Some type of contact manifolds

M. Tarafdar, A. Bhattacharyya and T. De

Abstract. The object of this paper is to show that a contact manifolds with characteristic vector field ξ, belonging to $k-$nullity distribution satisfying the conditions $R(\xi, X).\tilde{C} = 0$, $R(\xi, X).C = 0$, $R(\xi, X).W = 0$ and $R(\xi, X).Z = 0$, is $\eta-$Einstein, where $\tilde{C}$ is quasi conformal curvature tensor, C is conformal curvature tensor, W is the Weyl projective curvature tensor and Z is conharmonic curvature tensor. $R(\xi, X)$ is considered as a derivation of the tensor algebra at each point of the tangent space. Further three equivalent conditions are obtained when a contact manifold satisfies the above relations.

M.S.C. 2000: 53C25.
Key words: contact manifold, quasi-conformal curvature tensor, conformal curvature tensor, Weyl projective curvature tensor.

§ 1. Introduction

Contact Riemannian manifolds satisfying $R(X, \xi).R = 0$ where ξ belongs to the nullity distribution or similar conditions have been studied by various authors ([1], [3]). In 2001 M.Tarafdar and A.K.Sengupta have studied contact Riemannian manifold satisfying $R(\xi, X).\overline{C} = 0$ where $\overline{C}$ is a concircular curvature tensor in [5].

In the present paper we have considered contact manifolds with characteristic vector field ξ, belonging to k-nullity distribution satisfying the condition

$$R(\xi, X).\tilde{C} = 0, \quad R(\xi, X).C = 0, \quad R(\xi, X).W = 0 \text{ and } R(\xi, X).Z = 0, \tag{1.1}$$

where $\tilde{C}$ is quasi conformal curvature tensor, C is conformal curvature tensor, W is the Weyl projective curvature tensor, Z is conharmonic curvature tensor and $R(\xi, X)$ is considered as a derivation of the tensor algebra at each point of the tangent space and obtained some interesting results.

§ 2. Preliminaries

A contact manifold M^{2m+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M, induces a unique vector field ξ on M^{2m+1} satisfying $\eta(\xi) = 1$ and $d(\xi, X) = 0$ for every vector field X on M^{2m+1}. A Riemannian metric g is said to be

Differential Geometry - Dynamical Systems, Vol. 9, 2007, pp. 149-155.

associated with a contact manifold if there exists a tensor field φ of type (1,1) such that

$$d\eta(X,Y) = g(X,\varphi Y), \quad \eta(X) = g(X,\xi), \quad \varphi^2 = -I + \eta \otimes \xi. \tag{2.2}$$

The manifold M^{2m+1} with a contact metric structure (φ, ξ, η, g) is usually called a contact metric manifold ([2]). The $k-$nullity distribution ([4]) of a Riemannian manifold (M,g) for a real number k is a distribution

$$N(k) : p \to N_p(k) = \{Z \in T_p(M)/R(X,Y)Z = k[g(Z,Y)X - g(X,Z)Y]$$

for any $X, Y \in T_p(M)\}$; $M^{2m+1}(\varphi, \xi, \eta, g)$ is a contact metric manifold with ξ belonging to the $k-$nullity distribution, i.e.,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \tag{2.3}$$

From (2.3), we have

$$S(X,\xi) = 2mk\eta(X) \tag{2.4}$$

$$S(X,Y) = \sum g(R(e_i,X)Y, e_i), \tag{2.5}$$

S is the Ricci tensor and $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.

Also from (2.3), since $g(R(X,Y)\xi, Z) = g(R(\xi,Z)X, Y)$ it follows that

$$R(\xi,X)Z = k[g(Z,X)\xi - \eta(Z)X] \tag{2.6}$$

§ 3. The contact manifold satisfying $R(\xi,X).\tilde{C} = 0$

We consider a contact manifold M^{2m+1} satisfying

$$R(\xi,X).\tilde{C} = 0 \tag{3.7}$$

where the quasi conformal curvature tensor $\tilde{C}$ is defined as

(3.8)
$$\begin{aligned}\tilde{C}(X,Y)Z = \ & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & -\tfrac{r}{m}(\tfrac{a}{m-1} + 2b)[g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

where a and b are non-zero constants and r is he scalar curvature of the manifold.

From (3.8) it follows that

$$\tilde{C}(X,Y)Z = -\tilde{C}(Y,X)Z \tag{3.9}$$

$$g(\tilde{C}(X,Y)\xi, \xi) = 0 \tag{3.10}$$

(3.11)
$$\sum g(\tilde{C}(e_i,V)W,e_i) = \{a+b(2m-1)\}S(V,W) + [rb - 2r(\frac{a}{m-1}+2b)]g(V,W)$$

where $\{e_i\}$ is defined in (2.5).

