Projective curvature inheritance in an $NP - Fn$

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Abstract. The concept of projective curvature inheritance in Finsler space have been studied by S.P. Singh [6]. In the present investigation, our aim is to study the Projective curvature inheritance in an $NP - F_n$. Corresponding results for contra and concurrent vector fields are rendered intuitive.

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§ 1. Preliminaries

K.Yano [4] defined a set of parameters

$$\Pi^i_{kh} = G^i_{kh} - \frac{\dot{x}^i}{n+1} G^r_{khr}\ , \tag{1.1}$$

which form a connection called the normal projective connection. The functions Π^i_{kh}, G^i_{kh} andG^i_{jkh} are symmetric in their lower indices and are positively homogeneous of degree 0, 0 and -1 respectively in their $\dot{x}^i$'s. The functions $G^i_{jk}$are the Berwald's connection parameters. The derivatives $\dot{\partial}_j \Pi^i_{kh}$,denoted by Π^i_{jkh} is given by

$$\Pi^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + \dot{x}^i G^r_{jkhr}), \tag{1.2}$$

are symmetric in k and h only and are positively homogeneous of degree -1 in directional arguments. Therefore, the following relation which will be used in our discussion follow from (1.1) and (1.2)

$$\begin{cases} a) & \Pi^i_{kh}\dot{x}^k = \Pi^i_{hk}\dot{x}^k = G^i_h\ , \\ b) & \Pi^i_{ki} = G^i_{ki}\ , \\ c) & \dot{x}^j \Pi^i_{jkh} = 0\ , \\ d) & \Pi^i_{ikh} = \frac{2}{n+1} G^i_{ikh}\ , \\ e) & \Pi^i_{jki} = G^i_{jki} = \Pi^i_{jik}\ . \end{cases} \tag{1.3}$$

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The normal projective covariant derivative of a vector field $X^i(x,\dot{x})$ is defined by

$$\nabla_k X^i = \dot{\partial}_k X^i - (\dot{\partial}_j X^i)\Pi^j_{kh}\dot{x}^h + X^j \Pi^i_{jk}\ , \tag{1.4}$$

where $\partial_k = \partial/\partial x^k$, $\dot{\partial}_k = \partial/\partial \dot{x}^k$, and preserve the vector character of X^i.

In particular, this derivative vanishes for $\dot{x}^i$. The corresponding curvature tensor $N^i_{jkh}(x,\dot{x})$ as called by K.Yano, the normal projective curvature tensor, is given by

$$N^i_{jkh} = 2\{\partial_{[j}\Pi^i_{k]h} + \Pi^i_{lh[j}\Pi^l_{k]m}\dot{x}^m + \Pi^i_{l[j}\Pi^l_{k]h}\}. \tag{1.5}$$

Definition 1.1. The manifold F_n with normal projective connection parameters Π^i_{kh} and the normal projective curvature tensor N^i_{jkh} is termed as normal projective Finsler manifold and is usually denoted by $NP - F_n$.

It is worth mentioning that the normal projective curvature tensor is skew-symmetric in j, k indices and is a homogeneous function of degree 0 in $\dot{x}^i$'s, so in the light of (1.5) it is fairly easy to observe that

$$\begin{cases} a) & N^i_{jkh} = -N^i_{kjh}\ , \\ b) & \dot{\partial}_l N^i_{jkh}\dot{x}^l = 0\,. \end{cases} \tag{1.6}$$

The contraction of N^i_{jkh} with respect to i, j ; i, k and i, h give

$$\begin{cases} a) & N^i_{ikh} = N_{kh}\ , \\ b) & N^i_{jih} = -N^i_{ijh} = -N_{jh}\ , \\ b) & N^i_{jkh} = 2N_{[kj]}\ , \end{cases} \tag{1.7}$$

respectively, where $[kj]$ represent the skew-symmetric part.

