On the CR-submanifolds of Kaehler product manifolds

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Abstract. In this paper, the geometry of CR-submanifolds of a Kaehler product manifold is studied. Fundamental of these submanifolds are investigated such as CR-product, pseudo-umbilical and curvature-invariant. Finally, we show that there exists no totally umbilical CR-submanifold in a Kaehler product manifold $M = M_1(c_1) \times M_2(c_2)$.

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1 Introduction

The geometry of CR-submanifolds of a Kaehler is an interesting subject which was studied many geometers[2, 3, 9]. In particular, CR-Submanifolds of a Kaehler product manifold was studied in [9] by M. H. Shadid. He showed that CR-Submanifold is a Riemannian product manifold if it is $D^{\perp}$-totally geodesic. Moreover, he had some results which in related to the sectional and holomorphic curvatures of CR-submanifold and CR-submanifold is D-totally geodesic. Finally, necessary and sufficient conditions are given on minimal CR-submanifold of a Kaehler product manifold to be totally geodesic.

Also, the geometry of CR-Submanifold of any Kaehler manifold was studied in[1, 2] by A. Bejancu and in [3] by B.Y. Chen.

In this paper, we show that the distributions D and $D^{\perp}$ on CR-submanifold of a Kaehler product manifold are pseudo-umbilic(resp. curvature-invariant) if and only if CR-submanifold is pseudo umbilic (resp. curvature-invariant) and CR-submanifold of a Kaehler product manifold to be CR-product. Finally, we have researched totally umbilical proper CR-submanifold in a Kaehler product manifold $M = M_1(c_1) \times M_2(c_2)$.

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2 Preliminaries

Now, let M be an n-dimensional Riemannian manifold and $\bar{M}$ be a m-dimensional isometrically immersed submanifold in M. Then $\bar{M}$ becomes a Riemannian submanifold of M with Riemannian metric induced by the Riemannian metric on M. Also, we denote the Levi-Civita connections on $\bar{M}$ and M by $\bar{\nabla}$ and ∇, respectively, then the Gauss formulae is given by

$$\nabla_X Y = \bar{\nabla}_X Y + h(X,Y), \tag{2.1}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $h : \Gamma(T\bar{M}) \times \Gamma(T\bar{M}) \longrightarrow \Gamma(T\bar{M}^{\perp})$ is the second fundamental form of $\bar{M}$ in M. Furthermore, for any $X \in \Gamma(T\bar{M})$ and $\xi \in \Gamma(T\bar{M}^{\perp})$, we denote the tangential and normal part of $\nabla_X \xi$ by $A_\xi X$ and $\nabla_X^{\perp}\xi$, respectively, then the Weingarten formulae is given by

$$\nabla_X \xi = -A_\xi X + \nabla_X^{\perp}\xi, \tag{2.2}$$

where A_ξ is called the shape operator of $\bar{M}$ in M and $\nabla^{\perp}$ denote the normal connection in $\Gamma(T\bar{M}^{\perp})$. Moreover, from (2.1) and (2.2) we have

$$g(h(X,Y),\xi) = g(A_\xi X, Y), \tag{2.3}$$

for any $X, Y \in \Gamma(T\bar{M})$ and $\xi \in \Gamma(T\bar{M}^{\perp})$.

Definition 2.1. *Let $\bar{M}$ be a submanifold of any Riemannian manifold M. Then the mean curvature vector field H of $\bar{M}$ is defined by*

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

where $\{e_i\}$, $1 \leq i \leq n$, is a local orthonormal basis of $\Gamma(T\bar{M})$. If the submanifold $\bar{M}$ having one of conditions

$$h = 0, \quad h(X,Y) = g(X,Y)H, \quad g(h(X,Y),H) = \lambda g(X,Y), \quad H = 0,$$

$\lambda \in C^{\infty}(\bar{M},\mathbb{R})$ for any $X, Y \in \Gamma(T\bar{M})$, then it is called totally geodesic, totally umbilical, pseudo-umbilical and minimal submanifold in M, respectively[4].

