Timelike surfaces of revolution with Constant Mean Curvature in Minkowski 3-Space

Sungwook Lee and Jeffrey H. Varnado

Abstract. First, we study certain ODEs that characterize timelike surfaces of revolution with constant mean curvature in Minkowski 3-space. These ODEs are non-linear and it is very difficult to find their solutions explicitly. Numerical solutions to these ODEs can be found by well-known numerical methods such as Runge-Kutta's or Euler's methods. We obtain examples of such surfaces from the numerical solutions.

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Key words: constant mean curvature, minimal surfaces, Minkowski spacetime, rotational surfaces, timelike surfaces, worldsheets.

$\mathbb{S}$ 1. Introduction

Minkowski 3-space has more complicated and richer geometric structures compared with familiar Euclidean 3-space. In particular, Minkowski 3-space has 3 distinguished axes of rotation, namely, *spacelike*, *timelike*, and *lightlike axes* (or *null axes*). Hence, one can consider three different kinds of rotations; rotations about spacelike, timelike, and lightlike axes. Especially, the rotations about spacelike axes (the so-called *boosts* which are regarded as *rotations between space and time directions*) and the rotation about timelike axis (this is similar to the conventional rotation in Minkowski 3-space) are the parity and time preserving linear transformations of Minkowski 3-space. Moreover, these linear transformations are Lorentz isometries (the so-called *Lorentz transformations* in special relativity), i.e., they preserve the Minkowski metric. The rotations about spacelike and timelike axes form a group called the special Lorentz group $\mathrm{SO}(2,1)$. As is well known in special relativity, the Lorentz transformation is a natural consequence of Albert Einstein's famous postulate of the constancy of the speed of light in flat (vacuum) spacetime.

Surfaces of revolution in Euclidean 3-space, namely Delaunay surfaces, unduloids, and nodoids have been studied extensively for a long time and many examples of such surfaces have been obtained. Naturally, one may also study surfaces of revolution in Minkowski 3-space, analogously to the Euclidean case. However, there is a clear distinction between Lorentzian case and Euclidean case. As mentioned earlier, there are three distinguished axes of rotation, hence we need to consider surfaces of revolution

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about each of three axes. Furthermore, the metric of Minkowski 3-space is indefinite and this makes the geometry of surfaces therein more complicated.

In physics, the trajectory of a *massive* particle in spacetime is called a *worldline*. In geometry, a wordline $x(\lambda) = (x^\mu(\lambda))$ is also called a *timelike curve*. The velocity vector $\frac{dx^\mu}{d\lambda}$ satisfies $\langle x(\lambda), x(\lambda)\rangle = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0$, where $\langle\ ,\ \rangle$ is the inner product induced by the flat Lorentzian metric and $\eta_{\mu\nu}$ is the metric tensor of signature $(-,+,+)$. The arc-length of $x(\lambda)$ is defined to be

$$\Delta x := \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda.$$

This arc-length Δx is called *proper time* in physics, because it measures the actual time elapsed on a physical clock carried along the curve. In this article, we study a certain kind of surfaces, the so-called *timelike surfaces*, of revolution with constant mean curvature in Minkowski 3-space. Timelike surfaces of revolution are obtained by rotating a timelike profile curve about a spacelike axis, timelike axis, or a lightlike axis. Especially, we are interested in timelike surfaces of revolution with constant mean curvature (abbreviated as cmc). Timelike cmc surfaces are physically interesting because they are the solutions of nonhomogeneous wave equation $-\varphi_{tt}+\varphi_{xx} = \gamma^2 HN$, where γ is a positive function called the conformal factor, H is the mean curvature and N is the unit normal vector field of the timelike cmc surface $\varphi(t,x)$. In particular, timlike minimal surfaces (cmc $H = 0$) are the solutions of homogeneous wave equation $-\varphi_{tt} + \varphi_{xx} = 0$. One can imagine timelike surfaces as surfaces swept by propagating waves in spacetime. Timelike cmc $\neq 0$ surfaces can be regarded as the critical points of the energy functional

$$E(\varphi) = \int_{\mathbb{D}} ||d\varphi|| - k\int_{\mathbb{D}} H\langle N,\varphi\rangle dA$$

coupled with volume constraint. Here, we consider timelike cmc surfaces that bound certain volumes. Timelike cmc surfaces are also interesting in string theory point of view. In string theory, particles are considered to be tiny vibrating strings in spacetime. A string evolves in time while sweeping a surface in spacetime called a *worldsheet*. Hence, string worldsheets are in fact timelike surfaces. A closed string[1] is an object in the configuration space, that is homeomorphic to S^1. Timelike cmc $\neq 0$ surfaces may be interpreted as worldsheets that are swept by closed strings in spacetime. In physics, it is well-known that classical *Nambu-Goto* string (*bosonic* string) worldsheets are timelike minimal surfaces.

It should be remarked that L. McNertney completely classified timelike minimal surfaces in Minkowski space in her Ph.D. thesis [5] and R. López studied timelike minimal surfaces and timelike surfaces with constant mean curvature that are foliated by circles in [4]. In particular, he showed that if a timelike surface with non-zero constant mean curvature is foliated by circles in parallel planes, it must be rotational.

The main purpose of this article is 2-folds. First, we study certain ODEs that characterize timelike cmc surfaces of revolution in Minkowski 3-space. These ODEs are non-linear and it is very difficult to find their solutions explicitly. Next, we solve these ODEs numerically by well-known numerical methods such as Runge-Kutta's or

[1] In physics, gravitons are corresponded to closed strings.

Euler's methods. From the numerical solutions, we obtain examples of timelike cmc surfaces of revolution in Minkowski 3-space. In this article, the numerical calculations and graphics are done with the aid of MAPLE 9.5.

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§ 2. Timelike Surfaces in Minkowski 3-Space

Let $\mathbb{E}^3_1$ be Minkowski 3-space with linear coordinates ξ_0, ξ_1, ξ_2 and the standard Lorentzian metric $ds^2 = -(d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2$.