P.N. Pandey [5] has shown the following relationship between the normal projective curvature tensor N^i_{jkh} and the Berwald's curvature tensor H^i_{jkh}.

$$N^i_{jkh} = H^i_{jkh} - \frac{\dot{x}^i}{n+1}\dot{\partial}_l H^r_{khr}. \tag{1.8}$$

The covariant derivative gives rise to the commutation formula

$$2\nabla_{[j}\nabla_{k]}X^i = N^i_{jkh}X^h - (\dot{\partial}_l X^i)N^l_{jkh}\dot{x}^h \tag{1.9}$$

together with the normal projective curvature tensor N^i_{jkh}. In term of contracted tensor $N_{kh} = N^i_{ikh}$ there is defined a tensor

$$M_{kh} = -\frac{1}{n^2-1}(nN_{kh} + N_{hk}) \tag{1.10}$$

and the Weyl's projective curvature tensor W^i_{jkh} is given by K.Yano [4]

$$W^i_{jkh} = N^i_{jkh} + 2\{\delta^i_{[j}M_{k]h} - M_{[jk]}\delta^i_h\}. \tag{1.11}$$

The projective curvature tensor also satisfies

$$(1.12)\qquad \begin{cases} a) & \dot{\partial}_j \partial_l W^i_{jkh} = 0, \\ b) & W^i_{ikh} = -W^i_{kih} = 0, \\ c) & W^i_{khi} = 0. \end{cases}$$

The commutation formulae for any general tensor, involving the curvature tensor, are given as follows

$$(1.13)\qquad 2\nabla_{[k}\nabla_{h]}T^i_j = N^i_{khl}T^l_j - N^l_{khj}T^i_l - (\dot{\partial}_l T^i_j)N^l_{khm}\dot{x}^m,$$

$$(1.14)\qquad (\dot{\partial}_j \nabla_k - \nabla_k \dot{\partial}_j)T^i_h = \Pi^i_{jkl}T^l_h - \Pi^l_{jkh}T^i_l - \Pi^l_{jkm}\dot{x}^m(\dot{\partial}_l T^i_h).$$

The Lie-derivative of a tensor T^i_j and the connection coefficients Π^i_{jk} defined by an infinitesimal transformation

$$(1.15)\qquad \bar{x}^i = x^i + \varepsilon v^i(x)$$

are characterized by K.Yano [4]

$$(1.16)\qquad £T^i_j = v^h(\nabla_h T^i_j) - T^h_j(\nabla_h v^i) + T^i_h(\nabla_j v^h) + (\dot{\partial}_h T^i_j)(\nabla_s v^h)\dot{x}^s$$

and

$$(1.17)\qquad £\Pi^i_{jk} = \nabla_j \nabla_k v^i - N^i_{hjk}v^h + \Pi^i_{hjk}(\nabla_l v^h)\dot{x}^l$$

respectively.

The commutation formulae with respect to Lie-derivative and other for any tensor T^i_{jk} are givenby

$$(1.18)\qquad \begin{aligned} £(\nabla_l T^i_{jk}) - \nabla_l(£T^i_{jk}) &= (£\Pi^i_{lh})T^h_{jk} - (£\Pi^i_{jl})T^i_{rk} - (£\Pi^r_{kl})T^i_{jr} \\ &\quad -(£\Pi^r_{lm})\dot{x}^m(\dot{\partial}_l T^i_{jk}) \end{aligned}$$

and

$$(1.19)\qquad \dot{\partial}_l(£T^i_{jk}) - £(\dot{\partial}_l T^i_{jk}) = 0.$$

The Lie-derivative of the normal projective curvature tensor N^i_{kjh} is expressed in the form

$$(1.20)\qquad \begin{aligned} \nabla_k(£\Pi^i_{jh}) - \nabla_j(£\Pi^i_{kh}) &= £N^i_{kjh} + (£\Pi^r_{km})\dot{x}^m\Pi^i_{rjh} \\ &\quad - (£\Pi^r_{jm})\dot{x}^m\Pi^i_{rkh}. \end{aligned}$$

In view of the infinitesimal transformation(1.15) K.Yano [4] defined a projective motion, if there exists a homogeneous scalar function p of degree one in $\dot{x}^i$'s satisfying

$$(1.21)\qquad £\Pi^i_{jk} = 2\delta^i_{(j}p_{h)}, p_h = \dot{\partial}_h p$$

where (jh) represents the symmetric part. For the homogeneity of p_k and p_{jk}, they satisfy the conditions

$$a)p_k\dot{x}^k = p, b)p_{jk}\dot{x}^k = 0. \tag{1.22}$$

§ 2. Projective N-curvature inheritance

S.P. Singh [6] defined the projective H-curvature inheritance as an infinitesimal transformation with respect to which the Lie-derivative of Berwald's curvature tensor H^i_{jkh} satisfies a relation of the form

$$\pounds H^i_{jkh} = \alpha H^i_{jkh}, \tag{2.23}$$

where $\alpha(x)$ is non-zero scalar function.

