L^∞-uniqueness of Schrödinger operators on a Riemannian manifold

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Abstract. The main purpose of this paper is to study L^∞-uniqueness of Schrödinger operators and generalized Schrödinger operators on a complete Riemannian manifold. Also, we prove the $L^1(E, d\mu)$-uniqueness of weak solutions for the Fokker-Planck equation associated with this pre-generators.

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§ 1. Preliminaries

Let E be a Polish space equipped with a σ-finite measure μ on its Borel σ-field $\mathcal{B}$. It is well known that, for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^1(E, d\mu)$, its adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ is no longer strongly continuous on the dual topological space $L^\infty(E, d\mu)$ of $L^1(E, d\mu)$ with respect to the strong dual topology of $L^\infty(E, d\mu)$. In [15] Wu and Zhang introduce a new topology on $L^\infty(E, d\mu)$ for which the usual semigroups in the literature becomes C_0-semigroups. That is *the topology of uniform convergence on compact subsets of* $(L^1(E, d\mu), \| \,.\, \|_1)$, denoted by $\mathcal{C}(L^\infty, L^1)$. More precisely, for an arbitrary point $y_0 \in L^\infty(E, d\mu)$, a basis of neighborhoods with respect to $\mathcal{C}(L^\infty, L^1)$ is given by:

$$N(y_0; K, \varepsilon) := \left\{ y \in L^\infty(E, d\mu) \;\middle|\; \sup_{x\in K} |\langle x, y\rangle - \langle x, y_0\rangle| < \varepsilon \right\} \tag{1.1}$$

where K runs over all compact subsets of $(L^1(E, d\mu), \| \,.\, \|_1)$ and $\varepsilon > 0$.
If $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on $L^1(E, \mu)$ with generator $\mathcal{L}$, then $\{T^*(t)\}_{t\geq 0}$ is a C_0-semigroup on $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ with generator $\mathcal{L}^*$. Moreover, on can proove that $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ is complete and that the topological dual of $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ is $\left(L^1(E, d\mu), \| \,.\, \|_1\right)$.

Let $\mathcal{A} : \mathcal{D} \longrightarrow L^\infty(E, d\mu)$ be a linear operator with domain $\mathcal{D}$ dense in $(L^\infty(E, d\mu))$, with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. $\mathcal{A}$ is said to be a *pre-generator* in $(L^\infty(E, d\mu))$, if there exists some C_0-semigroup on $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ such that its generator $\mathcal{L}$ extends $\mathcal{A}$. We say that $\mathcal{A}$ is an *essential generator* in $(L^\infty(E, d\mu))$ (or

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$(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$-unique), if $\mathcal{A}$ is closable and its closure $\overline{\mathcal{A}}$ with respect to $\mathcal{C}(L^\infty,L^1)$ is the generator of some C_0-semigroup on $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$. This uniqueness notion was studied by Arendt [1], Eberle [5], Djellout [4], Röckner [10], Wu [13] and [14] and others in the Banach spaces setting and Wu and Zhang [15], Lemle and Wu [6] in the case of locally convex spaces.
The main result concerning $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$-uniqueness of pre-generators is (see [15] or [6] for much more general results)

Theorem 1.1. *Let $\mathcal{A}$ be a linear operator on $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$ with domain $\mathcal{D}$ (the test-function space) which is assumed to be dense in $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$. Assume that there is a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$ such that its generator $\mathcal{L}$ is an extension of $\mathcal{A}$ (i.e., $\mathcal{A}$ is a pre-generator in $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$). The following assertions are equivalents:*
(i) $\mathcal{A}$ is a $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$-essential generator (or $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$-unique);
(ii) the closure of $\mathcal{A}$ in $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$ is exactly $\mathcal{L}$ (i.e., $\mathcal{D}$ is a core for $\mathcal{L}$);
(iii) $\mathcal{A}^*=\mathcal{L}^*$ which is the generator of the dual C_0-semigroup $\{T^*(t)\}_{t\geq 0}$ on $(L^1(E,d\mu))$;
(iv) for some $\lambda>\omega$ ($\omega\geq 0$ is the constant in definition of C_0-semigroup $\{T(t)\}_{t\geq 0}$), the range $(\lambda I-\mathcal{A})(\mathcal{D})$ is dense in $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$;
(v) (Liouville property) for some $\lambda>\omega$, $\mathcal{K}er\,(\lambda I-\mathcal{A}^*)=\{0\}$ (i.e., if $y\in\mathcal{D}(\mathcal{A}^*)$ satisfies $(\lambda I-\mathcal{A}^*)y=0$, then $y=0$);
(vi) (uniqueness of solutions for the resolvent equation) for all $\lambda>\omega$ and all $y\in L^1(E,d\mu)$, the resolvent equation of $\mathcal{A}^*$