The covariant derivative of the second fundamental form h is defined by

$$(\nabla_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\bar{\nabla}_X Y, Z) - h(\bar{\nabla}_X Z, Y), \tag{2.4}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. Furthermore, the Gauss and Codazzi equations are ,respectively, given by

$$\begin{aligned} R(X,Y)Z &= \bar{R}(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) \\ &- (\nabla_Y h)(X,Z) \end{aligned} \tag{2.5}$$

and

$$\{R(X,Y)Z\}^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.6}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where R and $\bar{R}$ denote the Riemannian curvature tensors of M and $\bar{M}$, respectively. Also $\{R(X,Y)Z\}^{\perp}$ is the normal component of $R(X,Y)Z$. We recall that $\bar{M}$ is called curvature-invariant submanifold of M if $\{R(X,Y)Z\}^{\perp} = 0$ for any $X, Y, Z \in \Gamma(T\bar{M})$[6].

Now, let M be a differentiable and real 2n-dimensional manifold. An almost complex structure on M is a tensor field J of type (1,1) on M such that $J^2 = -I$. M is called an almost complex manifold if it has an almost complex structure. A Hermitian metric on an almost complex manifold M is a Riemannian metric g satisying

$$g(JX, JY) = g(X, Y),$$

for any $X, Y \in \Gamma(TM)$. Furthermore, M is called Kaehler manifold if the almost complex structure is parallel with respect to ∇, i.e., $(\nabla_X J)Y = 0$ for any $X, Y \in \Gamma(TM)$.

For each plane γ spanned orthonormal vectors X and Y in $\Gamma(TM)$ and for each points in M, we define the sectional curvature $K(\gamma)$ by

$$K(\gamma) = K(X \Lambda Y) = g(R(X,Y)Y, X).$$

If $K(\gamma)$ is constant for all planes γ in $\Gamma(TM)$ and for all points in M, then M is called a space of constant curvature or real space form. We denote by $M(c)$ a real space form of constant sectional curvature c, then the Riemannian curvature tensor of $M(c)$ is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for any $X, Y, Z \in \Gamma(TM)$.

Now, we consider a plane γ invariant by the complex structure J. In this case, we can choose a basis $\{X, JX\}$ in γ, where X is a unit vector in γ. The sectional curvature $K(\gamma)$ is denoted by $H(X)$ and it is called holomorphic sectional curvature of M determined by the unit vector X, then we have

$$H(X) = g(R(X, JX)JX, X).$$

If $H(X)$ is constant for all unit vectors in $\Gamma(TM)$ and for all points in M, then M is called a space form of constant holomorphic sectional curvature. In this case, the Riemannian curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX - g(Z,JX)JY \\ &+ g(X,JY)JZ\} \end{aligned} \tag{2.7}$$

for any $X, Y, Z \in \Gamma(TM)$, where c is the constant holomorphic sectional curvature of M[5].

3 Kaehler Product Manifolds

Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be a almost Hermitian manifolds with complex dimensional n_1 and n_2, respectively, and $M_1 \times M_2$ be a Riemannian product manifold of M_1 and M_2. We denote by P and Q the projection mappings $\Gamma(T(M_1 \times M_2))$ onto $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively, then we have $P + Q = I$, $P^2 = P$, $Q^2 = Q$ and $PQ = QP = O$. If we put $F = P - Q$, then it is easily to see that $F^2 = I$, where I denotes the identity mapping of $\Gamma(T(M_1 \times M_2))$. The Riemannian metric tensor of $M_1 \times M_2$ is given by

$$g(X, Y) = g_1(PX, PY) + g_2(QX, QY)$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. From the definition of g, we get M_1 and M_2 are totally geodesic submanifolds of Riemannian product manifold $M_1 \times M_2$. We denote the Levi-Civita connection on $M_1 \times M_2$ by ∇, then we have $\nabla P = \nabla Q = \nabla F = 0$(for the more detail, we refer to [8]).