In the present paper, we consider the infinitesimal transformation (1.15) which admits the projective motion in an $NP - F_n$. Now we define and study the cases under which the infinitesimal transformation (1.15) defines a projective N-curvature inheritance in $NP - F_n$.

Definition 2.2. In an $NP-F_n$, if the normal projective curvature tensor field N^i_{jkh} satisfies the relation

$$\pounds N^i_{jkh} = \alpha N^i_{jkh}\ , \tag{2.24}$$

where $\alpha(x)$ is non zero scalar function and $\pounds$ denotes Lie-derivative defined by the infinitesimal transformation (1.15), which admits the projective motion. The transformation (1.15) is called projective N-curvature inheritance in the light of (2.24).

Contracting with respect to the indices i and j (2.24) yields

$$\pounds N_{kh} = \alpha N_{kh}. \tag{2.25}$$

In view of 2.25, the projective inheritance is called the projective Ricci-like N-curvature inheritance. Employing (1.21) in the equation (1.20), we arrive at

$$\begin{aligned} 2\{\delta^i_{[j}(\nabla_{k]}p_h) + \delta^i_h(\nabla_{[k}p_{j]})\} &= \pounds N^i_{jkh} + 2\delta^r_{(k}p_{m)}\dot{x}^m\Pi^i_{rjh} \\ &\quad -2\delta^r_{(j}p_{m)}\dot{x}^m\Pi^i_{rkh}. \end{aligned} \tag{2.26}$$

In view of (1.22) , (1.3c) and (2.24) the above equation reduces to

$$\alpha N^i_{jkh} = 2\{\delta^i_{[j}(\nabla_{k]}p_h) + \delta^i_h(\nabla_{[k}p_{j]}) + p\Pi^i_{[jk]h}\}. \tag{2.27}$$

Above discussion leads us to the following theorem.

Theorem 2.1. *An $NP-F_n$, admitting projective N-curvature inheritance, the normal projective curvature tensor field can not be expressed in terms of homogeneous scalar function $p(x,\dot{x})$ given in the form (2.27).*

If we contract (2.27) with respect to indices i and j and make use of (1.3d), (1.3e) and (1.7a), it is fairly easy to arrive at

$$\alpha N_{kh} = n\nabla_k p_h - \nabla_h p_k - (\frac{n-1}{n+1})pG^i_{ikh}. \tag{2.28}$$

Thus, we can state:

Corollary 2.1. *An $NP-F_n$, admitting projective Ricci-like N-curvature inheritance, Ricci-like normal projective curvature tensor N_{kh} can not be expressed in terms of only homogeneous scalar function $P(x, \dot{x})$ given in the form* (2.28)*.*

Using (1.19) for N^i_{jkh}, we get

$$\dot{\partial}_l(\pounds N^i_{jkh}) = \pounds(\dot{\partial}_l N^i_{jkh}), \tag{2.29}$$

which in view of (2.24) and (2.29) reduces to

$$\alpha(\dot{\partial}_l N^i_{jkh}) = \pounds(\dot{\partial}_l N^i_{jkh}), \tag{2.30}$$

where α is scalar function.