$$(\lambda I-\mathcal{A}^*)z=y \tag{1.2}$$

has the unique solution $z=((\lambda I-\mathcal{L})^{-1})^*y=(\lambda I-\mathcal{L}^*)^{-1}y$;
(vii) (uniqueness of strong solutions for the Cauchy problem) for each $x\in\mathcal{D}(\overline{\mathcal{A}})$, there is a $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$-unique strong solution $v(t)=T(t)x$ of the Cauchy problem (or the Kolmogorov backward equation)

$$\begin{cases}\partial_t v(t)=\overline{\mathcal{A}}v(t)\\ v(0)=x\end{cases} \tag{1.3}$$

i.e., $t\mapsto v(t)$ is differentiable from $\mathbb{R}^+$ to $(L^\infty(E,d\mu),\mathcal{C}(L^\infty,L^1))$ and its derivative $\partial_t v(t)$ coincides with $\overline{\mathcal{A}}v(t)$;
(viii) (uniqueness of weak solutions for the dual Cauchy problem) for every $y\in L^1(E,d\mu)$, the dual Cauchy problem (the Kolmogorov forward equation or the Fokker-Planck equation)

$$\begin{cases}\partial_t u(t)=\mathcal{A}^*u(t)\\ u(0)=y\end{cases} \tag{1.4}$$

has a $(L^1(E,d\mu))$-unique weak solution $u(t)=T^*(t)y$. More precisely, there is a unique function $\mathbb{R}^+\ni t\longmapsto u(t)=T^*(t)y$ which is continuous from $\mathbb{R}^+$ to $(L^1(E,d\mu),\|\,.\,\|_1)$ such that

$$\langle x,u(t)-y\rangle=\int_0^t\langle\mathcal{A}x,u(s)\rangle\,ds\quad,\quad\forall x\in\mathcal{D}; \tag{1.5}$$

(ix) there is only one C_0-semigroup on $\left(L^\infty(E,d\mu),\mathcal{C}\left(L^\infty,L^1\right)\right)$ such that its generator extends $\mathcal{A}$.
(x) there is only one C_0-semigroupe on $(L^1(E,d\mu),\|\ \|_1)$ such that its generator is contained in A^*.

§ 2. L^∞-uniqueness of Schrödinger operators on a complete Riemannian manifold

Let M be a complete Riemannian manifold with volume measure dx. Denote by Δ the Laplace-Beltrami operator with domain $C_0^\infty(M)$. The $L^p(M,dx)$-uniqueness of C_0-semigroup generated by $(\Delta, C_0^\infty(M))$ is proven by Strichartz [12] for $1<p<\infty$, by Davies [3] for $p=1$ and by Li [7] for $p=\infty$.
In fact, Li [7] showed that if M is complete and has bounded geometry, then M satisfies L^1-Liouville theorem: if the Ricci curvature of M has a negative quadratic lower bound, i.e., if there exists a fixed point $0\in M$ and a constant $C>0$ such that

$$Ric(x)\geq -C\left(1+d(x,0)^2\right)\quad,\quad \forall x\in M \tag{2.6}$$

where $d(x,0)$ denotes distance from x to o, then every non-negative L^1-integrable subharmonic function must be constant. Under the same condition, Li also proved that for each $h\in L^1(M,dx)$, the heat diffusion equation

$$\begin{cases}\partial_t u(t)=\Delta u(t)\\ u(0)=h\end{cases} \tag{2.7}$$

has one $L^1(M,dx)$-unique weak solution.
Consider the Schrödinger operator $(\mathcal{A}, C_0^\infty(M))$

$$\mathcal{A}=\frac{\Delta}{2}-V \tag{2.8}$$

where $V\geq 0$. In that case, we have

Theorem 2.2. *Let M be a complete Riemmanian manifold such that its Ricci curvature has a negative quadratic lower bound. Then $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M,dx)$-unique with respect to the topology $\mathcal{C}\left(L^\infty,L^1\right)$.*

