Geodesic mappings between Kahler-Weyl spaces

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Abstract. In this paper, we defined the geodesic mappings between Kahler-Weyl spaces and obtain necessary and sufficient conditions for such a space to admit a nontrivial geodesic mapping onto another space of the same type.

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§ 1. Introduction

Geodesic mappings between Riemannian Spaces were studied by many authors (e.g., [9], [2], [4]). In [4], [5] and [6], the geodesic mappings from a Kahlerian space K_n onto a Riemannian space V_n were investigated by J.Mikes. In these works, the author provides geodesic mappings of Kahlerian spaces which do not preserve the structure. In [7], was given a nontrivial example of geodesic mapping between Kahlerian spaces.

In this work, we study in geodesic mappings between Kahler-Weyl spaces.

An n-dimensional differentiable manifold W_n is said to be a *Weyl space* if it has a conformal metric tensor g and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2 T_k g_{ij} = 0 \; , \tag{1.1}$$

where T_k denotes a covariant vector field. Under the renormalization

$$\widetilde{g} = \lambda^2 g \tag{1.2}$$

of the metric tensor g, T is transformed by the law

$$\widetilde{T}_k = T_k + \partial_k \ln \lambda,$$

where λ is a function defined on W_n [8].

Writing (1.1) in local coordinates and expanding it we find that

$$\partial_k g_{ij} - g_{mj} \Gamma^m_{ik} - g_{im} \Gamma^m_{jk} - 2 T_k g_{ij} = 0,$$

where Γ^i_{jk} are the connection coefficients of the form

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} - (\delta^i_j T_k + \delta^i_k T_j - g^{li} g_{jk} T_l). \tag{1.3}$$

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An object A defined on $W_n(g,T)$ is called a *satellite* of weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\widetilde{A} = \lambda^p A$$

under the renormalization of the metric tensor g_{ij} ([8], [3]).

The prolonged covariant derivative of a satellite A is defined by

$$\dot{\nabla}_k A = \nabla_k A - p\, T_k A\ . \tag{1.4}$$

We note that the prolonged covariant derivative preserves the weight.

Let W_n be a Weyl space of dimension $n = 2m$ and endowed with almost complex structure F_i^j, i.e.,

$$F_i^j F_j^k = -\delta_i^k. \tag{1.5}$$

W_n is called a Kahler-Weyl space, if the structure F_i^h satisfies the following relations

$$g_{ij} F_h^i F_k^j = g_{hk}\ , \tag{1.6}$$

$$\dot{\nabla}_k F_i^j = 0\ , \tag{1.7}$$

$$F_{ij} = g_{jk} F_i^k = -F_{ji} \quad , \quad F^{ij} = g^{ih} F_h^j = -F^{ji} \tag{1.8}$$

where the tensors F_{ij} and F^{ij} of weight $\{2\}$ and $\{-2\}$, respectively. Such a Weyl space will be denoted by KW_n.

Consider two Kahler-Weyl spaces KW_n and $K\bar{W}_n$ and a diffeomorfism $f : KW_n \to K\bar{W}_n$. If, under this mapping, the geodesics of the space KW_n map to the geodesics of the space $K\bar{W}_n$, then f is called a geodesic mapping.

§ 2. Geodesic mappings onto Kahler-Weyl spaces

In this section, we determine the necesary and sufficient conditions for a Kahlerian Weyl space KW_n to admit a nontrivial geodesic mapping onto a Kahlerian Weyl space $K\bar{W}_n$.

