Some results on CR-submanifolds of a Bochner Kaehler manifold

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Abstract. CR-submanifolds of a Kaehler manifold have been studied by Bejancu [1], [2] and are being studied by many others (see [3] for details). The Bochner curvature tensor of a Kaehler manifold is considered as a complex version of a Weyl conformal curvature tensor on a Riemannian manifold. A Kaehler manifold is called *Bochner-Kaehler manifold* if its Bochner curvature tensor vanishes. CR-submanifld of a Bochner-Kaehler manifold was studied by Shahid [8], [9]. The aim of this note is to study the properities of Ricci tensor and scalar curvature of CR-submanifolds of a Bochner Kaehler manifold.

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§ 1. Preliminaries

Let $\overline{M}$ be a complex n-dimensional Bochner Kaehler manifold with almost complex structure J, and M a real m-dimensional Riemannian manifold isometrically immersed in $\overline{M}$. A $(2p+q)$-dimensional manifold M of $\overline{M}$ is called a *CR-submanifold* if there exists a pair of orthogonal complementary distributions D and $D^{\perp}$ such that $JD = D$ and $JD^{\perp} \subset \nu$, where ν is the normal bundle of M and $\dim D = 2p$ and $\dim D^{\perp} = q$.

We denote by $\overline{\bigtriangledown}$, $\bigtriangledown$ and $\bigtriangledown^{\perp}$ the Riemannian connection on $\overline{M}$, M and the normal bundle respectively. They are related by the Gauss and Weingarten formulas as

$$\overline{\bigtriangledown}_X Y = \bigtriangledown_X Y + h(X,Y), \quad \overline{\bigtriangledown}_X Y = -A_N X + \bigtriangledown^{\perp}_X N \tag{1.1}$$

for all $X, Y \in TM$, $N \in \nu$ where $h(X,Y)$ and $A_N X$ are the second fundamental form and second fundamental tensor which are related by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{1.2}$$

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For any vector field X tangent to M, we put

$$X = PX + QX, \tag{1.3}$$

where PX and QX belongs to distribution D and $D^{\perp}$ respectively.

The Gauss equation is given by

$$\overline{R}(X,Y;Z,W) = R(X,Y;Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)). \tag{1.4}$$

The curvature tensor $\overline{R}$ of a Bochner-Kaehler manifold $\overline{M}$ is given by [8]

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= L(Y,Z)g(X,W) - L(X,Z)g(Y,W) + g(Y,Z)L(X,W) \\ &- g(X,Z)L(Y,W) + M(Y,Z)g(JX,W) \\ &- M(X,Z)g(JY,W) + M(X,W)g(JY,Z) \\ &- M(Y,W)g(JX,Z) - 2M(X,Y)g(JZ,W) \\ &- 2M(Z,W)g(JX,Y) \end{aligned} \tag{1.5}$$

where $\overline{R}(X,Y,Z,W) = g(\overline{R}(X,Y)Z,W)$,

$$L(Y,Z) = \frac{1}{2n+4}\overline{R}ic(Y,Z) - \frac{\overline{\rho}}{2(2n+2)(2n+4)}g(Y,Z), \tag{1.6}$$

$$M(Y,Z) = -L(Y,JZ), \tag{1.7}$$

$$L(Y,Z) = L(Z,Y), \quad L(Y,Z) = L(JY,JZ), \tag{1.8}$$

$\overline{R}ic$ and $\overline{\rho}$ are the Ricci tensor and scalar curvature of $\overline{M}$.

Definition 1.1 A CR-submanifold M is called *D-totally geodesic*(resp.*$D^{\perp}$-totally geodesic*) if $h(X,Y) = 0$(resp.$h(Z,W) = 0$) for all $X, Y \in D$ $(Z, W \in D^{\perp})$. M is called *mixed totally geodesic* if $h(X,Z) = 0$ for any $X \in D$ and $Z \in D^{\perp}$.