Hence we can state:

Lemma 2.1. *An $NP - F_n$, admitting projective N-curvature inheritance, the partial derivative of the normal projective curvature tensor satisfies the inheritance property (2.30).*

Contracting (2.30) with respect to indices i, $j$and then using (1.7a), we get

$$\alpha(\dot{\partial}_l N_{kh}) = \pounds(\dot{\partial}_l N_{kh}). \tag{2.31}$$

Accordingly, we can state:

Lemma 2.2. *An $NP - F_n$, admitting projective Ricci-like N-curvature inheritance, the partial derivative of the Ricci-like normal projective curvature tensor satisfies the inheritance property (2.31).*

Now using the commutation formula (1.13) for the normal projective curvature tensor N^i_{jkh}, we have

$$\begin{aligned} 2\nabla_{[l}\nabla_{m]}N^i_{jkh} &= N^i_{lmr}N^r_{jkh} - N^r_{lmj}N^i_{rkh} - N^r_{lmk}N^i_{jrh} \\ &\quad - N^r_{lmh}N^i_{jkr} - (\dot{\partial}_r N^i_{jkh})N^r_{lmn}\dot{x}^n. \end{aligned} \tag{2.32}$$

Applying Lie-derivative operator to both sides of the above equation, we get

$$\begin{aligned} 2\pounds\nabla_{[l}\nabla_{m]}N^i_{jkh} &= 2\alpha\{N^i_{lmr}N^r_{jkh} - N^r_{lmj}N^i_{rkh} - N^r_{lmk}N^i_{jrh} \\ &\quad - N^r_{lmh}N^i_{jkr} - (\dot{\partial}_r N^i_{jkh})N^r_{lmn}\dot{x}^n\} \end{aligned} \tag{2.33}$$

In view of (2.24) and Lemma 2.1 along with the fact $\pounds\dot{x}^i = 0$. The above equation together with (2.32) simplifies to yield

$$\pounds\nabla_{[l}\nabla_{m]}N^i_{jkh} = 2\alpha\nabla_{[l}\nabla_{m]}N^i_{jkh}. \tag{2.34}$$

Hence we can state:

Theorem 2.2. *An $NP - F_n$, admitting projective N-curvature inheritance, the relation (2.34) holds good.*

Contracting the equation (2.34) with respect to indices i, j and then using (1.7a), we obtain

$$\pounds\nabla_{[l}\nabla_{m]}N_{kh} = 2\alpha\nabla_{[l}\nabla_{m]}N_{kh}. \tag{2.35}$$

Hence we can state:

Corollary 2.2. *An $NP-F_n$, admitting projective Ricci-like N-curvature inheritance, the relation (2.35) holds good.*

In view of the relation (1.18) for N^i_{jkh}, and making use of the equation (2.24), we define

$$\begin{aligned}\pounds(\nabla_l N^i_{jkh}) - \nabla_l(\pounds N^i_{jkh}) \quad &= \quad (\pounds\Pi^i_{lr})N^r_{jkh} - (\pounds\Pi^r_{jl})N^i_{rkh} \\ &\quad - (\pounds\Pi^r_{kl})N^i_{jrh} - (\pounds\Pi^r_{hl})N^i_{jkr} \\ &\quad -(\pounds\Pi^r_{lm})\dot{x}^m(\dot{\partial}_r N^i_{jrh}),\end{aligned} \tag{2.36}$$

provided the gradient vector $\nabla_l\alpha = \alpha_l$ is zero.

Since the infinitesimal transformation (1.15) is a projective motion, in view of (1.22a) and (1.6b) it becomes

$$\begin{aligned}\pounds(\nabla_l N^i_{jkh}) - \alpha\nabla_l N^i_{jkh} \quad &= \quad \delta^i_l p_r N^r_{jkh} - p_j N^i_{lkh} \\ &\quad -p_k N^i_{jlh} - p_h N^i_{jkl} \\ &\quad -2p_l N^i_{jkh} - p(\dot{\partial}_l N^i_{jkh}).\end{aligned} \tag{2.37}$$

Transvecting (2.37) by $\dot{x}^l$, we find

$$\pounds(\nabla_l N^i_{jkh})\dot{x}^l - \alpha\nabla_l N^i_{jkh}\dot{x}^l = \dot{x}^l p_l N^i_{jkh} - p_j N^i_{lkh}\dot{x}^l - p_k N^i_{jlh}\dot{x}^l - 2pN^i_{jkh} \tag{2.38}$$

in the light of (1.22a) and (1.6b).