We define a mapping by $J = J_1P + J_2Q$ of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(T(M_1 \times M_2))$, then it is easily seen that $J^2 = -I$, $J_1P = PJ$, $J_2Q = QJ$ and $FJ = JF$. Thus J is an almost complex structure on $M_1 \times M_2$. Furthermore, if (M_1, J_1, g_1) and (M_2, J_2, g_2) are both almost Hermitian manifolds, then

$$\begin{aligned} g(JX, JY) &= g_1(PJX, PJY) + g_2(QJX, QJY) \\ &= g_1(J_1PX, J_1PY) + g_2(J_2QX, J_2QY) \\ &= g_1(PX, PY) + g_2(QX, QY) \\ &= g(X, Y) \end{aligned}$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. Thus $(M_1 \times M_2, J, g)$ is an almost Hermitian manifold. By direct calculations, we obtain

$$(\nabla_X J)Y = (\nabla_{PX} J_1)PY + (\nabla_{QX} J_2)QY + (\nabla_{QX} J_1)PY + (\nabla_{PX} J_2)QY. \tag{3.1}$$

If $(M_1 \times M_2, J, g)$ is a Kaehler manifold, then we have

$$\begin{aligned} (\nabla_X J)Y &= (\nabla_{PX} J_1)PY + (\nabla_{QX} J_2)QY + (\nabla_{QX} J_1)PY \\ &+ (\nabla_{PX} J_2)QY = 0, \end{aligned} \tag{3.2}$$

for any $X, Y \in \Gamma(T(M_1 \times M_2))$. Taking FX instead of X in (3.2), we obtain

$$\begin{aligned} (\nabla_X J)Y &= (\nabla_{PX} J_1)PY + (\nabla_{QX} J_2)QY - (\nabla_{QX} J_1)PY \\ &- (\nabla_{PX} J_2)QY = 0. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we derive $(\nabla_{PX} J_1)PY = (\nabla_{QX} J_2)QY = 0$, that is, (M_1, J_1, g_1) and (M_2, J_2, g_2) are Kaehler manifolds. We denote Kaehler product manifold by $(M_1 \times M_2, J, g)$ throughthout this paper.

Furthermore, if M_1 and M_2 are complex space forms with constant holomorphic sectional curvatures c_1, c_2 and denote them by $M_1(c_1)$ and $M_2(C_2)$,respectively, then the Riemannian curvature tensor R of Kaehler product manifold $M_1(c_1) \times M_2(c_2)$ is given by

$$\begin{aligned} R(X,Y)Z &= \frac{1}{16}(c_1+c_2)\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ + 2g(FY,Z)FX - g(FX,Z)FY + g(FJY,Z)FJX \\ &- g(FJX,Z)FJY + 2g(FX,JY)FJZ\} \\ &+ \frac{1}{16}(c_1-c_2)\{g(FY,Z)X - g(FX,Z)Y + g(Y,Z)FX - g(X,Z)FY \\ &+ g(FJY,Z)JX - g(FJX,Z)JY + g(JY,Z)FJX - g(JX,Z)FJY \\ &+ 2g(FX,JY)JZ + 2g(X,JY)JFZ\} \end{aligned} \tag{3.4}$$

for any $X, Y, Z \in \Gamma(T(M_1 \times M_2))$[6].