§ 2. Ricci tensor and scalar curvature of D-minimal and $D^{\perp}$-minimal CR-submanifold of Bochner-Kaehler manifold

Let M be an m-dimensional CR-submanifold of a Bochner Kaehler manifold $\overline{M}$. Let $\{E_1, ..., E_m\}$ be a local field of orthonarmal frames on M. Let $\{E_1, ..., E_p, E_{p+1} = JE_1, ..., E_{2p} = JE_p\}$ is a local frame field on D and $\{F_1, ..., F_q\}$ is a local frame field on $D^{\perp}$.

The mean curvature vector field H of M in $\overline{M}$ is defined by

$$H = \frac{1}{m}\{\sum_{i=1}^{2p} h(E_i, E_i) + \sum_{k=1}^{q} h(F_k, F_k)\}. \tag{2.1}$$

If $H = 0$ then M is said to be minimal. Also we have

$$H_D = \frac{1}{2p}\sum_{i=1}^{2p} h(E_i, E_i), \quad H_{D^\perp} = \frac{1}{q}\sum_{k=1}^{q} h(F_k, F_k). \tag{2.2}$$

Definition 2.1 A CR-submanifold M of a Bochner-Kaehler manifold $\overline{M}$ is called D-minimal (resp. $D^\perp$-minimal) if $H_D = 0$ (resp $H_{D^\perp} = 0$).

Now let $X, Y \in D$, $Z, W \in D^\perp$ and U, V be any vector field tangent to M. Then the Ricci tensor and scalar curvature are given by

$$S_D(U, V) = \sum_{i=1}^{2p} g(R(E_i, U)V, E_i), \tag{2.3}$$

$$S_{D^\perp}(U, V) = \sum_{k=1}^{q} g(R(F_k, U)V, F_k), \tag{2.4}$$

$$\rho_{DD} = \sum_{i=1}^{2p} S_D(E_i, E_i), \quad \rho_{DD^\perp} = \sum_{k=1}^{q} S_D(F_k, F_k), \tag{2.5}$$

$$\rho_{D^\perp D} = \sum_{i=1}^{2p} S_{D^\perp}(E_i, E_i), \quad \rho_{D^\perp D^\perp} = \sum_{k=1}^{q} S_{D^\perp}(F_k, F_k). \tag{2.6}$$

Using Gauss equation (1.4) and equation (1.3) in equation (1.5), the curvature tensor of M is given by

$$\begin{aligned} R(X,Y,Z,W) &= L(Y,Z)g(X,W) - L(X,Z)g(Y,W) + g(Y,Z)L(X,W) \\ &- g(X,Z)L(Y,W) + M(Y,Z)g(JPX,W) \\ &- M(X,Z)g(JPY,W) + g(JPY,Z)M(X,W) \\ &- g(JPX,Z)M(Y,W) - 2M(X,Y)g(JPZ,W) \\ &- 2M(Z,W)g(JPX,Y) + g(h(X,W),h(Y,Z)) \\ &- g(h(X,Z),h(Y,W)), \end{aligned} \tag{2.7}$$

for $X, Y, Z, W \in TM$.

Now we calculate Ricci tensor and scalar curvature for different choice of vector fields. For $X, Y \in D$ and $Z, W \in D^\perp$, equation (2.7) gives