Definition 1. *Let $\langle \cdot, \cdot \rangle$ denote the inner product induced by the Lorentzian metric. Let v be a vector in $\mathbb{E}^3_1$. Then v is said to be*

1. spacelike *if $\langle v, v \rangle > 0$,*
2. lightlike or null *if $\langle v, v \rangle = 0$, and*
3. timelike *if $\langle v, v \rangle < 0$.*

Let $\mathbb{D}$ be a simply-connected domain in $\mathbb{E}^2_1(t, x)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3) : \mathbb{D} \longrightarrow \mathbb{E}^3_1$ an immersion in $\mathbb{E}^3_1$. The immersion φ is said to be *timelike* if the induced metric on $\mathbb{D}$ is *Lorentzian*. This induced metric I determines a Lorentz conformal structure on $\mathbb{D}$. More specifically,

Definition 2. $\varphi(t, x) : \mathbb{D} \longrightarrow \mathbb{E}^3_1$ *is said to be* Lorentz conformal *if*

$$\langle \varphi_t, \varphi_x \rangle = 0, \tag{2.1}$$

$$-|\varphi_t| = |\varphi_x| = e^{\omega/2}, \tag{2.2}$$

where (t, x) is a local coordinate system in $\mathbb{D}$ and $\omega : \mathbb{D} \longrightarrow \mathbb{R}$ is a real-valued function from $\mathbb{D}$. Lorentz conformal timelike surfaces are called Lorentz surfaces.

If N is the unit normal vector field of a timelike immersion $\varphi : \mathbb{D} \longrightarrow \mathbb{E}^3_1$, then $\langle N, N \rangle = 1$, $\langle N, \varphi_t \rangle = \langle N, \varphi_x \rangle = 0$.

Definition 3. *Let $v = (v_0, v_1, v_2)$ and $w = (w_0, w_1, w_2)$ be two (spacelike or timelike) vectors in $\mathbb{E}^3_1$. Then the* Lorentzian cross product *is defined to be*

$$v \times w = \begin{vmatrix} -e_0 & e_1 & e_2 \\ v_0 & v_1 & v_2 \\ w_0 & w_1 & w_2 \end{vmatrix}, \tag{2.3}$$

where $e_0 = (1, 0, 0), e_1 = (0, 1, 0), e_2 = (0, 0, 1)$.

Remark 1. One can easily see that $v \times w$ is in fact a normal vector to both v and w.

Proposition 1. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{E}^3_1$ be an immersed (spacelike or timelike) surface in $\mathbb{E}^3_1$. Then*

$$||\varphi_t \times \varphi_x||^2 = \langle \varphi_t, \varphi_x \rangle - ||\varphi_t||^2 ||\varphi_x||^2. \tag{2.4}$$

Proof. Straightforward by direct computation. □

Let N denote a unit normal vector field of a timelike immersion $\varphi : \mathbb{D} \longrightarrow \mathbb{E}^3_1$. Let us define the following quantities to do some differential geometric computations:

$$E = \langle \varphi_t, \varphi_t \rangle, F = \langle \varphi_t, \varphi_x \rangle, G = \langle \varphi_x, \varphi_x \rangle$$
$$l = \langle \varphi_{tt}, N \rangle, m = \langle \varphi_{tx}, N \rangle, n = \langle \varphi_{xx}, N \rangle.$$

The equation (2.4) is also written as

$$||\varphi_t \times \varphi_x||^2 = F^2 - EG. \tag{2.5}$$

Remark 2. The unit normal vector field N of φ is given by

$$N = \frac{\varphi_t \times \varphi_x}{\sqrt{F^2 - EG}}. \tag{2.6}$$

The following formula is well-known in differential geometry of surfaces in Euclidean 3-space. The same formula still holds for (spacelike or timelike) surfaces in Minkowski 3-space.

Proposition 2. *The mean curvature H of a (spacelike or timelike) immersion $\varphi : M \longrightarrow \mathbb{E}^3_1$ is computed to be*

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)}. \tag{2.7}$$

Proof. The formula (2.7) is proved exactly the same way as in the Euclidean case such as, for instance, the elementary proof given in [6]. □

$\mathbb{S}$ 3. Timelike Surfaces of Revolution in $\mathbb{E}^3_1$

Timelike surfaces of revolution can be obtained by rotating timelike profile curves about timelike axis ξ_0, spacelike axes ξ_1, ξ_2, and lightlike axes $\xi_0 \pm \xi_1$, $\xi_0 \pm \xi_2$. In this section, we classify and characterize timelike surfaces of revolution in Minkowski 3-space.

Proposition 3. *There are three distinguished types of rotations in $\mathbb{E}^3_1$; rotations about timelike axis ξ_0, rotations about spacelikes axes ξ_1, ξ_2, rotations about null axes $\xi_0 \pm \xi_1$, $\xi_0 \pm \xi_2$. These rotations are determined by the following rotational matrices:*

1. *The matrix corresponds to the rotation about ξ_0 is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{pmatrix} \tag{3.8}$$

2. *The matrix corresponds to the rotation about* ξ_1 *is*

$$\begin{pmatrix} \cosh x & 0 & \sinh x \\ 0 & 1 & 0 \\ \sinh x & 0 & \cosh x \end{pmatrix}. \tag{3.9}$$

3. *The matrix corresponds to the rotation about* ξ_2 *is*

$$\begin{pmatrix} \cosh x & \sinh x & 0 \\ \sinh x & \cosh x & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

4. *The matrix corresponds to the rotation about* $\xi_0 + \xi_1$ *is*

$$\begin{pmatrix} 1 + \frac{x^2}{2} & -\frac{x^2}{2} & x \\ \frac{x^2}{2} & 1 - \frac{x^2}{2} & x \\ x & -x & 1 \end{pmatrix}. \tag{3.11}$$

5. *The matrix corresponds to the rotation about* $\xi_0 - \xi_1$ *is*

$$\begin{pmatrix} 1 + \frac{x^2}{2} & \frac{x^2}{2} & -x \\ -\frac{x^2}{2} & 1 - \frac{x^2}{2} & x \\ -x & -x & 1 \end{pmatrix}. \tag{3.12}$$