Proof. We prove the theorem by two steps.
step 1 $(\mathcal{A}, C_0^\infty(M))$ is a pre-generator on $\left(L^\infty(M,dx),\mathcal{C}\left(L^\infty,L^1\right)\right)$.
Let $(B_t)_{t\geq 0}$ be the Brownian Motion with values in the one-point compactification space $M\cup\partial$ defined on some filtered probabilities space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},(B_t)_{t\geq 0})$ with $\mathbb{P}_x(B_0=x)=1$ for any initial point $x\in M$. Let denote by

$$\tau_e=\inf\{t\geq 0\ |B_t=\partial\} \tag{2.9}$$

the explosion time. Consider the Feynman-Kac semigroup

$$P_t^V f(x)=\mathbb{E}^x 1_{[t<\tau_e]}f(B_t)e^{-\int_0^t V(B_s)ds}\quad. \tag{2.10}$$

Since $\left\{P_t^V\right\}_{t\geq 0}$is a C_0-semigroup on $L^1(M,dx)$, it is also a C_0-semigroup on $L^\infty(M,dx)$ with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. By Ito's formula, for every $f\in C_0^\infty(M)$ it follows that

$$f(B_t)e^{-\int\limits_0^t V(B_s)ds}-f(B_0)-\int\limits_0^t\left(\frac{\Delta}{2}-V\right)f(B_s)e^{-\int\limits_0^s V(B_r)dr}ds \tag{2.11}$$

is a local martingale. As it is bounded over bounded time intervals, it is a true martingale. Thus taking the expectation under $\mathbb{P}_x$ in the above formula, we find that

$$P_t^V f(x)-f(x)=\int\limits_0^t P_s^V\left(\frac{\Delta}{2}-V\right)f(x)ds \quad , \quad \forall t\geq 0 \tag{2.12}$$

which means that f belongs to the domain of the generator $\mathcal{L}_{(\infty)}^V$ of $\left\{P_t^V\right\}_{t\geq 0}$, i.e. $\mathcal{L}_{(\infty)}^V$ extend $\mathcal{A}$.

step 2 $(\mathcal{A},C_0^\infty(M))$ is $\left(L^\infty(M,dx),\mathcal{C}\left(L^\infty,L^1\right)\right)$-unique.

By Theorem 1.1, it follows that the operator $(\mathcal{A},C_0^\infty(M))$ is $L^\infty(M,dx)$-unique if and only if for some $\lambda>\omega$, the range $(\lambda I-\mathcal{A})\left(C_0^\infty(M)\right)$ is dense in $\left(L^\infty,\mathcal{C}\left(L^\infty,L^1\right)\right)$. It is enough to show that if $h\in L^1(M,dx)$ satisfies $(\lambda I-\mathcal{A})h=0$ in the sense of distribution, then $h=0$.

Let $h\in L^1(M,dx)$ such that for some $\lambda>\omega$

$$(\lambda I-\mathcal{A})h=0 \tag{2.13}$$

in the distribution sense. Then, by Kato's inequality, we have

$$\Delta|h|\geq sgn(h)\Delta h=2sgn(h)(\lambda+V)(h)=2(\lambda+V)|h|\geq 0 \quad . \tag{2.14}$$

Thus $|h|$ is a subharmonic function. By [7, Theorem 1, p. 447] it follows that $|h|$ is constant, therefore h is constant. Consequently $h=0$. □

Corollary 2.3. *If M is a complete Riemannian manifold such that its Ricci curvature has a negative quadratic lower bound, then for every $h\in L^1(M,dx)$, the Fokker-Planck equation*

$$\begin{cases}\partial_t u(t)=\left(\frac{\Delta}{2}-V\right)u(t)\\ u(0)=h\end{cases} \tag{2.15}$$

has one $L^1(M,dx)$-unique weak solution.