Contracting the equation (2.38) with respect to the indices i, j and there after using (1.7a),

we get

$$\pounds(\nabla_l N_{kh})\dot{x}^l - \alpha\nabla_l N_{kh}\dot{x}^l = -p_k N_{lh}\dot{x}^l - 2pN_{kh}. \tag{2.39}$$

But the contraction of (2.38) with respect to the indices i and h and in view of (1.7c) yields

$$\begin{aligned}\pounds(\nabla_l N_{[kj]})\dot{x}^l - \alpha\nabla_l N_{[kj]}\dot{x}^l \quad &= \quad \frac{1}{2}\dot{x}^i p_r N^r_{jki} - p_j N_{[kl]}\dot{x}^l \\ &\quad -p_k N_{[lj]}\dot{x}^l - 2pN_{[kj]}.\end{aligned} \tag{2.40}$$

Hence we can state:

Theorem 2.3. *An $NP - F_n$, admitting projective N-curvature inheritance, the relation (2.39) and (2.40) necessarily hold provided the gradient vector α_l is zero.*

R.B. Misra and F.M. Meher [1] have defined a Recurrent $NP - F_n$ as under:
In a non flat $NP - F_n$, if there exists a non zero vector field whose components λ_l are positively homogeneous of degree zero in directional arguments, such that the normal projective curvature tensor N^i_{jkh} satisfies

$$\nabla_l N^i_{jkh} = \lambda_l N^i_{jkh}, \tag{2.41}$$

then such a $NP - F_n$ is called a Recurrent $NP - F_n$ or briefly $RNP - F_n$. Contracting (2.41) with respect to the indices i and j, we have

$$\nabla_l N_{kh} = \lambda_l N_{kh}, \tag{2.42}$$

which shows that an $NP - F_n$ of recurrent curvature is also of Ricci recurrent curvature.

The covariant derivative of (2.41) with respect to x^m gives

$$\nabla_m \nabla_l N^i_{jkh} = \nabla_m \lambda_l N^i_{jkh} + \lambda_l \lambda_m N^i_{jkh}, \tag{2.43}$$

commutating (2.43) with respect to the indices m and l, we get

$$\nabla_{[m} \nabla_{l]} N^i_{jkh} = \nabla_{[m} \lambda_{l]} N^i_{jkh}. \tag{2.44}$$

Applying Lie-derivative operator to both sides of (2.44), we find

$$\pounds \nabla_{[m} \nabla_{l]} N^i_{jkh} = (\pounds \nabla_{[m} \lambda_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N^i_{jkh}. \tag{2.45}$$

In view of (2.24) and 2.2, the above equation reduces to

$$2\alpha \nabla_{[m} \nabla_{l]} N^i_{jkh} = (\pounds \nabla_{[m} \lambda_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N^i_{jkh}. \tag{2.46}$$

If we assume that $\pounds \nabla_{[m} \lambda_{l]} = -\alpha \nabla_{[m} \lambda_{l]}$, the equation (2.46) takes the form

$$\nabla_{[m} \nabla_{l]} N^i_{jkh} = 0. \tag{2.47}$$

Contracting (2.47) with respect to the indices$i$ and j, we arrive at

$$\nabla_{[m} \nabla_{l]} N_{kh} = 0. \tag{2.48}$$

Conversely, if (2.47) is true, the equation (2.46) yields

$$(\pounds \nabla_{[m} \nabla_{l]} + \alpha \nabla_{[m} \lambda_{l]}) N^i_{jkh} = 0. \tag{2.49}$$

Since the recurrent $NP - F_n$ is non-flat, (2.49) provides us

$$\pounds \nabla_{[m} \nabla_{l]} = -\alpha \nabla_{[m} \lambda_{l]}. \tag{2.50}$$

Accordingly we can state:

Theorem 2.4. *An $RNP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, the necessary and sufficient condition for $\nabla_{[m} \nabla_{l]} N^i_{jkh} = 0$ and $\nabla_{[m} \nabla_{l]} N_{kh} = 0$ to be true is that the recurrence vector λ_l satisfies the inheritance property (2.50).*

§ 3. Special cases

In this section, we wish to study three special cases of projective N-curvature inheritance in $NP - F_n$, $RNP - F_n$ and $SNP - F_n$ spaces.

(a). Contra field: In an $NP - F_n$, if the vector field $v^i(x)$ satisfies the relation

$$\nabla_j v^i = 0, \tag{3.51}$$

then the vector field $v^i(x)$ spans a contra field.