6. *The matrix corresponds to the rotation about* $\xi_0 + \xi_2$ *is*

$$\begin{pmatrix} 1 + \frac{x^2}{2} & x & -\frac{x^2}{2} \\ x & 1 & -x \\ \frac{x^2}{2} & x & 1 - \frac{x^2}{2} \end{pmatrix}. \tag{3.13}$$

7. *The matrix corresponds to the rotation about* $\xi_0 - \xi_2$ *is*

$$\begin{pmatrix} 1 + \frac{x^2}{2} & -x & \frac{x^2}{2} \\ -x & 1 & -x \\ -\frac{x^2}{2} & x & 1 - \frac{x^2}{2} \end{pmatrix}. \tag{3.14}$$

Proof. For instance, let A be a 3×3 rotational matrix corresponds to the rotation about the spacelike axis ξ_1. Then this rotation fixes the spacelike vector $e_1 = (0, 1, 0)$. The matrix A is a Lorentz transformation, i.e., it preserves the Lorentzian metric (*Lorentz isometry*). Hence, the matrix A can be found by solving the following equations simultanously:

1. $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
2. $A \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. $\det A = 1$.

The other rotational matrices are found in the same manner. □

Remark 3. The rotational matrices (3.9) and (3.10) are called the *Lorentz boosts*. These are regarded as rotations take place between space and time.

By the formula (2.7), we can easily prove the following theorem which characterizes timelike cmc surfaces of revolution in Minkowski 3-space.

Theorem 4. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{E}^3_1$ be an immersed timelike surface of revolution with constant mean curvature H in $\mathbb{E}^3_1$. Then φ is parameterized in one of the following ways.*

1. *φ is a timelike surface of revolution about timelike axis ξ_0:*

$$\varphi(t, x) = (t, h(t)\cos x, h(t)\sin x), \tag{3.15}$$

where $(t, 0, h(t))$ is a timelike profile curve in $\xi_0\xi_2$-plane with $h(t) > 0$.

$h(t)$ satisfies the differential equation

$$H = \frac{1}{2}\frac{\ddot{h}h + 1 - \dot{h}^2}{h(1 - \dot{h}^2)^{\frac{3}{2}}}. \tag{3.16}$$

Here, $\dot{h}$ stands for $\frac{dh(t)}{dt}$.

2. *φ is a timelike surface of revolution about spacelike axis ξ_1:*

$$\varphi(t, x) = (h(t)\cosh x, t, h(t)\sinh x), \tag{3.17}$$

where $(h(t), t, 0)$ is a timelike profile curve in $\xi_0\xi_1$-plane with $h(t) > 0$.

$h(t)$ satisfies the differential equation

$$H = -\frac{1}{2}\frac{\ddot{h}h + 1 - \dot{h}^2}{h(\dot{h}^2 - 1)^{\frac{3}{2}}}. \tag{3.18}$$

3. *φ is a timelike surface of revolution about spacelike axis ξ_2:*

$$\varphi(t, x) = (h(t)\cosh x, h(t)\sinh x, t), \tag{3.19}$$

where $(h(t), 0, t)$ is a timelike profile curve in $\xi_0\xi_2$-plane with $h(t) > 0$.

$h(t)$ satisfies the differential equation

$$H = \frac{1}{2}\frac{\ddot{h}h - \dot{h}^2 + 1}{h(\dot{h}^2 - 1)^{\frac{3}{2}}}. \tag{3.20}$$

For the rotations about null axes, we consider only $\xi_0 + \xi_1$ as our conventional null axis.

4. φ is a timelike surface of revolution about null axis $\xi_0 + \xi_1$:

$$\varphi(t,x) = (\varphi_0(t,x), \varphi_1(t,x), \varphi_2(t,x)), \tag{3.21}$$

where

$$\begin{aligned}
\varphi_0(t,x) &= t\left(1 + \frac{x^2}{2}\right) - \frac{1}{2}h(t)x^2,\\
\varphi_1(t,x) &= \frac{tx^2}{2} + h(t)\left(1 - \frac{x^2}{2}\right),\\
\varphi_2(t,x) &= tx - h(t)x.
\end{aligned}$$

The timelike profile curve is $(t, h(t), 0)$ with $t > h(t)$.

$h(t)$ satisfies the differential equation

$$H = \frac{1}{2}\frac{(t-h)\ddot{h} - (\dot{h}-1)(\dot{h}^2-1)}{(t-h)(1-\dot{h}^2)^{\frac{3}{2}}}. \tag{3.22}$$

§ 4. Timelike minimal surfaces of revolution in Minkowski 3-space

A Lorentz surface $\varphi : \mathbb{D} \longrightarrow \mathbb{E}^3_1$ is said to be *minimal* if $H = 0$. Timelike minimal surfaces of revolution in $\mathbb{E}^3_1$ can be easily found by solving the differential equations in Theorem 4 with $H = 0$.

It should be remarked that spacelike maximal surfaces and timelike minimal surfaces of revolution have been completely classified by L. McNertney in her Ph.D. thesis [5].

4.1 Timelike minimal surfaces of revolution about timelike axis

The mean curvature formula for timelike surfaces of revolution about ξ_0 axis is (3.16). Set $H = 0$. Then

$$\frac{1}{2}\frac{\ddot{h}h + 1 - \dot{h}^2}{h(1-\dot{h}^2)^{\frac{3}{2}}} = 0. \tag{4.23}$$

The quantities E, F, G are computed to be

$$E = -1 + \dot{h}^2,\ F = 0,\ G = h^2.$$

The conformality condition (2.2) is

$$1 - \dot{h}^2 = h^2.$$

This reduces (4.23) to the equation of a simple harmonic oscillator

$$\ddot{h}^2 + h = 0,$$

whose solution is

$$h(t) = C_1 \cos t + C_2 \sin t.$$

Hence, by (3.15) timelike minimal surfaces of revolution about ξ_0 axis is given by