Proof. By the Theorem 1.1 the $L^1(M,dx)$-uniqueness of weak solutions for the heat diffusion equation

$$\begin{cases}\partial_t u(t)=\left(\frac{\Delta}{2}-V\right)u(t)\\ u(0)=h\end{cases} \tag{2.16}$$

is equivalent to the $\left(L^\infty(M,dx),\mathcal{C}\left(L^\infty,L^1\right)\right)$-uniqueness of the Schrödinger operator $(\mathcal{A},C_0^\infty(M))$ which follows by the Theorem 2.2. □

§ 3. L^∞-uniqueness of generalized Schrödinger operators on a complete Riemannian manifold

Consider the generalized Schrödinger operator (or the Nelson's diffusion operator in stochastic mechanics see [14])

$$\mathcal{A} = \Delta - \nabla\phi \cdot \nabla \tag{3.17}$$

with domain $C_0^\infty(M)$ and an invariant measure $d\mu_\phi = e^{-\phi(x)}dx$, where Δ is the Laplace-Beltrami operator and ∇ is the gradient operator on M and $\phi \in C^\infty(M)$. The study of this operator is very important because it is well known that $\mathcal{A} = \Delta - \nabla\phi \cdot \nabla$ considered as a symmetric operator on $L^2(M, e^{-\phi(x)}dx)$ is unitary equivalent to the Schrödinger operator $H = \Delta - V$, with $V = \frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi$ considered on $L^2(M, dx)$. This unitary isomorphism $U : L^2(M, e^{-\phi(x)}dx) \to L^2(M, dx)$ is given by $Uf = e^{-\frac{\phi}{2}} f$. In the case where $M = \mathbb{R}^d$ or $M = D$ is an open domain in $\mathbb{R}^d$ for the diffusion operator $\mathcal{A} = \Delta + 2\frac{\nabla\phi}{\phi}\cdot \nabla$, $L^1(M, \phi^2 dx)$-uniqueness has been studied by Liskevitch [9], Stannat [11] and Wu [14].
The purpose of this section is to study $L^\infty(M, e^{-\phi(x)}dx)$-uniqueness of the operator $(\mathcal{A}, C_0^\infty(M))$ using some very recent result of Li [8] concerning the $L^1(M, e^{-\phi(x)}dx)$ Liouville theorem on a complete Riemannian manifold. Recall that for any constant $m \geq n = dim(M)$ the symmetric 2-tensor

$$Ric_{m,n}(\mathcal{A})(x) = Ric(x) + \nabla^2\phi(x) - \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m-n} \quad , \quad \forall x \in M \tag{3.18}$$

is called *the Bakry-Emery Ricci curvature* of the operator $\mathcal{A}$ (see [2]), with the convention that $m = n$ iff $\mathcal{A} = \Delta$.
The main result of this section is

Theorem 3.4. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists two constants $m > n$ and $C > 0$ such that*

$$Ric_{m,n}(\mathcal{A})(x) \geq -C\left(1 + d^2(x, 0)\right) \quad , \quad \forall x \in M \tag{3.19}$$

where $d(x, 0)$ is the distance from x to a fixed point $0 \in M$. Then $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$.

Proof. We prove the theorem by two steps.
step 1 $(\mathcal{A}, C_0^\infty(M))$ is a pre-generator on $\left(L^\infty(M, e^{-\phi(x)}dx), \mathcal{C}\left(L^\infty, L^1\right)\right)$.
Let $(W_t)_{t\geq 0}$ be the Riemannian Brownian motion on M. The coresponding stochastic differential equation

$$dX_t = \sqrt{2}dW_t - \nabla(X_t)dt \tag{3.20}$$

admits one unique martigale solution $(X_t)_{0\leq t<\tau_e}$, where τ_e is the explosion time. The transition semigroup $\{P_t\}_{t\geq 0}$ is given by the Feynman-Kac formula

$$P_t f(x) = \mathbb{E}^x f(X_t) 1_{[t<\tau_e]} \quad , \quad f \in C_0^\infty(M) \tag{3.21}$$

