Spacelike Darboux curves in Minkowski 3-space

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Abstract. For a given curve α on the surface M, if the tangent plane of surface M and the tangent plane of osculating sphere of the curve α coincide at every point of the curve, then α is called a Darboux curve. In this paper, we study spacelike Darboux curves on a non-degenerate surface in Minkowski 3-space.

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§ 1. Introduction

If the tangent plane of the osculating sphere of a curve α lying on a surface M and the tangent plane of surface M coincide at every point of $\alpha(t)$, then the curve α is called a Darboux curve on M. Darboux curves in the Euclidean space were studied by Saban [5] and were generalised by Ergin [1]. For a curve α on a surface in the Euclidean 3-space the function $\mathbb{D} = < \alpha''', n >= \kappa_n' - \kappa_g \tau_g$ is called Darboux function of α where n is normal vector field of surface, κ_n, κ_g and τ_g are normal curvature, geodesic curvature and geodesic torsion. Darboux function is equal to zero for Darboux curves.

Timelike Darboux curves on a timelike surface in Minkowski 3-space was examined by Ergin [2]. In this paper, we examine spacelike Darboux curves on a non-degenerate surface. Also, we define Darboux function of a spacelike curve in the Minkowski 3-space.

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle \vec{u}, \vec{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_1 v_2$ where $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\vec{u}$ in $\mathbb{E}_1^3$ is spacelike, lightlike or timelike if $\langle \vec{u}, \vec{u} \rangle_L > 0$, $\langle \vec{u}, \vec{u} \rangle_L = 0$ or $\langle \vec{u}, \vec{u} \rangle_L < 0$ respectively. Besides, we say that a timelike vector is future pointing or past pointing if the first compound of the a vector is positive or negative respectively. The norm of the vector $\vec{u} \in \mathbb{E}_1^3$ is defined by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle_L|}$. For any $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}_1^3$, the Lorentzian vector product $\vec{u} \wedge_L \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined as follows:

$$\vec{u} \wedge_L \vec{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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The hyperbolic and Lorentzian unit spheres are $H_0^2 = \left\{\vec{a} \in \mathbb{E}_1^3 : \langle \vec{a}, \vec{a} \rangle_L = -1\right\}$ and $S_1^2 = \left\{\vec{a} \in \mathbb{E}_1^3 : \langle \vec{a}, \vec{a} \rangle_L = 1\right\}$ respectively. There are two components of H_0^2 passing through $(1,0,0)$ and $(-1,0,0)$ a future pointing hyperbolic sphere and a past pointing hyperbolic unit sphere, and they are denoted by H_0^{2+} and H_0^{2-} respectively. A surface M is called spacelike or timelike if normal vector field at every point of M is timelike or spacelike respectively [4].

An arbitrary curve $\alpha : I \to \mathbb{E}_1^3$ is spacelike, timelike or null, if all of its velocity vectors $\alpha'(t)$ are respectively spacelike, timelike or null, for each $t \in I \subset \mathbb{E}$. Throughout this paper we shall assume all curves are parametrized by its arc lenght.

Let α be a non-lightlike curve and $\{T, N, B\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\left.\begin{array}{l} T^{'} = \kappa N \\ N^{'} = (\epsilon_B)\,\kappa T + \tau B \\ B^{'} = (\epsilon_T)\,\tau N \end{array}\right\} \text{and} \begin{bmatrix} T^{'} \\ N^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ (\epsilon_B)\,\kappa & 0 & \tau \\ 0 & (\epsilon_T)\,\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.1}$$

where $\epsilon_X =< X, X >_L$ and κ, τ are curvature function and torsion function respectively [3].

§ 2. Osculating sphere of a non-lightlike curve

Definition 1. The sphere having sufficiently close common four points at $\alpha(t) \in \alpha$ is called osculating sphere of α at the point $\alpha(t)$, where $\kappa \neq 0$ and $\tau \neq 0$.

Lemma 2. *Let* α be a non-lightlike curve in $\mathbb{E}_1^3$, then the radius and the center of the osculating sphere of α at $\alpha(t)$ are

$$r = \sqrt{(\epsilon_N)\,\frac{1}{\kappa^2} + (\epsilon_B)\left(\frac{\kappa^{'}}{\kappa^2 \tau}\right)^2} \text{ and } c = \alpha(t) + \frac{(\epsilon_T)\,(\epsilon_N)}{\kappa} N - \left(\frac{\kappa'}{\kappa^2 \tau}\right)(\epsilon_T)\,(\epsilon_B)\,B$$

where N and B are normal and binormal vector fields of the curve at $\alpha(t)$.

