limit of the sequence $h_z = \lim x \to z h_x$. Assume that at least one of these limits does not vanish. Then continuity of the action of $\mathbb{R}$ says us that $H_z \cong \mathbb{Z}$. If there exists only one limit equal to 0 consider the sequence of numbers $(h_n)_{n \in \mathbb{N}}$ and the sequence of elements $(x_n) \subset P$ such that $h_n x_{k \geq n} = x$. Then for each $n \in \mathbb{N}$ $h_n x_n = x_n \to z$ but by assumption $h_n z \neq z$, this contradicts continuity of the action of $\mathbb{R}$. Now let $R = \{h_x | x \in S \bigcap U(z), < h_x >= H_x\}$ be unbounded then the action of $\mathbb{R}$ does not generate foliation in the point z, since for any $h \in \mathbb{R}$, $hx \to x$ as $x \to z$.

Now apply construction of theorem 2 to the set $B \subset P \times \mathbb{R}$. □

§ 4. Existence of an Ehresmann connection orientability of the foliation F.

4.1 Elimination of non-triviality of B.

Our aim here is to construct a transversal $\tilde{P}$ such that the corresponding bundle $\tilde{B} \to \tilde{P}$ is trivial.

Assume that the fibration $B \to P$ is non-trivial. Then there exist $x \in P$, $a, b \in H_x$, $\psi \in \Phi_x$ such that $\psi(a) = a$, $\psi(b) \neq b$. Then the path $[a, b]$ under translation ψ passes to path $[a, \psi(b)]$. This means that there exists a natural mapping $f : P \to P$ of degree greater than one but evidently less than ∞. Then there exist points $\psi^i(b)$, $i = 2, \dots, K$. Then there exists a domain $[x, y) \subset P$ bijective to P, here $x, y \in L \bigcup P$.

Consider now union of all the curves γ_i, $i = 1, \dots K$, here γ_i connects $\psi^i(b)$ with $\psi^{i+1}(b)$ for $i = 1, \dots, K-1$ and γ_K connects $\psi^K(b)$ with b. Then union of all these curves translated along transversal P defines a torus $\mathbb{T}^2 \subset M$. Now the curve $\gamma = \bigcup\limits_{i=0}^{n-1} \phi^i(\gamma)$ intersects P in finite number of points. Then there exists a transversal $\tilde{P}$ intersecting γ in exactly one point.

4.2 Compact transversal

Here we assume that the transversal P is compact.

Then the mapping $\pi : B \to P$ is a fibration and the set Φ_x for any $x \in P$ possess a natural group structure. Our first goal is to find the other transversal P_1 such that the set Φ_y is a commutative group for any $y \in P_1$.

4.2.1 Elimination of finite cyclic subgroups

Here we construct the transversal P' whose group Φ' does not contain finite cyclic subgroups.

The main idea coincides with one given in the section 3.2.

Statement 11. *Assume that there exists the element $\phi \in \Phi_x$, $\phi : P \to P$ and the number $n \in \mathbb{N}$ such that $\phi^n = Id$. Then there exists a transversal P' such that the generators of Φ' are those of Φ with the exception of ϕ.*

Proof. Consider a curve $\gamma : [0,1] \to L$, $\gamma(0) = x$, $\gamma(1) = \phi(x)$ and define up to homotopy its translation along P into the curve starting at $\phi(x)$ and ending at $\phi^2(x)$. Let us denote this curve by $\phi(\gamma)$. Locally — in some neighbourhoods of the points x and $\phi(x)$ on the transversal P — this translation is correctly defined sine the mapping $\pi : B \to P$ is a fibration. Now, compactness of the transversal P gives us the possibility to extend this translation on all P. The question is whether the image of γ under ϕ^n is homotopic to γ. This holds true since the endpoints of γ and $\phi^n(\gamma)$ coincide by assumption and any loop in $\mathbb{R}^k = H$ is contractible. Next we deform the translation so that $\phi^n(\gamma) = \gamma$. This can be done since the set of points $x, \phi(x), \dots, \phi^{n-1}(x)$ is finite subset of P, hence each of them has a neighbourhood that does not intersect one of adjacent point.

So we have a set of curves $\gamma, \phi(\gamma), \dots \phi^{-1}(\gamma)$, whose union on H bounds some 2-dimensional domain (a set homeomorphic to the disk D^2).

