On point-line displacement in Minkowski 3-space

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Abstract. The purpose of this paper is to develop point-line displacements of a line, given by Yi Zhang and Kwun-Lon Ting [6] in R^3, in R^3_1. Firstly, we define dual split quaternions and give some algebraic properties of them. Also we discuss the screw motion and some geometrical applications of the screw operator in Minkowski 3-space. Moreover, we investigate point-line displacement, using the properties of screw operator.

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1 Introduction

Quaternion algebra, enunciated by Hamilton, has a significant role recently in several areas of the physical science; namely in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physic and quaternionic formulation of equation of motion in theory of relativity. Some algebraic properties of screw operators given by Agrawal [1].

In Kinematics, the motions of various objects are of major concern. An object may be expressed in the form of a rigid body, a line, a plane, or their combinations. Point-line is the combination of an oriented line and an endpoint on the line. Many tools used in manufacturing, such as milling, welding, drilling, grinding, screwing, painting, spraying, laser jet, high-speed water jet, etc., can be represented by point-lines in Kinematics.

In this paper, we obtain a point-line operator in Minkowski 3-space. The operator offers a simple and unique geometrical interpretation for point-line displacements.

2 Dual numbers and dual split quaternions

A brief summary of dual numbers and dual split quaternions is presented in this section for an easy of reference and to provide the necessary background for the mathematical formulations to be developed in this paper.

In this paper, a dual number A has the form $a + \varepsilon a^*$ where a and a^* are real numbers and ε is the dual symbol subjected to the rules

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$$\varepsilon \neq 0, \quad 0\varepsilon = \varepsilon 0 = 0, \quad 1\varepsilon = \varepsilon 1 = \varepsilon, \quad \varepsilon^2 = 0.$$

A split quaternion q is an expression of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k,$$

where a_0, a_1, a_2 and a_3 are real numbers, and i, j, k are split quaternionic units which satisfy the non-commutative multiplication rules

$$i^2 = -1, \ j^2 = k^2 = 1 ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j, \tag{2.1}$$

(See [2] for split quaternions).
Similarly, a dual split quaternion Q is written as

$$Q = Q_0 + Q_1 i + Q_2 j + Q_3 k,$$

where Q_0, Q_1, Q_2 and Q_3 are dual numbers.
As a consequence of this definition, a dual split quaternion Q can also be written as

$$Q = q + \varepsilon q^* = (q, q^*),$$

where $q = a_0 + a_1 i + a_2 j + a_3 k$ and $q^* = a_0^* + a_1^* i + a_2^* j + a_3^* k$ are, respectively, real and dual split quaternion components. The multiplication of split quaternionic units with dual symbol is commutative; i.e. $\varepsilon i = i\varepsilon$, and so on. Therefore, it is a matter of indifference whether one writes $Q_1 i$ or iQ_1, and so on. Owing to the properties of the eight units equality, additions, and subtraction of dual split quaternion are governed by the rules of ordinary algebra.

The tree dual split quaternionic units (i, j and k) are orthogonal unit vector with respect to scalar product defined below. Further, under a proper semi-orthogonal transformation, these units preserve the definition of split quaternionic units given in equation (2.1) and the definition of scalar product. For this reason i, j and k are identified as an orthogonal triad of unit vectors in Minkowski 3-space. It is useful, therefore, to define the following terms:
Dual number part of Q :

$$S_Q = Q_0.$$

Dual vector part of Q :

$$V_Q = Q_1 i + Q_2 j + Q_3 k.$$

There is also another product rule for dual split quaternions. Given two dual split quaternion be $P = S_P + V_P$ and $Q = S_Q + V_Q$, then

$$PQ = S_P S_Q + g(V_P, V_Q) + S_P V_Q + S_Q V_P + V_P \times V_Q \tag{2.2}$$

where

$$g(V_P, V_Q) = -P_1 Q_1 + P_2 Q_2 + P_3 Q_3,$$

and

$$V_P \times V_Q = (Q_2P_3 - P_2Q_3)\, i + (Q_1P_3 - P_1Q_3)\, j + (P_1Q_2 - Q_1P_2)\, k.$$