We suppose that $K(X\Lambda Y)$ be a sectional curvature of $M_1 \times M_2$ determined by orthonormal vectors X and Y. By using (3.4) we have

$$\begin{aligned} K(X\Lambda Y) &= \frac{1}{16}(c_1+c_2)\{1 + 3g(X,JY)^2 + 2g(FY,Y)g(FX,X) - g(FX,Y)^2 \\ &+ 3g(X,JFY)^2\} + \frac{1}{16}(c_1-c_2)\{g(FY,Y) + g(FX,X) \\ &+ 6g(FJX,Y)g(JX,Y)\}. \end{aligned} \tag{3.5}$$

Similarly, if $H(X)$ is the holomorphic sectional curvature of Kaehler product manifold $M_1 \times M_2$ determined by the unit vectors X and JX, then from (3.4) we conclude

$$\begin{aligned} H(X) = K(X\Lambda JX) &= \frac{1}{16}(c_1+c_2)\{4 + 5g(FX,X)^2\} \\ &+ \frac{1}{2}(c_1-c_2)\{g(FX,X)\}. \end{aligned} \tag{3.6}$$

4 CR-Submanifolds of a Kaehler Product Manifold

Definition 4.1. *Let $\bar{M}$ be an isometrically immersed submanifold of a Kaehler manifold. $\bar{M}$ is said to be CR-submanifold of M is there exists a differentiable distribution*

$$D : x \longrightarrow D_x \subseteq T_x\bar{M}$$

on $\bar{M}$ satisfying the following conditions:
(1) D is holomorphic(invariant) with respect to J,i.e., $J(D_x) = D_x$ for each $x \in \bar{M}$.
(2) The complementary orthogonal distribution

$$D^{\perp} : x \longrightarrow D_x^{\perp} \subset T_x\bar{M}$$

is totally real(anti-invariant) with respect to J, i.e., $J(D_x^{\perp}) \subset T_x\bar{M}^{\perp}$ for each $x \in \bar{M}$[3].

We denote by p and q the dimensional of the distributions D and $D^{\perp}$, respectively. In particular, $q = 0$(resp. $p = 0$) then CR-submanifold $\bar{M}$ is called holomorphic(resp. totally real) of M. A proper CR-submanifold is a CR-submanifold which is neither a holomorphic submanifold nor a totally real submanifold.

Now, let $\bar{M}$ be a submanifold of Kaehler product manifold $M_1 \times M_2$. For any vector X tangent to $\bar{M}$, we set $X = PX + QX$, where PX and QX belong to the distributions D and $D^{\perp}$, respectively. Thus $\bar{M}$ becomes a proper CR-submanifold of Kaehler product manifold $M_1 \times M_2$. In this case, D is a holomorphic subbundle of $\Gamma(TM_1)$ and $D^{\perp}$ is a totally real subbundle of $\Gamma(TM_2)$. Such the CR-submanifolds of Kaehler product manifold was introduced in [9].

Definition 4.2. *A CR-submanifold $\bar{M}$ of a Kaehler manifold M is said to be CR-product if it is a Riemannian product of a holomorphic distribution D and totally real distribution $D^{\perp}$ of $\bar{M}$.*

The notion of CR-Product in Kaehler manifolds was introduced in[3].

Theorem 4.1. *Let $\bar{M}$ be a proper CR-submanifold of Kaehler product manifold $M_1 \times M_2$ such that the holomorphic distribution D and totally real distribution $D^{\perp}$ belong to M_1 and M_2, respectively, then $\bar{M}$ is CR-product.*

Proof. We denote by g induced metric tensor on $\bar{M}$ as well as on M. Moreover, denote the Levi-Civita connection by $\bar{\nabla}$ on $\bar{M}$, then we have

$$\begin{aligned} g(\bar{\nabla}_{PX}PY, QZ) &= g(\nabla_{PX}PY, QZ) = g_1(\nabla_{PX}PY, PQZ) + g_2(\nabla_{PX}QPY, QZ) \\ &= 0. \end{aligned}$$

for any $X, Y \in \Gamma(T\bar{M})$. Thus $\bar{\nabla}_{PX}PY \in \Gamma(D)$. Similary we have $\bar{\nabla}_{QX}QY \in \Gamma(D^{\perp})$. Hence, the distributions D and $D^{\perp}$ are integrable and their leaves are totally geodesic in $\bar{M}$. This complete the proof. □