Now we are able to use the method introduced for crystallographic groups.

Let us first divide the transversal P into intervals isomorphic to $P/ < \phi >$. Consider a continuous map $\rho : P \to \mathbb{R}^k$ such that $\rho(x) = 0$, $\rho(\phi(x)) = h^{-1}$, here $\phi(x) = hx$, etc. It is possible again since the set $x, \phi(x), \dots, \phi^{n-1}(x)$ is finite one, or the parts representing continuous sections of $P/ < \phi > \to P$ are nontrivial intervals. This gives us a desired transversal $P' = \rho(P)$. The rest of the considerations coincide with that of the main statement proof of section 3.2. □

4.2.2 Construction of the transversal P'' with Abel group Φ''.

In this section we assume that the transversal P is such that the translation group Φ does not contain finite cyclic (hence just finite) subgroups.

Consider the relation $\prod\limits_{i=1}^{N} a_{\alpha_i}^{\beta_i} = 1$, here for $j_1, \dots, j_k \in \overline{1,N}$

$$\sum_{\alpha_{j_i}=\alpha_i} \beta_j = n_j \neq 0, i = \overline{1,k}$$

Let us simplify further expressions denoting elements a_{α_i} by b_i, $i = \overline{1,k}$.

Consider a group $\Phi_1 \subset \Phi$ generated by b_i, $i = \overline{1,k}$ and the infinite cyclic group $< b_1 > \subset \Phi_1$, generated by the element b_1. hen the set $S = \Phi_1 / < b_1 >$ is finite. Consider a set of representatives of the equivalence classes $S = \Phi' / < b_1 > S = \{e, c_1, \dots, c_m\}$, here e is the neutral element of Φ. Now we can construct a disk D with the boundary consisting of curves connecting points $x, c_1x, \dots, c_mx, x$.

Let us construct the new transversal as follows: Fix a point $x_0 \in D$, $x_0 \neq x, c_1x, \dots, c_mx, x$ and the set of its images under translations of D by elements of

$< b_1 >$. Let us construct a continuous mapping $\phi : P \to M$. First put $\phi(x) = \phi(c_1 x) = \ldots = \phi(c_m x) = x_0$. Then we extend this mapping to the whole transversal P. Consider now $P' = \phi(P)$. This again can be done since $Sx \subset P$ is finite.

The resulting curve P' is a transversal since we can assume that the diameter of the translated disk D is bounded from above.

The action of the image of transformations $c_1, \ldots, c_m \in \Phi$ under mapping ϕ will all coinside with that of the image of $x_0 \to b_1 x_0$.

Recall now that the element $a \in \Phi_x$ is characterised by two components, namely,

1) the image $a(x) \in L \bigcap P$ of a fixed point $x \in P$;

2) the curve on P connecting x with $a(x) \in P$.

Thus any edge of the disk D gives rise to some (not one) curve on P. Hence the boundary of D generates a loop in P. It is clear that the image of this loop under the mapping ϕ is also a loop on P' simply because image of all the vertices of D is one point.

After all the commutation relations satisfying 4.2.2 are thus eliminated we must introduce the relations of the type $\prod_{i=1}^{N} a_{\alpha_i}^{\beta_i} = 1$, here for any $j = \overline{1, N}$

$$\sum_{\alpha_j = \alpha_i} \beta_j = 0.$$

So we must either introduce relations $aba^{-1}b^{-1} = 1$ or show that the group Φ_x does not contain free noncyclic subgroups.

The latter of the statements appears to be better suited for the solution of our problem.

Statement 12. *The group Φ does not contain free noncyclic subgroups.*

Proof. Assume the contrary, i.e. that the group under consideration contains at least one such subgroup A. Let us denote basic elements of this group as a, b. Consider any Riemannian metric d' on M.

As in theorem 2 construct a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\forall \alpha, \beta, a \in \Phi_x$ $d(\phi(a\alpha x), \phi(a\beta x)) = c(\alpha, \beta)$.

Since the set $L \bigcap P \subset M$ is discrete there exist the leaf L', the point $x \in L'$ and the element $a \in H_x$ such that $d(x, ax) = \min\{d(y, by) | y \in P, b \in H_y\}$.