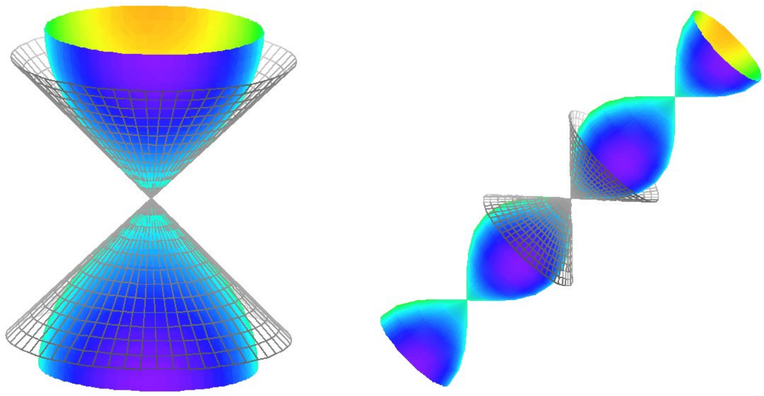
$$\varphi(t, x) = (t, (C_1 \cos t + C_2 \sin t) \cos x, (C_1 \cos t + C_2 \sin t) \sin x).$$

Timelike minimal surface of revolution about timelike axis is called *timelike catenoid with timelike axis*.

Figure 1 shows an example of timelike catenoid with timelike axis with $C_1 = 0, C_2 = 1$, i.e.,

$$\varphi(t, x) = (t, \sin t \cos x, \sin t \sin x).$$

The graphics of timelike catenoid with timelike axis is made with the *light cone* so it helps to see how it is positioned in Minkowski 3-space and where the rotation takes place.



(a) (b)

Figure 1: Timelike catenoid with timelike axis

4.2 Timelike maximal surfaces of revolution about spacelike axes in Minkowski 3-space

In this subsection, we consider timelike minimal surfaces of revolution about ξ_2 axis.

Set $H = 0$ in the equation (3.20). Then

$$\frac{1}{2} \frac{\ddot{h}h - \dot{h}^2 + 1}{h((\dot{h}^2 - 1)^{\frac{3}{2}}} = 0. \tag{4.24}$$

The quantities E, F, G are computed to be

$$E = -\dot{h}^2 + 1, \ F = 0, \ G = h^2.$$

The conformality condition (2.2) implies then

$$\dot{h}^2 - 1 = h^2.$$

This reduces the equation (4.24) to the equation of a simple harmonic oscillator

$$\ddot{h} - h = 0,$$

whose solution is

$$h(t) = C_1 e^t + C_2 e^{-t}.$$

Hence, by (3.19) timelike minimal surfaces of revolution about ξ_2 axis is given by

$$\varphi(t,x) = ((C_1 e^t + C_2 e^{-t})\cosh x, (C_1 e^t + C_2 e^{-t})\sinh x, t).$$

Timelike minimal surface of revolution about spacelike axis is called *timelike catenoid with spacelike axis*.

Figure 2 shows an example of timelike catenoid with spacelike axis with $C_1 = \frac{1}{2}$ and $C_2 = -\frac{1}{2}$, i.e.,

$$\varphi(t,x) = (\sinh t \cosh x, \sinh t \sinh x, t).$$

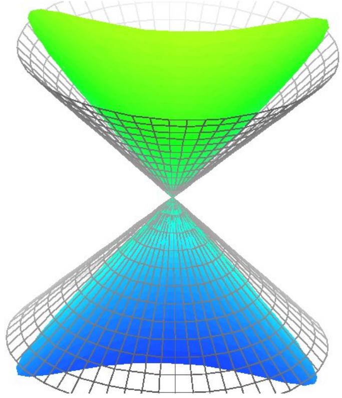


Figure 2: Timelike catenoid with spacelike axis

4.3 Timelike minimal surfaces of revolution about null axes

In this subsection, we consider spacelike maximal surfaces of revolution about the null axis $\xi_0 + \xi_1$.

In the equation (3.22), set $H = 0$. Then

$$\frac{1}{2}\frac{(t-h)\ddot{h} - (\dot{h}-1)(\dot{h}^2-1)}{(t-h)(1-\dot{h}^2)^{\frac{3}{2}}} \tag{4.25}$$

The quantities E, F, G are computed to be

$$E = -1 + \dot{h}^2,\ F = 0,\ G = (t-h)^2.$$

The conformality condition (2.2) is

$$1 - \dot{h}^2 = (t - h)^2.$$

This reduces (4.25) to the equation

$$\ddot{h} + (\dot{h} - 1)(t - h) = 0,$$

whose solution is

$$h(x) = t - C_1 \tanh\left[\frac{C_1}{2}(t - C_2)\right].$$

Take $C_1 = 2$ and $C_2 = 0$. Then by (3.21) the coordinate functions $\varphi_0, \varphi_1, \varphi_2$ of timelike minimal surface of revolution about $\xi_0 + \xi_1$ axis are given by

$$\begin{aligned}
\varphi_0(t, x) &= t\left(1 + \frac{x^2}{2}\right) - \left(\frac{t}{2} - \tanh t\right)x^2, \\
\varphi_1(t, x) &= \frac{tx^2}{2} + (t - 2\tanh t)\left(1 - \frac{x^2}{2}\right), \\
\varphi_2(t, x) &= tx - (t - 2\tanh t)x.
\end{aligned}$$

Note that $t > h(t)$ if $t > 0$.

Figure 3 shows the graphics of this surface.

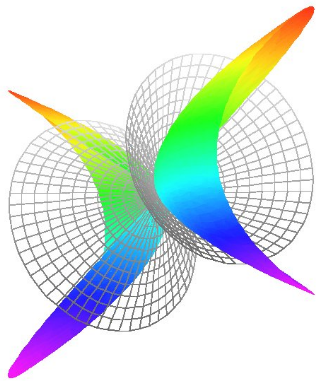


Figure 3: Timlike minimal surface of revolution about $\xi_0 + \xi_1$

§ 5. Timelike surfaces of revolution in Minkowski 3-space with non-zero constant mean curvature

In this section, using *calculus of variation* we derive simpler first-order ODEs whose solutions are timelike profile curves of timelike cmc $H \neq 0$ surfaces of revolution. These equations are still non-linear and it is very difficult to find explicit solutions. Since they are first-order ODEs, well-known simple numerical methods such as Runge-Kutta's method or Euler's method can be employed to find numerical solutions. From the numerical solutions we obtain examples of timelike cmc $\neq 0$ surfaces in Minkowski 3-space.