We know that

(3.12)
$$(R(\xi,Y).\tilde{C})(U,V)W = R(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(R(\xi,Y)U,V)W - \tilde{C}(U,R(\xi,Y)V)W - \tilde{C}(U,V)R(\xi,Y)W$$

By virtue of (3.7), it follows from (3.12) that

(3.13)
$$g(R(\xi,Y)\tilde{C}(U,V)W,\xi) - g(\tilde{C}(R(\xi,Y)U,V)W,\xi) - g(\tilde{C}(U,R(\xi,Y)V)W,\xi) - g(\tilde{C}(U,V)R(\xi,Y)W,\xi) = 0$$

i.e.,

$$\eta(R(\xi,Y)\tilde{C}(U,V)W) - \eta(\tilde{C}(R(\xi,Y)U,V)W) - \eta(\tilde{C}(U,R(\xi,Y)V)W) - \eta(\tilde{C}(U,V)R(\xi,Y)W) = 0. \tag{3.14}$$

Now by using (2.6) and (3.9) we get

$$\eta(R(\xi,Y)\tilde{C}(U,V)W) = k[\tilde{C}(U,V)W,Y - \eta(Y)\eta(\tilde{C}(U,V)W)], \tag{3.15}$$

where $\tilde{C}(U,V)W,Y = g(\tilde{C}(U,V)W,Y)$,

$$\eta(\tilde{C}(R(\xi,Y)U,V)W) = k[g(U,Y)\eta(\tilde{C}(\xi,V)W) - \eta(U)\eta(\tilde{C}(Y,V)W)] \tag{3.16}$$

$$\eta(\tilde{C}(U,R(\xi,Y)V)W) = k[\eta(\tilde{C}((U,\xi)W))g(V,Y) - \eta(V)\eta(\tilde{C}((U,Y)W))] \tag{3.17}$$

$$\eta(\tilde{C}(U,V)R(\xi,Y)W) = k[g(Y,W)\eta(\tilde{C}(U,V)\xi) - \eta(W)\eta(\tilde{C}(U,V)Y)] \tag{3.18}$$

Using (3.15), (3.16), (3.17) and (3.18) in (3.14) we have

(3.19)
$$\tilde{C}(U,V)W,Y - \eta(Y)\eta(\tilde{C}(U,V)W) - g(U,Y)\eta(\tilde{C}(\xi,V)W) + \eta(U)\eta(\tilde{C}(Y,V)W) - \eta(\tilde{C}(U,\xi)W)g(V,Y) + \eta(V)\eta(\tilde{C}(U,Y)W) - \eta(\tilde{C}(U,V)\xi)g(Y,W) + \eta(W)\eta(\tilde{C}(U,V)Y)] = 0.$$

Putting $Y = U = e_i$, where $\{e_i\}$ is defined in (2.5) and using (3.9), (3.10) in (3.19) and after some brief calculations we obtain

$$g(\tilde{C}(e_i,V,W)e_i) - 2m\eta(\tilde{C}(\xi,V)W) + \eta(W)g(\tilde{C}(e_i,V)e_i,\xi) = 0 \tag{3.20}$$

Using (2.4), (2.5), (2.6), (3.8) and (3.11) in (3.20) we get

(3.21)
$$S(V,W) = \frac{(4m^2bk + 2mak - 7rb)}{(a-b)} g(V,W) + \frac{(rb - 4m^2bk - 2mbk)}{(a-b)} \eta(V)\eta(W),$$

which shows that M^{2m+1} is $\eta-$Einstein manifold provided $a \neq b$.

Thus we can state,

Theorem 1. Let $M^{2m+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the k-nullity distribution satisfying $R(\xi, X).\tilde{C} = 0$ provided $a \neq b$. Then M^{2m+1} is $\eta-$Einstein where $\tilde{C}$ is a quasi conformal curvature tensor.

We know that if $a = 1,\ b = \frac{-1}{2m-1}$ in $\tilde{C}$, then it reduces to conformal curvature C. Therefore putting $a = 1,\ b = \frac{-1}{2m-1}$ in (3.21) we obtain

$$S(V,W) = (\frac{7r}{2m} - k)g(V,W) + (2mk + k - \frac{r}{2m})\eta(V)\eta(W) \tag{3.22}$$

Thus we have the following

Corollary 1. *A contact manifold satisfying $R(\xi, X).C = 0$ is $\eta-$Einstein, where C is the conformal curvature tensor.*

§ 4. Contact manifolds satisfying $R(\xi, X).W = 0$

Again we consider M^{2m+1} satisfying

$$R(\xi, X).W = 0, \tag{4.23}$$

where the Weyl projective curvature tensor W is defined as

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2m}[S(Y,Z)X - S(X,Z)Y] \tag{4.24}$$

From (4.24) it follows that

$$W(X,Y)Z = -W(Y,X)Z \tag{4.25}$$

$$g(W(X,Y)\xi, \xi) = 0 \tag{4.26}$$

$$\sum g(W(e_i, V)W, e_i) = 0 \tag{4.27}$$

where $\{e_i\}$ is defined in (2.5).