Since $\{P_t\}_{t\geq 0}$ is a C_0-semigroup on $(L^1(M, e^{-\phi(x)}dx), \|\;\|_1)$, it is also a C_0-semigroup on $L^\infty(M, e^{-\phi(x)}dx)$ with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. By Ito's formula, for every $f \in C_0^\infty(M)$

$$f(X_t) - f(x) - \int_0^t \mathcal{A}f(X_t)ds \tag{3.22}$$

is a local martingale. As it is bounded over bounded time intervals, it is a true martingale. By taking the expectation under $\mathbb{P}_x$ in the above formula, we find that

$$P_t f(x) - f(x) = \int_0^t P_s \mathcal{A} f(x) ds \quad , \quad \forall t \geq 0 \tag{3.23}$$

which means that f belongs to the domain of the generator $\mathcal{L}_{(\infty)}$ of $\{P_t\}_{t\geq 0}$ i.e., $\mathcal{L}_{(\infty)}$ extend $\mathcal{A}$.

step 2 $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. By Theorem 1.1, it follows that the operator $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique if and only if for some $\lambda > \omega$, the range $(\mathcal{A}-\lambda I)\left(C_0^\infty(M)\right)$ is dense in $\left(L^\infty, \mathcal{C}\left(L^\infty, L^1\right)\right)$. It is enough to show that if $u \in L^1(M, e^{-\phi(x)}dx)$ satisfies $(\mathcal{A} - \lambda I)u = 0$ in the sense of distribution, then $u = 0$.

Let $u \in L^1(M, e^{-\phi(x)}dx)$ such that

$$(\mathcal{A} - I)u = 0 \tag{3.24}$$

in the distribution sense, i.e.

$$\langle u, (\mathcal{A} - I)f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(M). \tag{3.25}$$

Then from

$$\int_M u\Delta f\, e^{-\phi(x)}dx - \int_M u\nabla\phi\nabla f\, e^{-\phi(x)}dx = \int_M uf\, e^{-\phi(x)}dx \tag{3.26}$$

we will obtain the following equality

$$-\int_M \nabla u\nabla f\, e^{-\phi(x)}dx - \int_M u\nabla\phi\nabla f\, e^{-\phi(x)}dx = \int_M uf\, e^{-\phi(x)}dx \tag{3.27}$$

for all positive function $f \in H^{1,2}(M, e^{-\phi(x)}dx)$ with compact support. Now on can follow Eberle [5, proof of Theorem 2.5, Step 2] to show that (an inequality of Kato's type)

$$-\int_M \nabla|u|\nabla f\, e^{-\phi(x)}dx - \int_M |u|\nabla\phi\nabla f\, e^{-\phi(x)}dx \geq \int_M |u|f\, e^{-\phi(x)}dx \tag{3.28}$$

for all positive function $f \in H^{1,2}(M, e^{-\phi(x)}dx)$ with compact support. But this is equivalent to

$$\int_M |u|\mathcal{A}f\, e^{-\phi(x)}dx \geq \int_M |u|f\, e^{-\phi(x)}dx \geq 0 \quad . \tag{3.29}$$

That means $|u|$ is a $\mathcal{A}$-subharmonic function. By the Theorem [8, Theorem 7.1], it follows that $|u|$ must be a constant function and then $u = 0$. By the Theorem 1.1 it follows that $(\mathcal{A}, C_0^\infty(M))$ is $L^\infty(M, e^{-\phi(x)}dx)$-unique. □

Corollary 3.5. *In the hypothesis of Theorem 3.4, for every $h \in L^1(M, dx)$, the Fokker-Planck equation*

$$\begin{cases} \partial_t u(t) = \mathcal{A}^* u(t) \\ u(0) = h \end{cases} \tag{3.30}$$

has one $L^1(M, dx)$-unique weak solution.

Proof. The assertion follows by Theorem 1.1 and the Theorem 3.4. □

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