5.1 Timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about the timelike axis ξ_0

In this subsection, we obtain an example of timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about timelike axis.

Let $t := \xi_0$ and $x := \xi_1$. Let $x = h(t)$ be a timelike profile curve of a timelike $\mathsf{cmc} \neq 0$ surface of revolution about ξ_0. Then the area functional with volume constraint is given by

$$\begin{aligned} J(x, \dot{x}, t) &= S - \lambda V \\ &= \int_{t_1}^{t_2} (2\pi h\sqrt{1 - \dot{h}^2} - \lambda\pi h^2) dt. \end{aligned}$$

Here, S is the area functional, V is volume constraint, and λ is a coupling constant called the *Lagrangian multiplier.*

Remark 4. Since $x = h(t)$ is a timelike profile curve, the metric on the profile curve is $-dt^2 + dx^2 < 0$. Hence, the line element of the profile curve is given by the *proper time*[2] $\sqrt{dt^2 - dx^2} = \sqrt{1 - \dot{h}^2} dt$.

Let $f(x, \dot{x}, t) = 2\pi h\sqrt{1 - \dot{h}^2} - \lambda\pi h^2$. Then J has an extremum if and only if $f(x, \dot{x}, t)$ satisfies the *Euler-Lagrange equation*

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - \dot{x}\frac{\partial f}{\partial \dot{x}}\right) = 0. \tag{5.26}$$

This equation is equivalent to

$$h^2 \pm \frac{2ah}{\sqrt{1 - \dot{h}^2}} = \pm b^2. \tag{5.27}$$

Figure 4 shows a numerical solution of

$$h^2 - \frac{2ah}{\sqrt{1 - \dot{h}^2}} = b^2 \tag{5.28}$$

with $a = 1, b = 2$. The initial condition is $h(0) = -1.2$.

By rotating the profile curve in figure 4 about ξ_0 axis, we obtain the timelike $\mathsf{cmc} \neq 0$[3] surface of revolution in figure 5.

5.2 Timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about spacelike axes

In this subsection, we obtain an example of timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about spacelike axes.

[2]Physically, proper time is the actual time elapsed on a physical clock carried along the curve.

[3]At the moment, we are not able to determine the value of the constant mean curvature. We will discuss how to determine the mean curvature in the next section.

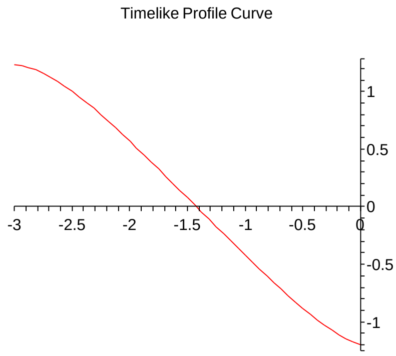


Figure 4: Numerical solution of (5.28) with $a = 1, b = 2$ and the initial condition $h(0) = -1.2$

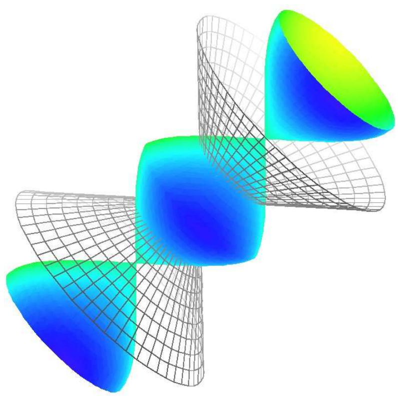


Figure 5: Timelike $\mathsf{cmc} \neq 0$ surface of revolution about ξ_0 axis in Minkowski 3-space

Let $x := \xi_0$ and $t := \xi_1$. Let $x = h(t)$ be a timelike profile curve. Then the area functional with volume constraint is given by

$$J(x, \dot{x}, t) = \int_{t_1}^{t_2} (2\pi h \sqrt{\dot{h}^2 - 1} - \lambda \pi h^2) dt.$$

Note that the line element of $x = h(t)$ is given by $\sqrt{dx^2 - dt^2} = \sqrt{\dot{h}^2 - 1} dt$ in this case.

Let $f(x, \dot{x}, t) = 2\pi h \sqrt{\dot{h}^2 - 1} - \lambda \pi h^2$. Then J has an extremum if and only if $f(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation (5.26) which turns out to be

$$h^2 \pm \frac{2ah}{\sqrt{\dot{h}^2 - 1}} = \pm b^2. \tag{5.29}$$

Figure 6 shows a numerical solution of

$$h^2 - \frac{2ah}{\sqrt{h^2-1}} = b^2 \qquad (5.30)$$

with $a = 1, b = 2$. The initial condition is $h(0) = -1.8$.