Definition 4.3. *A CR-submanifold of a Kaehler manifold is said to be mixed-geodesic submanifold if $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$.*

Theorem 4.2. *Let $\bar{M}$ be a CR-submanifold of a Kaehler product manifold $M_1 \times M_2$ such that the distributions D and $D^{\perp}$ belong to M_1 and M_2, respectively. Then $\bar{M}$ is mixed geodesic submanifold.*

Proof.

$$g(h(PX, QY), \xi_1) = g(\nabla_{PX}QY, \xi_1) = -g(\nabla_{PX}\xi_1, QY) = 0$$

and

$$g(h(PX, QY), \xi_2) = g(\nabla_{QY}PX, \xi_2) = -g(\nabla_{QY}\xi_2, PX) = 0,$$

for any $X, Y \in \Gamma(T\bar{M})$ and ξ_1 and ξ_2 are the normal vector fields to D and $D^{\perp}$ in M_1 and M_2, respectively. Thus we conclude that $h(PX, QY) = 0$ which gives our assertion. □

Furthermore, we obtain

$$h(X,Y) = h_1(PX,PY) + h_2(QX,QY), \tag{4.1}$$

for any $X, Y \in \Gamma(T\bar{M})$, where it easily to see that h_1 and h_2 are the second fundamental forms of D and $D^\perp$ in M_1 and M_2, respectively.

Now, let $\{e_1, e_2, ..., e_p, e_{p+1}, ..., e_{n_1}, e^1, e^2, ..., e^q, e^{q+1}, ..., e^{n_2}\}$ be a local field orthonormal frames on $M_1 \times M_2$ such that $\{e_1, e_2, ..., e_p\}$(resp. $\{e^1, e^2, ..., e^q\}$) be a local field orthonormal frames on D(resp. $D^\perp$) and $\{e_{p+1}, ..., e_{n_1}, e^{q+1}, ..., e^{n_2}\}$ be a local field normal frames to $\bar{M}$ in $M_1 \times M_2$.

We denote the mean curvature vector field of $\bar{M}$ in $M_1 \times M_2$ by H, then we can easily to see that

$$mH = \sum_{i=p+1}^{n_1} trh_1 e_i + \sum_{j=q+1}^{n_2} trh_2 e^j = pH_1 + qH_2, \quad m = p + q \tag{4.2}$$

where H_1 and H_2 denote the mean curvature vector fields D and $D^\perp$ in M_1 and M_2, respectively.

Next, we will give the mean theorems of this paper.

Theorem 4.3. *Let $(M_1 \times M_2, J, g)$ be Kaehler product manifold and $\bar{M}$ be a proper CR-submanifold of $M_1 \times M_2$ such that the distributions D and $D^\perp$ belong to M_1 and M_2, respectively. Then $\bar{M}$ is pseudo-umbilical submanifold of $M_1 \times M_2$ if and only if D and $D^\perp$ are pseudo-umbilical distributions of M_1 and M_2, respectively.*

Proof. We suppose that $\bar{M}$ is pseudo-umbilical submanifold of $M_1 \times M_2$, then there exists a smooth function on $\bar{M}$ such that

$$g(h(X,Y), H) = \lambda g(X,Y) \tag{4.3}$$

for any $X, Y \in \Gamma(T\bar{M})$. Taking $e_1, e_2, ..., e_p$ for $X = Y$ in (4.3), we get

$$\begin{aligned} g(\sum_{i=1}^{p} h(e_i, e_i), H) &= \lambda \sum_{i=1}^{p} g(e_i, e_i) \\ g(pH_1, H) &= \lambda p \Longrightarrow \lambda = g(H_1, H). \end{aligned}$$