Proof. Let α be a non-lightlike curve in $\mathbb{E}_1^3$. If we take a function $F(t)$ as

$$F(t) =< \alpha(t) - c, \alpha(t) - c >_L \pm r^2$$

where r and c are radius and center of the osculating sphere respectively. Since, the osculating sphere must be 4-point contact with α, then we have

$$F(t) = F'(t) = F''(t) = F'''(t) = 0.$$

If we use $T' = \kappa N$ for non-lightlike curves, we find

$$\begin{array}{l} F'(t) = 0 \Longrightarrow T \perp \alpha(t) - c \\ F''(t) = 0 \Longrightarrow < N, \alpha(t) - c >_L = \dfrac{-(\epsilon_T)}{\kappa} \\ F'''(t) = 0 \Longrightarrow < B, \alpha(t) - c >_L = \dfrac{\kappa'}{\tau \kappa^2}\,(\epsilon_T)\,. \end{array}$$

Considering the equations $<\alpha-c,N>_L=a\epsilon_N$ and $<\alpha-c,B>_L=a\epsilon_B$ for $\alpha(t)-c=aN+bB$, we state the center and radius of the osculating sphere of a non-lightlike curve as

$$c=\alpha(t)+\frac{(\epsilon_T)(\epsilon_N)}{\kappa}N-\left(\frac{\kappa'}{\kappa^2\tau}\right)(\epsilon_T)(\epsilon_B)B \tag{2.2}$$

and

$$r=\sqrt{(\epsilon_N)\frac{1}{\kappa^2}+(\epsilon_B)\left(\frac{\kappa^{'}}{\kappa^2\tau}\right)^2}.$$

□

Theorem 3. Let α be a non-lightlike curve in $\mathbb{E}_1^3$ and the center of osculating sphere at $\alpha(t)$ be c then,

$$<\alpha^{'''}(t),\alpha(t)-c>_L=0.$$

Proof. If we use Frenet formulas (1.1) , we find

$$\alpha'''=\kappa'N+\kappa(\epsilon_B\kappa T+\tau B). \tag{2.3}$$

And using the equation

$$\alpha(t)-c=-\frac{(\epsilon_T)(\epsilon_N)}{\kappa}N+\frac{\kappa^{'}}{\kappa^2\tau}(\epsilon_T)(\epsilon_B)B$$

we obtain

$$\begin{aligned}<\alpha^{'''}(t),\alpha(t)-c>_L &=-\frac{\kappa'(\epsilon_T)(\epsilon_N)^2}{\kappa}+\frac{\kappa^{'}}{\kappa^2\tau}(\epsilon_T)(\epsilon_B)^2\\ &=-\frac{\kappa'(\epsilon_T)}{\kappa}+\frac{\kappa'(\epsilon_T)}{\kappa^2}\kappa=0.\end{aligned}$$

□

Definition 4. Let $\alpha: I\rightarrow M$ be a non-lightlike curve on a regular non-degenerate surface M. If the tangent plane of the osculating sphere of α coincides with the tangent plane of M at the every point $\alpha(t)$, then the curve α is called a Darboux curve on M.

By the Definition 4, it is obvious that $\alpha(t)-c=\lambda n$, $\lambda\in\mathbb{R}$. So, it can expressed following theorem considering also Theorem 3.

Theorem 5. Let $\alpha: I\rightarrow M$ be a non-lightlike curve on a regular non-degenerate surface M in Minkowski 3-space. If α is a Darboux curve on surface M, then $<\alpha'''(t),n(t)>_L=0$ for every $t\in I$.

§ 3. Spacelike Darboux curves on spacelike surfaces

Let M be a spacelike surface with the unit timelike normal vector n in $\mathbb{E}_1^3$. If α is a spacelike curve on $M \subset \mathbb{E}_1^3$, then we have the Frenet frame $\{T, N, B\}$ and Darboux frame $\{T, n, g\}$ with spacelike vector $g = T \wedge_L n$ along the curve α. Since α is a spacelike curve, T is a spacelike vector and one of the vectors N and B must be timelike. So, we will examine relations between these frames of the curve with respect to normal of the spacelike curve is timelike or not.