Consider these points $x, ax \in L \bigcap P$. We can deform action of H on M so that in some neighbourhood $U(x) = (x_1, x_2) \subset P$ the distance between points $y(t), a(t)y(t)$, $y(t) \in U(x)$ in the leaf $L_{y(t)}$ becomes constant. Recall that $B \to P$ is a fibration. So this procedure can be extended to the whole transversal P. Next consider the leaf L_1, the point $x_1 \in L$ and the element $a_1 \in H_x$ such that $d(x_1, a_1 x_1) = \min\{d(y, by) | y \in P, b \in H_y$, b does not belong to the connected component of a in the fibration B $\}$. Repeat deformation of the group action with these elements as instead of x, ax. And so on. Note that no one of these processes nor any combination of them produce singular action of the group H.

At thesame time we have an equality $d(x, b^{-n}ab^n x) = d(b^n x, ab^n x) = d(x, ax)$ by construction of the new group action. Since group A is free the set of points $b^{-n}ab^n x$ is infinite. Then the set Ax is not discrete. This contradiction completes the proof. □

4.2.3 Finiteness of elimination processes.

According to the results of the previous sections we can say that the group Φ_x does not contain either commutative finite or free noncyclic subgroups for any $x \in P$. Let us show now that this group is also finitely generated.

Statement 13. *The commutation relations elimination process of section 4.2.2 is finite.*

Proof. Assume the contrary. Then the sequence of coverings of the previous sections $P \to P_1 \to P_2 \to \dots$ is infinite, here each transversal P_i is obviously diffeomorphic to $\mathbb{S}^{\mathbb{K}}$. Hence, any neighbourhod of arbitrary point of M contains closed transversal. Contradiction with the definition of foliated manifold. $\square$

Corollary 4. *Group Φ is finitely generated.*

Proof. First note that statement 13 implies that there is finite number of commutation relations.

Thus the only problem is finiteness of basic set of Φ. Again assume the contrary. Then for any leaf L and any point $x \in L \bigcap P$ there must exist a sequence $x_n \subset L$, $d(x_n, x_m) \to 0$, $(n, m \to \infty)$. Indeed, recall first that d_L is Φ-invariant. Then the basic set of Φ provides us with infinite number of distinct points in L. Consider now the diameter of the net $\Sigma_k = \bigcup\limits_{i=1}^{k} \bigcup\limits_{j \in \mathbb{Z}} a_i^j x$. Note that diameter $d_n = \operatorname{diam}(\Sigma_n)$ is monotone decreasing as $n \to \infty$. Consider $R = \inf_n(d_n)$. Then by definition of infimum there is infinite number of points $(a_{n_k} x)_{k=1}^{\infty}$ such that $R < d(x, a_{n_k} x) < 2R$. Then the set $(a_{n_k} x)_{k=1}^{\infty}$ has an accumulation point, hence $R = 0$. A contradiction with the assumption of closedness of $P \subset M$. $\square$

4.2.4 Modification of the existing commutation relations

Now the translation group Φ of the transversal P is finitely generated and does not possess commutation relations of the type 4.2.2. Let us denote this transversal by P_0.

This transversal is such that the commutation relations of its group Φ_0 are only of the type $a_i^{l_{ij}} a_j^{l_{ji}} = a_j^{l_{ji}} a_i^{l_{ij}}$.

Now put $l_i = LCM(l_{i,j})_{j=1}^{N}$, here N equals number of basic elements of Φ_0. Consider again a polygon in L with vertices $x, a_i x, \dots, a^{l_i} x \subset L$. We must apply to it the procedure of section 4.2.2. Let us denote the result transversal by P'. Note that it is l_i-covered by P.

Thus we can assume that $\forall a, b \in \Phi'$, $ab = ba$.

4.2.5 Existence of the Ehresmann connection

The main construction idea is as in note 1.

Statement 14. *If for some $x \in P$ group Φ_x is commutative then its action on $H_x\{x\} = L_x \bigcap P$ is free.*

Proof. Assume the contrary, i.e. $\exists h \in H_x$ and $a \in \Phi_x$, $a(x) \neq x$, but $a(hx) = hx$. But then by transitivity of the action of Φ_x $\exists \psi \in \Phi_x$, such that $\psi(x) = hx$. Hence $\psi \circ a \circ \psi^{-1}(x) = x$. But the group Φ_x is commutative so $\psi \circ a \circ \psi^{-1} = a$, thus $a(x) = x$. Contradiction with the assumption. Hence $a = id \in \Phi_x$. □

1) By statement 8, $B \to P_0$ is trivial bundle and the group Φ can be identified with H_x.