In the same way, taking $e^1, e^2, ..., e^q$ for $X = Y$ in (4.3) we obtain

$$\begin{aligned} g(\sum_{j=1}^{q} h(e^j, e^j), H) &= \lambda \sum_{j=1}^{q} g(e^j, e^j) \\ g(qH_2, H) &= \lambda q \Longrightarrow \lambda = g(H_2, H). \end{aligned}$$

Also, taking into account of (4.2) we get

$$\lambda = \frac{p}{m} g_1(H_1, H_1) = \frac{q}{m} g_2(H_2, H_2).$$

Now, we take PX and PY instead of X and Y in (4.3), respectively, then we obtain

$$\begin{aligned} g(h_1(PX,PY),H) &= \frac{p}{m}g(H_1,H_1)g(PX,PY) \\ g(h_1(PX,PY),H_1) &= g(H_1,H_1)g(PX,PY). \end{aligned}$$

It follows that the holomorphic distribution D is pseudo-umbilical in M_1. In the same way, choosing QX and QY instead of X and Y in (4.3), respectively, then we derive

$$\begin{aligned} g(h_2(QX,QY),H) &= \frac{q}{m}g(H_2,H_2)g(QX,QY) \\ g(h_2(QX,QY),H_2) &= g(H_2,H_2)g(QX,QY). \end{aligned}$$

Thus the totally real distribution $D^{\perp}$ is pseudo-umbilical in M_2.

Conversely, let D and $D^{\perp}$ be pseudo-umbilical distributions in M_1 and M_2, respectively, then we have

$$g_1(h_1(PX,PY),H_1) = g_1(H_1,H_1)g_1(PX,PY)$$

and

$$g_2(h_2(QX,QY),H_2) = g_2(H_2,H_2)g_2(QX,QY)$$

for any $X,Y \in \Gamma(T\bar{M})$. From (4.1), (4.2) and using the projections

$$P : \Gamma(T(M_1 \times M_2)) \longrightarrow \Gamma(TM_1)$$

and

$$Q : \Gamma(T(M_1 \times M_2)) \longrightarrow \Gamma(TM_2)$$

we obtain

$$\frac{m}{p}g_1(h_1(PX,PY),PH) = \frac{m^2}{p^2}g_1(PH,PH)g_1(PY,PX)$$

and

$$\frac{m}{q}g_2(h_2(QX,QY),QH) = \frac{m^2}{q^2}g_2(QH,QH)g_2(QY,QX)$$

which give us

$$g_1(h_1(PX,PY),PH) = \frac{m}{p}g_1(PH,PH)g_1(PY,PX) \tag{4.4}$$

and

$$g_2(h_2(QX,QY),QH) = \frac{m}{q}g_2(QH,QH)g_2(QY,QX). \tag{4.5}$$

Moreover, we have

$$\begin{aligned} g(H,H) &= \frac{p^2}{m^2}g(H_1,H_1)+\frac{q^2}{m^2}g_2(H_2,H_2) \\ &= \frac{p^2}{m^2}g(H_1,H_1)+\frac{q}{m}\frac{p}{m}g(H_1,H_1) \\ &= \frac{p}{m}g(H_1,H_1)\{\frac{p}{m}+\frac{q}{m}\} \\ &= \frac{p}{m}g(H_1,H_1)=\frac{q}{m}g(H_2,H_2). \qquad (4.6) \end{aligned}$$

To sum up (4.4), (4.5) and using (4.6) we obtain

$$\begin{aligned} g_1(Ph(X,Y),PH)+g_2(Qh(X,Y),QH) &= \frac{m}{p}g_1(PH,PH)g_1(PX,PY) \\ &+ \frac{m}{q}g_2(QH,QH)g_2(QX,QY) \\ g(h(X,Y),H) &= g(H,H)g(X,Y), \end{aligned}$$

for any $X,Y\in\Gamma(T\bar{M})$. Thus $\bar{M}$ is pseudo-umbilical submanifold of $M_1\times M_2$. □