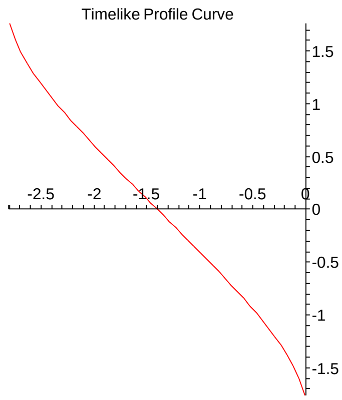
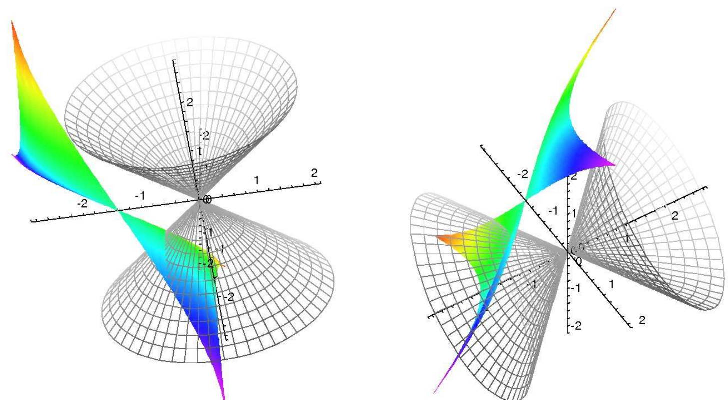


Figure 6: Numerical solution of (5.30) with $a = 1, b = 2$ and the initial condition $h(0) = -1.8$

By rotating the profile curve in figure 6 about ξ_1 axis, we obtain the timelike $\mathsf{cmc} \neq 0$ surface of revolution in figure 7.



(a) (b)

Figure 7: Timelike $\mathsf{cmc} \neq 0$ surface of revolution about ξ_1 axis in Minkowski 3-space

5.3 Timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about null axes

In this subsection, we obtain an example of timelike $\mathsf{cmc} \neq 0$ surfaces of revolution about null axes.

This case is tricky because it is difficult to set up area functional in terms of the rectangular coordinates t, x.

Let $t := \xi_0$ and $x := \xi_1$. Let $u = t + x$ and $v = -t + x$. Then (u, v) is the *null coordinate system.* We are able to set up the area functional with volume constraint in terms of null coordinates.

Let $v = g(u)$ be a timelike profile curve. If we assume that the profile curve can be written as $x = h(t)$ in terms of rectangular coordinates (t, x), then the line element $d\ell$ is given by

$$\begin{aligned} d\ell &= \sqrt{dt^2 - dx^2} \\ &= \sqrt{-dudv} \\ &= \sqrt{-g'(u)}du. \end{aligned}$$

Here, $g^{'}$ denotes $\frac{dg}{du}$. So, the area functional with volume constraint is

$$J(v, v^{'}, u) = \int_{u_1}^{u_2} (2\pi g\sqrt{-g'} - \lambda\pi g^2)du.$$

Let $f(v, v^{'}, u) = 2\pi g\sqrt{-g'} - \lambda\pi g^2$. Then J has an extremum if and only if $f(v, v^{'}, u)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial u} - \frac{d}{du}\left(f - v^{'}\frac{\partial f}{\partial v'}\right) = 0.$$

This is equivalent to

$$g^2 \pm ag\sqrt{-g'} = \pm b^2. \tag{5.31}$$

Now, we want to convert the equation (5.31) back in terms of rectangular coordinate system (t, x). In particular, the equation

$$g^2 - ag\sqrt{-g'} = b^2$$

with $a = 1$ and $b = 2$ is converted to

$$\frac{-1 + \dot{x}}{1 + \dot{x}} = -\frac{[(-t + x)^2 - 4]^2}{(-t + x)^2}. \tag{5.32}$$

Note that the LHS of this equation is the same as $g^{'}(u) = \frac{dg}{du}$.

Figure 8 shows a numerical solution of (5.32) with the initial condition $x(0) = 0$.

By rotating the profile curve in figure 8 about $\xi_0 + \xi_1$ axis, we obtain the timelike $\mathsf{cmc} \neq 0$ surface of revolution in figure 9.

§ 6. More on timelike $\mathsf{cmc} \neq 0$ surfaces of revolution in Minkowski 3-space

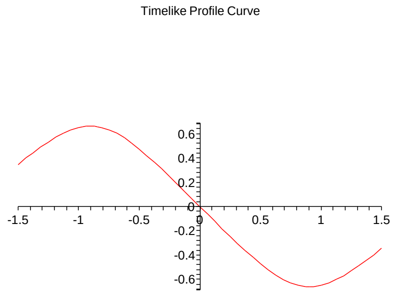


Figure 8: Numerical solution of (5.32) with the initial condition $x(0) = 0$

In the previous section, we obtained some examples of timelike cmc $\neq 0$ surfaces of revolution in Minkowski 3-space. Although we know that those surfaces have nonzero constant mean curvature, we were not able to determine the value of constant mean curvature.

In this section, we study how to determine the mean curvature of timelike cmc $\neq 0$ surfaces of revolution in Minkowski 3-space.

Proposition 5. *If $h(t)$ satisfies the equation* (5.27) *or* (5.29)*, then the mean curvature H of the resulting surface of revolution is $H = \pm\frac{1}{2a}$.*

Proof. Assume that $h(t)$ satisfies the equation (5.27). Differentiating (5.27), we get

$$\frac{1}{2a} = \pm\frac{1}{2}\frac{\ddot{h}h + 1 - \dot{h}^2}{h(1-\dot{h}^2)^{\frac{3}{2}}}.$$

It follows immediately from (3.16) that $H = \pm\frac{1}{2a}$. □

By proposition 5, we are now able to determine the mean curvatures of surfaces in figures 5 and 7. For both surfaces $a = 1$ was used, so both surfaces have the constant mean curvature[4] $H = \pm\frac{1}{2}$.

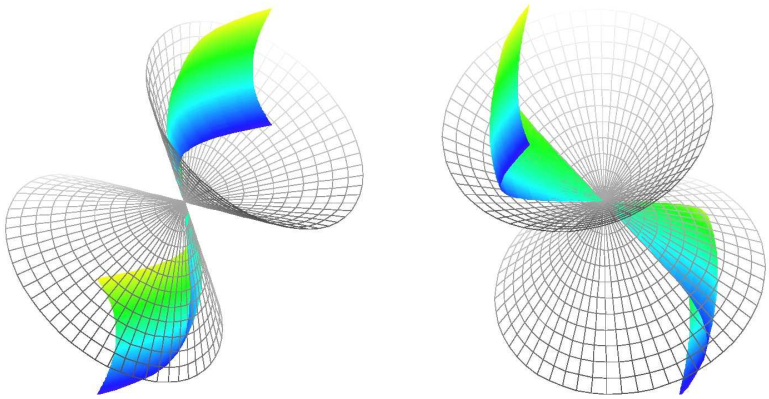
Remark 5. The authors are unable to determine the mean curvature of timelike surfaces of revolution about null axes such as in figure 9. The reason is that the equation (5.31) was obtained in terms of the null coordinate system (u, v), while the equation (3.22) was obtained in terms of the rectangular coordinate system (t, x), and there is no obvious conversion between these two equations.