Here we consider a special infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon v^i(x), \nabla_j v^i = 0, \tag{3.52}$$

which admits a projective motion in $NP - F_n$. It is assumed that the relation (2.24) is also satisfied in $NP - F_n$, then the transformation (3.52) defines a projective N-curvature inheritance. Employing (1.21) in (1.17), we obtain

$$2\delta^i_{(j}p_{k)} = \nabla_j \nabla_k v^i + N^i_{hjk} v^h + \Pi^i_{hjk}(\nabla_l v^h)\dot{x}^l. \tag{3.53}$$

If $v^i(x)$ spans a contra field, then (3.53) assumes the following

$$N^i_{hjk} v^h = 2\delta^i_{(j}p_{k)} \tag{3.54}$$

The covariant differentiation of (3.54) with respect to x^l together with (3.51) yields

$$-(\nabla_l N^i_{jhk})v^h = \delta^i_j \nabla_l p_k + \delta^i_k \nabla_l p_j, \tag{3.55}$$

where we have taken into account (1.6a). Commutating (3.55) with respect to the indices l and k,
we get

$$2\nabla_{[k} N^i_{|jh|l]} v^h = 2\{\delta^i_{[k}\nabla_{l]} p_j + \delta^i_j \nabla_{[l} p_{k]}\}, \tag{3.56}$$

where the index in two parallel bars is unaffected when we consider skew-symmetric parts.

Since (3.52) defines a projective N-curvature inheritance therefore in view of (2.27), the equation (3.56) assumes the form

$$2\nabla_{[k} N^i_{|jh|l]} v^h = \alpha N^i_{klj} - 2p\Pi^i_{[kl]j}. \tag{3.57}$$

Contracting (3.57) with respect to indices i, j and making use of (1.3e), (1.7a) and (1.7c), we have

$$\nabla_{[k} N_{|h|l]} v^h = \alpha N_{[lk]} \tag{3.58}$$

Hence we can state:

Theorem 3.5. *An $NP - F_n$, admitting projective N-curvature and projecyive Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ spans contra field, the relations (3.57) and (3.58) hold good.*

Now if the space under consideration is a recurrent $NP - F_n$, then in view of (1.17), the equation (3.55) becomes

$$-\lambda_l N^i_{jhk} v^h = \delta^i_j \nabla_l p_k + \delta^i_k \nabla_l p_j. \tag{3.59}$$

Commutating (3.59) with respect to indices l and k, we get

$$2\lambda_{[k} N^i_{|jh|l]} v^h = 2\{\delta^i_{[k} \nabla_{l]} p_j + \delta^i_j \nabla_{[l} p_{k]}\}. \tag{3.60}$$

We now assume that the transformation (3.52) defines a projective N-curvature inheritance in $RNP - F_n$ also. Then in view of (2.27), the equation (3.60) is defined in the form

$$2\lambda_{[k} N^i_{|jh|l]} v^h = \alpha N^i_{klj} - 2p\Pi^i_{[kl]j}. \tag{3.61}$$

Contracting (3.61) with respect to indices i and j, we get

$$\lambda_{[k} N^i_{|h|l]} v^h = \alpha N^i_{[lk]}, \tag{3.62}$$

where we have taken into account (1.3e), (1.7a) and (1.7c).

Thus, we can state:

Theorem 3.6. *An $RNP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ spans a contra field then the relations (3.61) and (3.62) necessarily holds.*

Next, we assume that the transformation (3.52) define a projective N-curvature inheritance in $SNP - F_n$ R.B. Misra , N. Kishore and P.N. Pandey [2]. Then the equation (3.57) is define in the form

$$\alpha N^i_{klj} = 2p\Pi^i_{[kl]j}, \tag{3.63}$$

Hence we can state:

Theorem 3.7. *An $SNP - F_n$, admitting projective N-curvature inheritance, if the vector field $v^i(x)$ spans a contra field then the relation (3.63) necessarily holds.*

Contracting (3.63) with respect to indices i and j and using the equations (1.3e) and (1.7c), we have

$$\alpha N_{lk} = \alpha N_{kl} \tag{3.64}$$

Thus, we can state:

Corollary 3.3. *In an $SNP - F_n$, admitting projective Ricci-likeN-curvature inheritance, if the vector field $v^i(x)$ spans a contra field, the relations (3.64) necessarily symmetric.*

(b). Concurrent field: In an $NP - F_n$, if the vector field $v^i(x)$ satisfies the relation

$$\nabla_j v^i = c\delta^i_j, \tag{3.65}$$

where c is non zero constant then the vector field $v^i(x)$ determines a concurrent field.