Theorem 4.4. *Let $(M_1\times M_2,J,g)$ be Kaehler product manifold and $\bar{M}$ be a proper CR-submanifold of $M_1\times M_2$ such that the distributions D and $D^\perp$ belong to M_1 and M_2, respectively. Then $\bar{M}$ is curvature-invariant submanifold of $M_1\times M_2$ if and only if D and $D^\perp$ are curvature-invariant distributions of M_1 and M_2, respectively.*

Proof. By R we denote Riemannian curvature tensor of Kaehler product manifold $M_1\times M_2$. From $\nabla F=0$ and the properties of R, we can easily to se $R(PX,QY)=0$ for any $X,Y\in\Gamma(T(M_1\times M_2))$. By using the first Bianchi identity and $\nabla P=\nabla Q=0$, by direct calculations we obtain

$$R(X,Y)Z=R_1(PX,PY)PZ+R_2(PX,PY)PZ \qquad (4.7)$$

for any $X,Y,Z\in\Gamma(T\bar{M})$, where R_1 and R_2 are also Riemannian curvature tensors of M_1 and M_2, respectively. Thus from (2.5), (2.6) and (4.7) we obtain

$$\begin{aligned} (\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z) &= (\nabla_{PX}h_1)(PY,PZ)-(\nabla_{PY}h_1)(PX,PZ) \\ &+ (\nabla_{QX}h_2)(QY,QZ)-(\nabla_{QY}h_2)(QX,QZ) \end{aligned}$$

for any $X,Y,Z\in\Gamma(T\bar{M})$. Thus we arrive

$$(\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z)=0$$

if and only if

$$(\nabla_{PX}h_1)(PY,PZ)-(\nabla_{PY}h_1)(PX,PZ)=0$$

and

$$(\nabla_{QX}h_1)(QY,QZ)-(\nabla_{QY}h_2)(QX,QZ)=0,$$

which proves our assertion. □

Theorem 4.5. *Let $(M_1 \times M_2, J, g)$ be Kaehler product manifold and $\bar{M}$ be a proper CR-submanifold of $M_1 \times M_2$ such that the distributions D and $D^\perp$ belong to M_1 and M_2, respectively. There exists no totally umbilical proper CR-submanifold in Kaehler product manifold $M = M_1(c_1) \times M_2(c_2)$ with $c_1 + c_2 \neq 0$.*

Proof. We suppose that $\bar{M}$ is a totally umbilical proper CR-submanifold of Kaehler product manifold $M = M_1(c_1) \times M_2(c_2)$. By using (2.4) we obtain

$$(\nabla_X h)(Y, Z) = g(Y, Z)\nabla_X^\perp H \tag{4.8}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. On the other hand, we have

$$K(X, Y, Z, V) = g(Y, Z)g(\nabla_X^\perp H, V) - g(X, Z)g(\nabla_X^\perp H, V) \tag{4.9}$$

for any $V \in \Gamma(T\bar{M}^\perp)$. Taking $X = Z \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$, then $JX \in \Gamma(D)$ and $JY \in \Gamma(T\bar{M}^\perp)$. Since the ambient space Keahler manifold and taking into account of (4.9) , we get

$$K(X, Y, JX, JY) = g(Y, JX)g(\nabla_X^\perp H, JY) - g(X, JX)g(\nabla_Y^\perp H, JY) = 0,$$

that is,

$$K(X, Y, JX, JY) = K(X, Y, X, Y) = 0,$$

where K denote the Riemannian-Christoffel curvature tensor of Kaehler product manifold $M_1(c_1) \times M_2(c_2)$. On the other hand, from (3.5) we have

$$K(X, Y, X, Y) = -\frac{1}{16}(c_1 + c_2)$$

for any orthonormal vector fields $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. This is a contradiction. This completes the proof of the theorem. □

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