We know that

(4.28)
$$\begin{aligned}(R(\xi,Y).W)(U,V)P = \ & R(\xi,Y)W(U,V)P - W(R(\xi,Y)U,V)P - W(U,R(\xi,Y)V)P \\ & - W(U,V)R(\xi,Y)P.\end{aligned}$$

By virtue of (4.23), (4.28) takes the form

$$\eta(R(\xi,Y)W(U,V)P)-\eta(W(R(\xi,Y)U,V)P)-\eta(W(U,R(\xi,Y)V)P)$$
$$-\eta(W(U,V)R(\xi,Y)P)=0 \tag{4.29}$$

Using (2.6) and (4.25), we get

$$\eta(R(\xi,Y)W(U,V)P)=k[g(W(U,V)P,Y)-\eta(Y)\eta(W(U,V)P)] \tag{4.30}$$

$$\eta(W(R(\xi,Y)U,V)P)=k[g(U,Y)\eta(W(\xi,V)P)-\eta(U)\eta(W(Y,V)P)] \tag{4.31}$$

$$\eta(W(U,R(\xi,Y)V)P)=k[\eta(W(U,\xi)P)g(V,Y)-\eta(V)\eta(W((U,V)P))] \tag{4.32}$$

$$\eta(W(U,V)R(\xi,Y)P)=k[g(P,Y)\eta(W(U,V)\xi)-\eta(P)\eta(W(U,V)Y)] \tag{4.33}$$

Using (4.30), (4.31), (4.32) and (4.33) in (4.29), we have

(4.34)
$$g(W(U,V)P,Y)-\eta(Y)\eta(W(U,V)P)-g(U,Y)\eta(W(\xi,V)P)+\eta(U)\eta(W(Y,V)P)$$
$$-\eta(W(U,\xi)P)g(V,Y)+\eta(V)\eta(W(U,Y)P)-g(P,Y)\eta(W(U,V)\xi)$$
$$+\eta(P)\eta(W(U,V)Y)=0.$$

Putting $Y=U=e_i$, where $\{e_i\}$ is defined in (2.5) and using (2.4), (2.5), (2.6), (4.24), (4.25), (4.26) and (4.27) in (4.34) and after some brief calculations we have

$$S(V,P)=2mkg(V,P)+\{(4m+1)k-2m(1+\frac{r}{4m^2})\}\eta(V)\eta(P), \tag{4.35}$$

which shows that M^{2m+1} is $\eta-$Einstein manifold. Hence we can state the following

Theorem 2. *Let $M^{2m+1}(\varphi,\xi,\eta,g)$ be a conctact metric manifold with ξ belonging to the k-nullity distribution satisfying $R(\xi,X).W=0$. Then M^{2m+1} is $\eta-$Einstein manifold where W is Weyl projective curvature tensor.*

§ 5. Contact manifolds satisfying $R(\xi,X).Z=0$

Again we consider the manifold M^{2m+1} satisfying

$$R(\xi,X).Z=0, \tag{5.36}$$

where conharmonic curvature tensor Z is defined as

(5.37)
$$Z(X,Y)W=R(X,Y)W-\frac{1}{2m-1}[S(Y,W)X-S(X,W)Y+g(Y,W)QX-g(X,W)QY]$$

From (5.37) it follows that

$$Z(X,Y)W=-Z(Y,X)W \tag{5.38}$$

$$g(Z(X,Y)\xi,\xi) = 0 \tag{5.39}$$

$$\sum g(Z(e_i,V)W,e_i) = -\frac{r}{2m-1}g(V,W) \tag{5.40}$$

where $\{e_i\}$ is defined in (2.5). We know that

(5.41)
$$(R(\xi,Y).Z)(U,V)W = R(\xi,Y)Z(U,V)W - Z(R(\xi,Y)U,V)W - Z(U,R(\xi,Y)V)W- \\ -Z(U,V)R(\xi,Y)W.$$

By virtue of (5.36) it follows from (5.41) that

$$\eta(R(\xi,Y)Z(U,V)W) - \eta(Z(R(\xi,Y)U,V)W) - \eta(Z(U,R(\xi,Y)V)W)- \\ -\eta(Z(U,V)R(\xi,Y)W) = 0. \tag{5.42}$$