Let θ be the hyperbolic angle between N and n.

i) If the principal normal of the spacelike curve is timelike:

In this case, the relationships between $\{T, N, B\}$ and $\{T, n, g\}$ are as follows:

$$\begin{bmatrix} N \\ B \\ T \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ g \\ T \end{bmatrix} \tag{3.4}$$

and

$$\begin{bmatrix} n \\ g \\ T \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 \\ -\sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N \\ B \\ T \end{bmatrix}. \tag{3.5}$$

By differentiating (3.5) and using (3.4) and Frenet formulas, we obtain

$$\left.\begin{aligned} \frac{dT}{dt} &= (\kappa\cosh\theta)\, n + (\kappa\sinh\theta)\, g \\ \frac{dg}{dt} &= (-\kappa\sinh\theta)\, T + \left(\tau - \frac{d\theta}{dt}\right) n \\ \frac{dn}{dt} &= (\kappa\cosh\theta)\, T + \left(\tau - \frac{d\theta}{dt}\right) g \end{aligned}\right\}. \tag{3.6}$$

If we represent $\kappa\cosh\theta$, $\kappa\sinh\theta$ and $\tau - \dfrac{d\theta}{dt}$ with the symbols κ_n, κ_g and τ_g respectively, then the equations in (3.6) can be written as

$$\begin{bmatrix} T' \\ n' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_n & \kappa_g \\ \kappa_n & 0 & \tau_g \\ -\kappa_g & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}. \tag{3.7}$$

Now, let's calculate Darboux function of the curve. For this, using (2.3) and (3.5) and considering N is a timelike vector we find

$$\begin{aligned} < \alpha''', n >_L &= -\kappa'\cosh\theta - \kappa\tau\sinh\theta \\ &= -\kappa_n' - \kappa_g\tau_g. \end{aligned}$$

Therefore, *Darboux function* $\mathbb{D}$ *of a spacelike curve* α with timelike normal is $\mathbb{D} = -\kappa_n' - \kappa_g\tau_g$. Consequently, we can express following theorem.

Theorem 6. Let $\alpha : I \to M$ be a spacelike curve with timelike principal normal on a regular spacelike surface M in Minkowski 3-space. If α is a Darboux curve on M, then the Darboux function $\mathbb{D} = -\kappa_n' - \kappa_g\tau_g$ is zero for every $t \in I$.

ii) If the principal normal of the spacelike curve is spacelike:
In the case B is timelike, relations between frames become

$$\begin{bmatrix} B \\ T \\ N \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} n \\ T \\ g \end{bmatrix} \tag{3.8}$$

and

$$\begin{bmatrix} n \\ T \\ g \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & -\sinh\theta \\ 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} B \\ T \\ N \end{bmatrix}. \tag{3.9}$$

So, we get derivative formulas as

$$\begin{bmatrix} T' \\ n' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ \kappa_g & 0 & \tau_g \\ -\kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}.$$

Therefore, using (2.3) and (3.9) and considering B is a timelike vector, Darboux function of the curve is found as

$$\begin{aligned} < \alpha''', n >_L &= -\kappa' \sinh\theta - \kappa\tau\cosh\theta \\ &= -\kappa_g' - \kappa_n\tau_g. \end{aligned}$$

Thus, *Darboux function* $\mathbb{D}$ *of a spacelike curve* α with spacelike principal normal is $\mathbb{D} = -\kappa_g' - \kappa_n\tau_g$ and we can express following theorem.

Theorem 7. Let $\alpha : I \to M$ be a spacelike curve with timelike binormal on a regular spacelike surface M in Minkowski 3-space. If α is a Darboux curve on M, then the Darboux function $\mathbb{D} = -\kappa_g' - \kappa_n\tau_g$ vanish for every $t \in I$.

§ 4. Spacelike Darboux curves on timelike surfaces

Let M be a timelike surface with the unit spacelike normal vector n in $\mathbb{E}_1^3$. If α is a spacelike curve on $M \subset \mathbb{E}_1^3$, then we have the Frenet frame $\{T, N, B\}$ and Darboux frame $\{T, n, g\}$ with timelike vector $g = -T \wedge_L n$ along the curve α. Since α is a spacelike curve, T is a spacelike vector and one of the vectors N and B must be timelike. So, we will examine relations between these frames of the curve with respect to normal of the spacelike curve is timelike or not.