$$
\begin{aligned}
S_D(X,Y) &= \sum_{i=1}^{2p} g(R(E_i,X)Y,E_i) \\
&= \sum_{i=1}^{2p} \{L(X,Y)g(E_i,E_i) - L(E_i,Y)g(X,E_i) \\
&+ g(X,Y)L(E_i,L_i) - g(E_i,Y)L(X,E_i) \\
&+ M(X,Y)g(JPE_i,E_i) - M(E_i,Y)g(JPX,E_i) \\
&+ M(E_i,E_i)g(JPX,Y) - M(X,E_i)g(JPE_i,Y) \\
&- 2M(E_i,X)g(JPY,E_i) - 2M(Y,E_i)g(JPE_i,X) \\
&+ g(h(E_i,E_i),h(X,Y)) - g(h(E_i,Y),h(X,E_i))\}, \\
&= 2(p+2)L(X,Y) + \sum_{i=1}^{2p} g(X,Y)L(E_i,E_i) \sum_{i=1}^{2p} g(JX,Y)L(E_i,JE_i) \\
&+ \sum_{i=1}^{2p} \{g(2pH_D,h(X,Y)) - g(h(E_i,Y),h(X,E_i))\}. \qquad (2.8)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
S_D(X,Z) &= \sum_{i=1}^{2p} g(R(E_i,X)Z,E_i) \\
&= 2(p+1)L(X,Z) + g(2pH_D,h(X,Z)) - \sum_{i=1}^{2p} g(h(E_i,Z),h(X,E_i)) \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
S_D(Z,W) &= \sum_{i=1}^{2p} g(R(E_i,Z)W,E_i) \\
&= L(Z,W)2p + \sum_{i=1}^{2p} g(Z,W)L(E_i,E_i) + g(2pH_D,h(Z,W)) \\
&- \sum_{i=1}^{2p} g(h(E_i,W),h(Z,E_i)). \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
S_{D^\perp}(Z,W) &= \sum_{k=1}^{q} g(R(F_k,Z)W,F_k) \\
&= (q-2)L(Z,W) + \sum_{k=1}^{q} g(Z,W)L(F_k,F_k) \\
&+ g(qH_{D^\perp},h(Z,W)) - \sum_{k=1}^{q} g(h(F_k,W),h(Z,F_k)) \qquad (2.11)
\end{aligned}
$$

$$\begin{aligned} S_{D^\perp}(X,Z) &= \sum_{k=1}^{q} g(R(F_k,X)Z,F_k) \\ &= L(X,Z)(q-1) + g(qH_{D^\perp}, h(X,Z)) \\ &- \sum_{k=1}^{q} g(h(F_k,Z),h(X,F_k)). \end{aligned} \tag{2.12}$$

$$\begin{aligned} S_{D^\perp}(X,Y) &= \sum_{k=1}^{q} g(R(F_k,X)Y,F_k) \\ &= L(X,Y)q + \sum_{k=1}^{q}\{g(X,Y)L(F_k,F_k) - g(JX,Y)L(F_k,JF_k)\} \\ &+ g(qH_{D^\perp},h(X,Y)) - \sum_{k=1}^{q} g(h(F_k,Y),h(X,F_k))\}. \end{aligned} \tag{2.13}$$