Let KW_n and $K\bar{W}_n$ be Kahler-Weyl spaces endowed with the connections ∇ and $\bar{\nabla}$, respectively. It is known that, the curve

$$L : x^i = x^i(t) \tag{2.9}$$

is a geodesic of an afine space if and only if

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \rho(t)\frac{dx^i}{dt} \tag{2.10}$$

holds, where $\rho(t)$ is a certain function of t [1]. Suppose that $f : KW_n \to K\bar{W}_n$ is a geodesic mapping. Then, by f, the geodesic L of KW_n maps to the geodesic $\bar{L}$ of $K\bar{W}_n$ and

$$\frac{d^2x^i}{dt^2} + \bar{\Gamma}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \bar{\rho}(t)\frac{dx^i}{dt} \tag{2.11}$$

holds, where $\bar{x}^i = x^i$. Substracting (2.10) and (2.11) we find

$$2\psi_j \frac{dx^j}{dt}\frac{dx^i}{dt} = (\bar{\Gamma}^i_{jk} - \Gamma^i_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt}, \tag{2.12}$$

which implies that

$$\psi_j \delta^i_k + \psi_k \delta^i_j = \bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$$

or,

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j, \tag{2.13}$$

where $2\psi_j \frac{dx^j}{dt} = \bar{\rho(t)} - \rho(t)$.

It can be easily seen that the equation (2.11) holds under the condition (2.13). On the other hand, contracting (2.13) with respect to i and k, we get

$$\bar{\Gamma}^i_{ji} - \Gamma^i_{ji} = (n+1)\psi_j$$

from which follows

$$\psi_j = \frac{1}{n+1}\Big[\partial_j \ln\sqrt{\frac{\bar{g}}{g}} - nP_j\Big],$$

where $P_j = \bar{T}_j - T_j$. If $\psi_j = 0$, then P_j is a gradient which implies the complementary vector fields $\bar{T}_k$ and T_k are gradient fields. In this case, two Kahler-Weyl spaces reduce to Riemannian spaces with Kahlerian structure and the geodesic mapping are said to be trivial or affine. Hence, we consider the case $\psi_j \neq 0$, i.e., the geodesic mapping is non-trivial. Then we can state the following

Theorem 2.1. *A Kahler-Weyl space KW_n admits a non-trivial geodesic mapping onto a Kahler-Weyl space $K\bar{W}_n$, if and only if the following holds*

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \psi_j \delta^i_k + \psi_k \delta^i_j, \tag{2.14}$$

where $\psi_j = 1/(n+1)\Big[\partial_j \ln\sqrt{\frac{\bar{g}}{g}} - nP_j\Big]$.

Substituting the condition (2.13) into (1.1) we obtain

$$\begin{aligned} \dot{\nabla}_k \bar{g}_{ij} &= \nabla_k \bar{g}_{ij} - 2T_k \bar{g}_{ij} \\ &= 2(\psi_k + P_k)\bar{g}_{ij} + \bar{g}_{kj}\psi_i + \bar{g}_{ik}\psi_j. \end{aligned} \tag{2.15}$$

On the other hand, by the formula (1.7) we have

$$\begin{aligned} \dot{\nabla}_k \bar{F}^h_i &= \nabla_k \bar{F}^h_i \\ &= \bar{F}^h_k \psi_i - \delta^h_k \bar{F}^a_i \psi_a \ , \end{aligned} \tag{2.16}$$

which implies that the structure $\bar{F}^j_i$ satisfies the condition

$$\dot{\bar{\nabla}}_k \bar{F}^j_i = 0 \ , \tag{2.17}$$

and guarantees that $\bar{F}_i^j$ is the complex structure of the Kahler-Weyl space $K\bar{W}_n$. As result, we have proved

Theorem 2.2. *The KW_n admits a nontrivial geodesic mapping onto the Kahler-Weyl space $K\bar{W}_n$, if and only if the following conditions hold*

$$\begin{aligned} a)\dot{\nabla}_k \bar{g}_{ij} &= 2(\psi_k + P_k)\bar{g}_{ij} + \bar{g}_{kj}\psi_i + \bar{g}_{ik}\psi_j \ , \\ b)\dot{\nabla}_k \bar{F}_i^h &= \bar{F}_k^h \psi_i - \delta_k^h \bar{F}_i^a \psi_a \ . \end{aligned}$$

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