Let θ be the hyperbolic angle between N and n respectively.

i) If principal normal of spacelike curve is timelike:
In this case, the relationships between $\{T, N, B\}$ and $\{T, n, g\}$ are as follows:

$$\begin{bmatrix} N \\ B \\ T \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g \\ n \\ T \end{bmatrix} \tag{4.10}$$

and

$$\begin{bmatrix} g \\ n \\ T \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 \\ -\sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N \\ B \\ T \end{bmatrix}. \tag{4.11}$$

So, we obtain derivative formulas as

$$\begin{bmatrix} T' \\ n' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}. \tag{4.12}$$

Therefore, using (2.3) and (4.11) and considering N is a timelike vector, Darboux function of the curve is found as

$$\begin{aligned} <\alpha''', n>_L &= \kappa' \sinh\theta + \kappa\tau\cosh\theta \\ &= \kappa_g' + \kappa_n\tau_g. \end{aligned}$$

Thus, *Darboux function* $\mathbb{D}$ *of a spacelike curve* α with timelike principal normal is $\mathbb{D} = \kappa_g' + \kappa_n\tau_g$ and we can express following theorem.

Theorem 8. Let $\alpha : I \to M$ be a spacelike curve with timelike principal normal on a regular timelike surface M in Minkowski 3-space. If α is a Darboux curve on M, then the Darboux function $\mathbb{D} = \kappa_g' + \kappa_n\tau_g$ is zero for every $t \in I$.

ii) If principal normal of spacelike curve is spacelike:
In this case, B is timelike and relations are

$$\begin{bmatrix} B \\ T \\ N \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} g \\ T \\ n \end{bmatrix} \tag{4.13}$$

and

$$\begin{bmatrix} g \\ T \\ n \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & -\sinh\theta \\ 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} B \\ T \\ N \end{bmatrix}. \tag{4.14}$$

So, we get derivative formulas as

$$\begin{bmatrix} T' \\ n' \\ g' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_n & \kappa_g \\ -\kappa_n & 0 & \tau_g \\ \kappa_g & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ g \end{bmatrix}. \tag{4.15}$$

Therefore, using (2.3) and (4.14) and considering B is a timelike vector, Darboux function of the curve is found as

$$<\alpha''', n>_L \quad = \kappa' \cosh\theta + \kappa\tau\cosh\theta = \kappa_n' + \kappa_g\tau_g.$$

Thus, *Darboux function* $\mathbb{D}$ *of a spacelike curve* α with timelike binormal is $\mathbb{D} = \kappa_n' + \kappa_g\tau_g$ and we can express following theorem.

Theorem 9. Let $\alpha : I \to M$ be a spacelike curve with timelike binormal on a regular timelike surface M in Minkowski 3-space. If α is a Darboux curve on M, then the Darboux function $\mathbb{D} = \kappa_n' + \kappa_g\tau_g$ is zero for every $t \in I$.

Corollary 10. Let α be a timelike curve on a timelike surface, If a is a Darboux curve, then $\mathbb{D} = \kappa_n' - \kappa_g\tau_g = 0$ where $\kappa_n = \kappa\cos\theta$, $\kappa_g = \kappa\sin\theta$, $\tau_g = \tau - \dfrac{d\theta}{dt}$, and θ is the angle between N and n.

Corollary 11. Let α be a spacelike curve on a reguler non-degenerate surface M, If a is a Darboux curve, then

i) $\mathbb{D} = -\kappa_n' - \kappa_g \tau_g = 0$ for a spacelike curve with timelike principal normal on spacelike surface.

ii) $\mathbb{D} = -\kappa_g' - \kappa_n \tau_g = 0$ for a spacelike curve with timelike binormal on spacelike surface.

iii) $\mathbb{D} = \kappa_g' + \kappa_n \tau_g = 0$ for a spacelike curve with timelike principal normal on timelike surface.

iv) $\mathbb{D} = \kappa_n' + \kappa_g \tau_g = 0$ for a spacelike curve with timelike binormal on timelike surface

where $\kappa_n = \kappa \cosh \theta$, $\kappa_g = \kappa \sinh \theta$, $\tau_g = \tau - \dfrac{d\theta}{dt}$, and θ is the hyperbolic angle between N and n.

Non-lightlike Darboux curves in Minkowski n-space are examined in [6]. Also, it is proved that Darboux function $\mathbb{D}$ is zero for all curves on a hyperplane or a hypersphere in E_1^n.

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