The following lemma can be easily proved by a direct computation.

Lemma 6. *1. The mean curvature equation* (3.16) *is equivalent to*

$$\frac{d}{dt}\left(Hh^2 - \frac{h}{\sqrt{1-\dot{h}^2}}\right) = 0. \tag{6.33}$$

[4] The sign of mean curvature depends on the orientation of a surface, or equivalently the orientation of normal vector field of a surface.



(a) (b)

Figure 9: Timelike $\mathsf{cmc} \neq 0$ surface of revolution about $\xi_0 + \xi_1$ axis in Minkowski 3-space

2. *The mean curvature equation* (3.18) *is equivalent to*

$$\frac{d}{dt}\left(Hh^2 - \frac{h}{\sqrt{\dot{h}^2 - 1}}\right) = 0. \tag{6.34}$$

3. *The mean curvature equation* (3.20) *is equivalent to*

$$\frac{d}{dt}\left(Hh^2 + \frac{h}{\sqrt{\dot{h}^2 - 1}}\right) = 0. \tag{6.35}$$

Theorem 7. 1. *If a timelike surface of revolution about timelike axis with $H \neq 0$ satisfies*

$$Hh^2 - \frac{h}{\sqrt{1 - \dot{h}^2}} = 0,$$

then it is part of the pseudosphere[5] S_1^2 *of radius* $\frac{1}{H}$, *and hence it is totally umbilic.*

2. *If a timelike surface of revolution about spacelike axis ξ_1 or ξ_2 with $H \neq 0$ satisfies*

$$Hh^2 \mp \frac{h}{\sqrt{\dot{h}^2 - 1}} = 0,$$

then it is part of the pseudosphere S_1^2 of radius $\frac{1}{H}$, and hence it is totally umbilic.

[5]The pseudosphere is also called *de Sitter* 2*-space*.

Proof. Assume that the profile curve $h(t)$ satisfies the equation

$$Hh^2 - \frac{h}{\sqrt{1-\dot{h}^2}} = 0,$$

i.e.,

$$\frac{dh}{dt} = \pm\frac{\sqrt{H^2h^2-1}}{Hh}.$$

This is a simple separable equation and its solution is

$$h(t) = \pm\frac{\sqrt{H^2(t+d)^2+1}}{H},$$

where d is a constant.

Note that $\varphi(t,x) = (t+d, h(t)\cos x, h(t)\sin x)$ is a timelike cmc $H \neq 0$ surface of revolution about timelike axis with the profile curve $h(t)$. Now,

$$\begin{aligned}||\varphi||^2 &= -(t+d)^2 + h(t)^2\cos^2 x + x^2 + h(t)^2\sin^2 x\\ &= \frac{1}{H^2}.\end{aligned}$$

Hence, φ is part of the pseudosphere S_1^2 centered at $(d,0,0)$ with radius $\frac{1}{H}$. Since S_1^2 is totally umbilic, so is φ.

The other cases are proved in the same manner. □

Remark 6. It is well-known that a totally umbilic timelike surface of constant mean curvature H in Minkowski 3-space is part of timelike plane ($H = 0$) or part of the psudosphere S_1^2 of radius $\frac{1}{H}$.

Example 1. $\varphi(t,x) = (t, \sqrt{t^2+1}\cos x, \sqrt{t^2+1}\sin x)$ is a totally umbilic timelike surface of revolution about timelike axis with $H = 1$. See figure 10. The graphics of $\varphi(t,x)$ was drawn with the pseudosphere (grey wire-framed) centered at $(0,0,0)$ with radius 1.

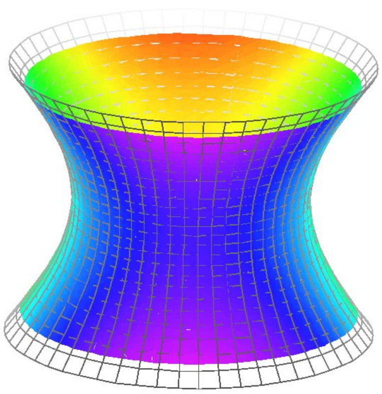


Figure 10: $\varphi(t,x) = (t, \sqrt{t^2+1}\cos x, \sqrt{t^2+1}\sin x)$

Example 2. $\psi(t,x) = (\sqrt{t^2-1}\cosh x, t, \sqrt{t^2 1}\sinh x)$ is a totally umbilic timelike surface of revolution about spacelike axis ξ_1 with $H = 1$. See figure 11. The graphics of $\psi(t,x)$ was drawn with the pseudosphere (grey wire-framed) centered at $(0,0,0)$ with radius 1.

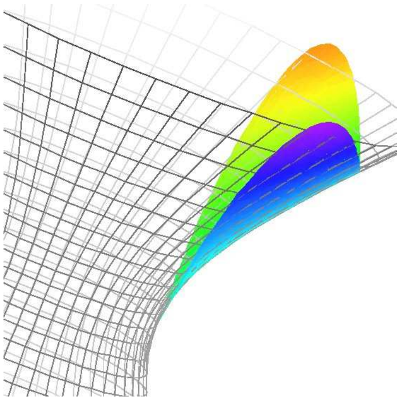


Figure 11: $\psi(t,x) = (\sqrt{t^2-1}\cosh x, t, \sqrt{t^2 1}\sinh x)$

Remark 7. The equations (6.33), (6.34), and (6.35) can be also used to obtain examples of timelike cmc surfaces of revolution including timelike minimal surfaces of revolution.

The following proposition is easily obtained by a direct calculation.