2) Φ is a commutative free group without finite subgroups by 4.2.2. Thus $\Phi \cong \mathbb{Z}^k$.

3) The proof of statement 12 implies that $\Phi \cong \mathbb{Z}^k$ is isometrically embedded into$H \cong \mathbb{R}^n$. Hence $k \leq n$.

1)-3) combined tell us that the group Φ meets conditions of theorem 3.

Theorem 5. *Let a manifold M with foliation F meet the following conditions:*

1) codim$F = 1$.

2) *The foliation F is generated by locally free action of* $\mathbb{R}^n$.

3) *There exists a compact connected transversal on (M, F).*

Then there exist a transversal homotopic to H-complementary one and Ehresmann connection on (M, F).

4.3 Noncompact transversal on transversally orientable foliation

Our aim is to prove statement similar to the last one of the previous section under slightly different conditions, namely

1) codim$F = 1$.

2) The foliation F is generated by locally free action of the group $\mathbb{R}^n$.

3) (M, F) is transversally orientable.

Main idea of solution is to reduce the problem to the one considered in section 4.2.

Statement 15. *Let (M, F) be transversally orientable. Then there exists a transversal P on M such that $B \to P$ is a fibre bundle.*

Proof. Construction of this transversal is similar to one of the proof of statement 10. □

Now consider two cases.

1) There exist only noncompact transversals on (M, F).

2) There exists at least one compact transversal on (M, F). It holds true for compact manifold M [15].

The second case clearly coincides with one considered in the previous section.

Statement 16. *Let the foliation F on a manifold M be transversally orientable. Assume that there exist only noncompact transversals on (M, F).*

Then the following holds true. $\forall (p_n)_{n \in \mathbb{N}}$, $p_n \to \pm\infty$ *($n \to \infty$),* $\nexists p_0$, $p_n \to p_0$ *as* $n \to \infty$.

Proof. Assume the contrary. Then application of the procedure from the proof of lemma 1 [15] in the coordinate chart adapted to the foliation of the neighbourhood of $p_0 \in P$ produces a compact transversal of (M, F). □

Now let us compactify a transversal $P \cong \mathbb{R}$ in the usual way to get $\mathbb{S}^1 \cong P \bigcup \{\infty\}$. This can be done on M by statement 16.

1) Deform the action of the group H on M so that $\forall h \in H \lim\limits_{p\to\infty} hp = \lim\limits_{p\to-\infty} hp$.

2) Assume that there exists a limit $\lim\limits_{p\to\pm\infty} h(p)$ and $|h_p| \searrow \lim\limits_{p\to\pm\infty} h(p)$ for any element of H_x, $x \in P$, here $h(p) \in H_p$ is an element of the connected component of $H \in H_p \times p \subset B$.

3) Define action of H in the $\{\infty\} \in M$ as follows:

$$h\{\infty\} = \lim_{p\to\infty} hp$$

Note that then $\forall h(p) \in H_p$, $\lim\limits_{p\to\infty} h(p)\{\infty\} = \{\infty\}$.

Thus we get a manifold $M' = M \bigcup H\{\infty\}$ such that the action of H on it is locally free, hence this action generates a foliation F' on it. At the same time (M', F') possess a compact transversal $P \bigcap \{\infty\}$. The coordinate chart in the neighbourhood of $\{\infty\} \times 0 \in M'$ is a cartesian product $(P \setminus K) \times B_{\min\{|h|/2 | h\in H_\infty\}}(0)$ with the natural coordinates. The coordinate chart of any point $\{\infty\} \times h$ is then $(P \setminus K) \times B_{\min\{|h-h'|/2 | h'\in H_\infty\}}(h)$

Thus again M, F, P meets conditions of the main statement of the previous section. Hence we get a corollary of theorem 6.

Theorem 6. *Let a manifold M with foliation F meet following conditions:*

1) $\mathrm{codim} F = 1$.

2) *Foliation F is generated by action of $\mathbb{R}^n$.*

3) (M, F) *is transversally orientable.*

Then there exist a transversal homotopic to H-complimentary and an Ehresmann connection on (M, F).

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