Now for scalar curvatures of CR-submanifold M, we have

$$\begin{aligned} \rho_{DD} &= \sum_{i=1}^{2p} S_D(E_i,E_i) = \sum_{i,j=1}^{2p}\{2(p+2)L(E_i,E_i) \\ &+ g(E_i,E_i)L(E_i,E_i) - g(JE_i,E_i)L(E_i,JE_i)\} \\ &+ \sum_{i,j=1}^{2p}\{g(h(E_i,E_i),h(E_j,E_j) - g(h(E_i,E_j),h(E_j,E_i))\} \\ &= (4p+4)\sum_{i}^{2p} L(E_i,E_i) + 4p^2 g(H_D,H_D) \\ &- \sum_{i,j=1}^{2p} g(h(E_i,E_j),h(E_j,E_i)\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \rho_{D^\perp D} &= \sum_{i=1}^{2p} S_{D^\perp}(E_i,E_i) \\ &= \sum_{i=1}^{2p}\sum_{k=1}^{q}\{L(E_i,E_i)q + g(E_i,E_i)L(F_k,F_k) \\ &- g(JE_i,E_i)L(F_k,JF_k) \\ &+ g(h(F_k,F_k),h(E_i,E_i) - g(h(F_k,E_i),h(E_i,F_k))\} \\ &= \sum_{i=1}^{2p}\sum_{k=1}^{q}\{L(E_i,E_i)q + 2pL(F_k,F_k) \\ &+ 2pqg(H_{D^\perp},H_D) - g(h(F_k,E_i),h(E_i,F_k))\}. \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
\rho_{DD^\perp} &= \sum_{k=1}^{q} S_D(F_k,F_k)\\
&= \sum_{i=1}^{2p}\sum_{k=1}^{q}\{L(F_k,F_k)2p + g(F_k,F_k)L(E_i,E_i)\\
&+ g(h(E_i,E_i),h(F_k,F_k) - g(h(E_i,F_k),H(F_k,E_i))\}\\
&= \sum_{i=1}^{2p}\sum_{k=1}^{q}\{L(F_k,F_k)2p + qL(E_i,E_i)\\
&+ 2pqg(H_D,H_{D^\perp}) - g(h(E_i,F_k),h(F_k,E_i))\}.
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
\rho_{D^\perp D^\perp} &= \sum_{k=1}^{q} S_{D^\perp}(F_k,F_k)\\
&= \sum_{k=1}^{q}\{L(F_k,F_k)(q-2) + g(F_k,F_k)L(F_k,F_k)\}\\
&+ \sum_{k,j=1}^{q}\{g(h(F_k,F_k),h(F_j,F_j) - g(h(F_k,F_j),h(F_j,F_k)\}\\
&= \sum_{k=1}^{q}\{L(F_k,F_k)(2q-2)\\
&+ \sum_{k,j=1}^{q} q^2 g(H_{D^\perp},H_{D^\perp}) - g(h(F_k,F_j),h(F_j,F_k))\}.
\end{aligned}
\tag{2.17}
$$

From (2.15)and (2.16) we observe

$$\rho_{DD^\perp} = \rho_{D^\perp D}.$$

Thus we have

Proposition 2.1. *Let M be a D-minimal submanifold of a Bochner-Kaehler manifold $\overline{M}$. Then*

(i) $S_D(X,Y) - 2(p+2)L(X,Y) - \sum_{i=1}^{2p} g(X,Y)L(E_i,E_i) + \sum_{i=1}^{2p} g(JX,Y)L(E_i,JE_i)$ *is negative semi-definite for $X,Y \in D$.*

(ii) $S_D(Z,W) - 2pL(Z,W) - \sum_{i=1}^{2p} g(Z,W)L(E_i,E_i)$ *is negative semi-definite for $Z,W \in D^\perp$.*

(iii) $\rho_{DD} \leq (4p+4)\sum_{i=1}^{2p} L(E_i,E_i)$.

(iv) $\rho_{DD^\perp} \leq \sum_{k=1}^{q} 2pL(F_k,F_k) + q\sum_{i=1}^{2p} L(E_i,E_i)$.

Proof. From (2.8) and (2.10) we have

$$\begin{aligned} S_D(X,Y) - 2(p+2)L(X,Y) \quad - & \quad \sum_{i=1}^{2p}\{g(X,Y)L(E_i,E_i) + g(JX,Y)L(E_i,JE_i)\} \\ = & \quad -\sum_{i=1}^{2p} g(h(E_i,Y),h(X,E_i)). \end{aligned}$$

$$S_D(Z,W) - 2pL(Z,W) - \sum_{i=1}^{2p} g(Z,W)L(E_i,E_i) = -\sum_{i=1}^{2p} g(h(E_i,W),h(Z,E_i))$$

Also from equation (2.14) we have

$$\rho_{DD} = (4p+4)\sum_{i=1}^{2p} L(E_i,E_i) - \sum_{i=1}^{2p} g(h(E_i,E_j),h(E_j,E_i)).$$