Proposition 8. *1. The general solution to the equation* (6.33) *with* $H = 0$ *is*

$$h(t) = \pm\frac{1}{a}\sin(at \pm b),$$

where $a \neq 0$ *and* b *are constants. The timelike surface obtained by rotating* $h(t)$ *about timelike axis is conformal if and only if* $a = \pm 1$.

Notice that this solution with $a = \pm 1$ *coincides with the solution we found in subsection 4.1.*

2. The general solution to the equation (6.34) *or* (6.35) *with* $H = 0$ *is*

$$h(t) = \pm\frac{1}{a}\sinh(at \pm b),$$

where $a \neq 0$ *and* b *are constants. The timelike surface obtained by rotating* $h(t)$ *about spacelike axis* ξ_1 *or* ξ_2 *is conformal if and only if* $a = \pm 1$.

Notice that this solution with $a = \pm 1$ *coincides with the solution we found in subsection 4.2.*

The equation

$$Hh^2 - \frac{h}{\sqrt{1-\dot{h}^2}} = c \tag{6.36}$$

with $c \neq 0$, or equivalently

$$\frac{dh}{dt} = \pm \frac{\sqrt{(Hh^2 - c)^2 - h^2}}{Hh^2 - c}$$

is a separable first-order ODE. However, the integration involves elliptic functions with $c \neq 0$. Hence, numerical methods still need to be employed to obtain examples of timelike cmc $\neq 0$ surfaces of revolution about timelike axis using the equation (6.36). Figure 12 shows numerical solution to (6.36) with $H = c = 1$ and the initial condition $h(0) = -0.55$.

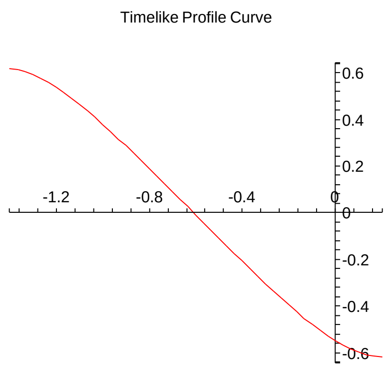


Figure 12: Numerical solution to $\frac{dh}{dt} = \frac{\sqrt{h^4 - 3h^2 + 1}}{h^2 - 1}$ with initial condition $h(0) = -0.55$.

Figure 13 shows timelike cmc 1 surface that is obtained by rotating $h(t)$ in figure 12 about timelike axis.

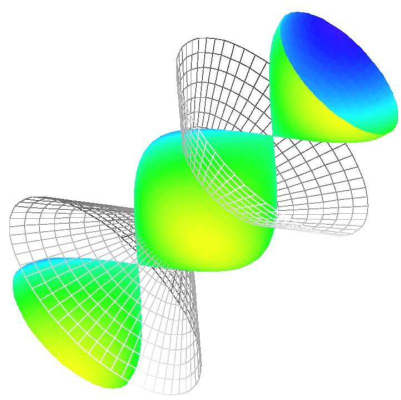


Figure 13: Timelike cmc 1 surface of revolution about timelike axis

Remark 8. The graphics in figures 1, 5, and 13 all show similar cylindrical shapes foliated by circles in parallel spacelike planes. Hence, one may wonder if this is the only case for timelike cmc surfaces of revolution about timelike axis. It turns out that this is the only case. In [4] (Proposition 1 on page 519), López proved that the cylinder $\xi_1^2+\xi_2^2=C^2$ is the only timelike cmc surfaces of revolution about timelike axis in Minkowski 3-space. Moreover, it is also a cmc surface of revolution with respect to the Euclidean metric.

Similarly, timelike cmc $\neq 0$ surfaces of revolution about spacelike axis ξ_1 or ξ_2 can be obtained by solving the equations

$$Hh^2 \mp \frac{h}{\sqrt{\dot{h}^2-1}} = c \tag{6.37}$$

with $c \neq 0$ or equivalently,

$$\frac{dh}{dt} = \pm\frac{\sqrt{(Hh^2-c)^2+h^2}}{Hh^2-c}.$$

Figure 14 shows numerical solution to the equation (6.37) with $H=c=1$ and the initial condition $h(0)=-0.9$.

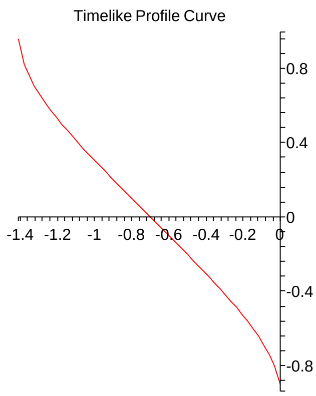


Figure 14: Numerical solution to $\frac{dh}{dt} = \frac{\sqrt{h^4-h^2+1}}{h^2-1}$ with initial condition $h(0)=-0.9$

Figure 15 shows timelike cmc 1 surface that is obtained by rotating $h(t)$ in figure 14 about ξ_1 axis.

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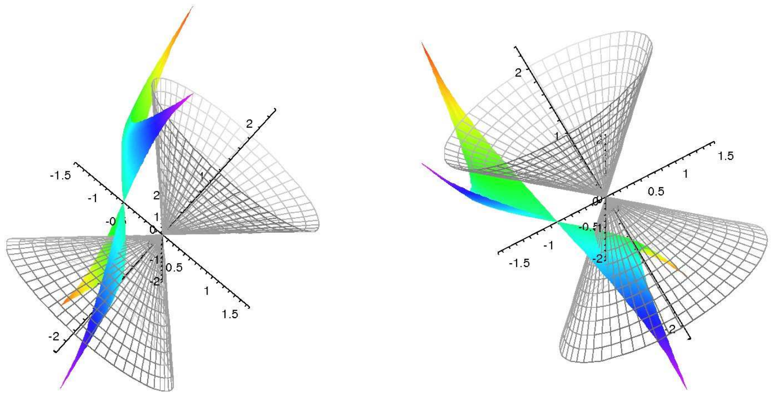

(a) (b)

Figure 15: Timelike cmc 1 surface of revolution about ξ_1 axis

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