Now by using (2.6) and (5.38) we get

$$\eta(R(\xi,Y)Z(U,V)W) = k[g(Z(U,V)W,Y) - \eta(Y)\eta(Z(U,V)W)] \tag{5.43}$$

$$\eta(Z(R(\xi,Y)U,V)W) = k[g(U,Y)\eta(Z(\xi,V)W) - \eta(U)\eta(Z(Y,V)W)] \tag{5.44}$$

$$\eta(Z(U,R(\xi,Y)V)W) = k[\eta(Z(U,\xi)W)g(V,Y) - \eta(V)\eta(Z((U,V)W))] \tag{5.45}$$

$$\eta(Z(U,V)R(\xi,Y)W) = k[g(Y,W)\eta(Z(U,V)\xi) - \eta(W)\eta(Z(U,V)Y)] \tag{5.46}$$

Using (5.43), (5.44), (5.45) and (5.46) in (5.42), we have

(5.47)
$$g(Z(U,V)W,Y) - \eta(Y)\eta(Z(U,V)W) - g(U,Y)\eta(Z(\xi,V)W) + \eta(U)\eta(Z(Y,V)W)- \\ -\eta(Z(U,\xi)W)g(V,Y) + \eta(V)\eta(Z(U,Y)W) - g(Y,W)\eta(Z(U,V)\xi)+ \\ +\eta(W)\eta(Z(U,V)Y) = 0.$$

Putting $Y = U = e_i$, where $\{e_i\}$ is defined in (2.5) and using (2.4), (2.5), (2.6), (5.37), (5.38), (5.39) and (5.40) in (5.47) and after some brief calculations we have

$$S(V,W) = (\frac{r}{2m} - k)g(V,W) + \{(2m+1)k - \frac{r}{2m}\}\eta(V)\eta(W), \tag{5.48}$$

which shows that the manifold is $\eta-$Einstein manifold. Thus we state,

Theorem 3. *If $M^{2m+1}(\varphi,\xi,\eta,g)$ be a conctact metric manifold with ξ belonging to the k-nullity distribution satisfying $R(\xi,X).Z = 0$, then it is $\eta-$Einstein manifold where Z is conharmonic curvature tensor.*

Now in a recent paper M.M.Tripathi have proved in [6] that if a contact manifold M^{2m+1} be a non-Sasakian η-Einstein (k,μ) manifold then following conditions are equivalent.

(a) M^{2m+1} is flat and 3-dimensional.
(b) M^{2m+1} is Ricci-Semisymetric.
(c) M^{2m+1} is ξ-Ricci-Semisymetric.

Using the above result and Th 1., Th 2. and Th 3., we can state,

Theorem 4. *Let M^{2m+1} be a contact manifold with ξ belonging to the k-nullity distribution and non-Sasakian manifold, satisfying $R(\xi, X).\tilde{C} = 0$, $R(\xi, X).W = 0$ and $R(\xi, X).Z = 0$. Then the following conditions are equivalent.*

(a) M^{2m+1} is flat and 3-dimensional.
(b) M^{2m+1} is Ricci-Semisymmetric.
(c) M^{2m+1} is ξ-Ricci-Semisymmetric.

References

[1] C. Baikoussis, T. Koufogiorgos, *On a type of contact manifolds*, J. Geom. 46, 1-2 (1993), 1-9.

[2] D.E. Blair,*Contact manifold in Riemannian Geometry*, Lecture Notes in Mathematics 509, Springer-Verlag, Berlin, 1976.

[3] D. Perrone, *Contact Riemannian manifolds satisfying $R(X, \xi).R = 0$*, The Yokohama Mathematical Journal 39 (1992), 141-149.

[4] S. Tanno, *Ricci Curvatures of contact Riemannian manifolds*, Tohoku Math. J. 40 (1988), 441-448.

[5] M. Tarafdar, A.K. Sengupta, *On a type of contact manifold*, An. St. Univ. "Al.I.Cuza" Iasi Tom XLVII, s.I a, Matematica (2001), f.1., 73-78.

[6] M.M. Tripathi, *Classification of ξ-Ricci-semisymmetric*, Balkan Journal of Geometry and Its Applications 11 (2006), 134-142.

Authors' addresses:

M.Tarafdar
Department of Pure Mathematics, University of Calcutta,
700019 Kolkata, India.
e-mail: manjushamajumdar@hotmail.com

A.Bhattacharyya
Department of Mathematics, Jadavpur University,
Kolkata-32, India.
e-mail: bhattachar1968@yahoo.co.in

T.De
Department of Mathematics,
Jadavpur University, Kolkata-32, India.
e-mail: tapan97@indiatimes.com