Hence

$$\rho_{DD} \leq (4p+4)\sum_{i=1}^{2p} L(E_i,E_i).$$

Similarly from equation (2.16), we have

$$\rho_{DD^\perp} = 2p\sum_{k=1}^{q} L(F_k,F_k) + q\sum_{i=1}^{2p} L(E_i,E_i) - \sum_{i=1}^{2p}\sum_{k=1}^{q} g(h(E_i,F_k),h(F_k,E_i).$$

Hence

$$\rho_{DD^\perp} \leq 2p\sum_{k=1}^{q} L(F_k,F_k) + q\sum_{i=1}^{2p} L(E_i,E_i).$$

□

Proposition 2.2. *Let M be $D^\perp$-minimal CR-submanifold of a Bochner-Kaehler manifold $\overline{M}$. Then*

(i) $S_{D^\perp}(Z,W) - L(Z,W)(q-2) - g(Z,W)\sum_{k=1}^{q} L(F_k,F_k)$ *is negative semi-difinite for* $Z,W \in D^\perp$.

(ii) $S_{D^\perp}(X,Y) - L(X,Y)q - g(X,Y)\sum_{k=1}^{q} L(Fk,F_k) + g(JX,Y)\sum_{k=1}^{q} L(F_k,JF_K)$ *is negative semi-definite for* $X,Y \in D$.

(iii) $\rho_{D^\perp D^\perp} \leq (2q-2)\sum_{k=1}^{q} L(F_k,F_k)$.

(iv) $\rho_{D^\perp D} \leq q\sum_{i=1}^{2p} L(E_i,E_i) + 2p\sum_{k=1}^{q} L(F_k,F_k)$.

Proposition 2.3. *Let M be a D-minimal CR-submanifold of a Bochner-Kaehler manifold $\overline{M}$. Then M is D-totally geodesic if and only if M satisfies one of the following condition :*

(i) $S_D(X,Y) = 2(p+2)L(X,Y) + \sum_{i=1}^{2p}\{g(X,Y)L(E_i,E_i) - g(JX,Y)L(E_i,JE_i)\}$ *for* $X,Y \in D$

(ii) $\rho_{DD} = (4p+4)\sum_{i=1}^{2p} L(E_i,E_i)$.

Proof. Suppose that M is D-totally geodesic i.e, $h(X,Y) = 0$ for all $X,Y \in D$. So from (2.8) and (2.14) we get

$$S_D(X,Y) = 2(p+2)L(X,Y) + \sum_{i=1}^{2p}\{g(X,Y)L(E_i,E_i) - g(JX,Y)L(E_i,JE_i)\},$$

$$\rho_{DD} = (4p+4)\sum_{i=1}^{2p} L(E_i,E_i).$$

Converse part follows easily from equation (2.8) and equation (2.14). □

Proposition 2.4. *Let M be $D^{\perp}$-minimal CR-submanifold of Bochner-Kaehler manifold $\overline{M}$. Then M is $D^{\perp}$-totally geodesic if and only if M satisfies one of the following condition :*

(i) $S_{D^{\perp}}(Z,W) = L(Z,W)(q-2) + \sum_{k=1}^{q} g(Z,W)L(F_k,F_k)$, *for* $Z,W \in D^{\perp}$,

(ii) $\rho_{D^{\perp}D^{\perp}} = (2q-2)\sum_{k=1}^{q} L(F_k,F_k)$.

Proof. Suppose that M is $D^{\perp}$-totally geodesic. i.e $h(Z,W) = 0$ for all $Z,W \in D^{\perp}$
Also we have $H_{D^{\perp}} = 0$. So from (2.11) and (2.17), we have
$S_{D^{\perp}}(Z,W) = L(Z,W)(q-2) + \sum_{k=1}^{q} g(Z,W)L(F_k,F_k)$
$\rho_{D^{\perp}D^{\perp}} = (2q-2)\sum_{k=1}^{q} L(F_k,F_k)$.
Converse part follows easily from equation (2.11) and equation (2.17). □

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