In this section, we consider the infinitesimal transformation.

$$\bar{x}^i = x^i + \varepsilon v^i(x), \nabla_j v^i = c\delta^i_j, \tag{3.66}$$

which admits a projective motion and defines a projective N-curvature inheritance in$NP - F_n$. The covariant derivative of (3.54) with respect to x^l and using (3.65), we get

$$\nabla_l N^i_{hjk} v^h + cN^i_{ljk} = \delta^i_j \nabla_l p_k + \delta^i_k \nabla_l p_j, \tag{3.67}$$

where we have taken into account (1.6a).

Commutating (3.67) with respect to indices l and j, we get

$$2\nabla_{[l} N^i_{|h|j]k} v^h + 2cN^i_{[lj]k} = \alpha N^i_{jlk} - 2p\Pi^i_{[jl]k}, \tag{3.68}$$

where we have made us of equations (1.6a) and (2.27).

Contracting (3.68) with respect to indices i and j and using the equations (1.3d), (1.3e), (1.7a) and (1.7b), we get

$$\alpha N_{lk} = -\{v^h(\nabla_l N_{hk} + \nabla_l N^i_{hlk}) + 2cN_{lk} + (\frac{n-1}{n+1})pG^i_{ilk}\}. \tag{3.69}$$

Hence we can state:

Theorem 3.8. *In an $NP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ determines a concurrent field then the relations (3.68) and (3.69) necessarily hold.*

Let us assume that the space under consideration is a $RNP - F_n$ and the transformation (3.66) defines a projective N-curvature inheritance in it. In this case the relation (3.67) assumes the form

$$\lambda_l N^i_{hjk} v^h + cN^i_{ljk} = \delta^i_j \nabla_l p_k + \delta^i_k \nabla_l p_j. \tag{3.70}$$

Commutating (3.20) with respect to indices l and j, we get

$$\lambda_{[l} N^i_{|h|j]k} v^h + cN^i_{[lj]k} = \delta^i_{[j} \nabla_{l]} p_k + \delta^i_k \nabla_{[l} p_{j]}. \tag{3.71}$$

In view of (2.27), (3.71) reduces to

$$2\lambda_{[l} N^i_{|h|j]k} v^h + 2cN^i_{[lj]k} = \alpha N^i_{ljk} - 2p\Pi^i_{[jl]k}. \tag{3.72}$$

Contracting (3.72 with respect to indices i and j and thereafter using the equations (1.3d), (1.7a) and (1.7b), we obtain

$$\alpha N_{lk} = -\{v^h(\lambda_l N_{hk} + \lambda_i N^i_{hlk}) + 2cN_{lk}\}. \tag{3.73}$$

Accordingly we can state:

Theorem 3.9. *In an $RNP - F_n$, admitting Projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$, determines a concurrent field then the relations (3.72) and (3.73) hold good.*

Next, we assumes that the transformation (3.66) defines a projective N-curvature inheritance in $SNP - F_n$.Then the equation (3.68) and (3.69) can be written in the form

$$\alpha N^i_{jlk} = 2cN^i_{[lj]k} + 2p\Pi^i_{[jl]k}. \tag{3.74}$$

Contracting (3.74) with respect to indices i and j and thereafter using the equations (1.3d), (1.3e), (1.7a) and (1.7b) gives

$$\alpha N_{lk} = -\{2cN_{lk} + (\frac{n-1}{n+1})pG^i_{ilk}\}. \tag{3.75}$$

Thus, we can state:

Theorem 3.10. *In an $SNP - F_n$, admitting projective N-curvature and projective Ricci-like N-curvature inheritance, if the vector field $v^i(x)$ determines a concurrent field then the relations (3.74) and (3.75) hold good.*

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