Hamiltonian conjugate of Q :

$$\overline{Q} = Q_0 - (Q_1 i + Q_2 j + Q_3 k) = S_Q - V_Q = \overline{q} + \varepsilon\overline{q^*}. \tag{2.3}$$

Norm of Q :

$$N_Q = Q\overline{Q} = \overline{Q}Q = Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2.$$

Reciprocal of Q :

$$Q^{-1} = \frac{\overline{Q}}{N_Q}. \tag{2.4}$$

Unit quaternion:

$$N_Q = 1.$$

3 Some properties of Minkowski 3-space

In this section, we give some useful definitions and propositions about Minkowski space.

Definition 3.1. [3] R^n with the metric tensor

$$g(\vec{u}, \vec{v}) = -\sum_{i=1}^{k} u_i v_i + \sum_{j=k+1}^{n} u_j v_j, \;\; \vec{u}, \vec{v} \in R^n, \;\; 0 \le k \le n$$

is called semi Euclidean space and is denoted by $R_k^n$where k is called the index of the metric. The resulting semi Euclidean space R_k^n reduced to R^n if $k = 0$. For $n \ge 1$, R_1^n is called Minkowski n-space.

Definition 3.2. [3] Let R_k^n be a semi Euclidean space furnished with a metric tensor g. A vector $\vec{w} \in R_k^n$ is called

spacelike if $g(\vec{w}, \vec{w}) > 0$ or $\vec{w} = 0$,

null if $g(\vec{w}, \vec{w}) = 0$ and $\vec{w} \neq 0$,

timelike if $g(\vec{w}, \vec{w}) < 0$,

the norm $\|\vec{w}\|$ of a vector $\vec{w} \in R_k^n$ is $|g(\vec{w}, \vec{w})|^{\frac{1}{2}}$, two vectors $\vec{w}_1$ and $\vec{w}_2$ in R_k^n are said to be orthogonal , if $g(\vec{w}_1, \vec{w}_2) = 0$.

Theorem 3.3. *[4] Let R_1^3 be a Minkowski 3-space furnished with a metric tensor $g(\vec{u},\vec{v}) = -u_1v_1 + u_2v_2 + u_3v_3$, $\vec{u},\vec{v} \in R^3$. Then we have the following:*

i. Every orthonormal set of three vectors is a basis for R_1^3.

ii. Every orthonormal basis has two spacelike vectors and one timelike vector.

iii. For every unit spacelike or unit time like vector $\vec{v}$, there is an orthonormal basis containing $\vec{v}$.

iv. For any $\vec{u} \in R^3$,

$$\vec{u} = \sum_{i=1}^{3} g(u, w_i)\ g(w_i, w_i) w_i$$

if $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthonormal basis.

Lemma 3.4. *Let $\vec{u}$ and $\vec{v}$ be vectors in the Minkowski 3-space.*

i. If $\vec{u}$ and $\vec{v}$ are future pointing (or past pointing) timelike split vectors, then $\vec{u}\times\vec{v}$ is a spacelike vector, $g(\vec{u},\vec{v}) = -\|\vec{u}\|\,\|\vec{v}\|\cosh\theta$ and $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sinh\theta$, where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.

ii. If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|g(\vec{u},\vec{v})| < \|\vec{u}\|\,\|\vec{v}\|$, then $\vec{u}\times\vec{v}$ is timelike, $g(\vec{u},\vec{v}) = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ and $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

iii. If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the inequality $|g(\vec{u},\vec{v})| > \|\vec{u}\|\,\|\vec{v}\|$, then $\vec{u}\times\vec{v}$ is spacelike, $|g(\vec{u},\vec{v})| = \|\vec{u}\|\,\|\vec{v}\|\cosh\theta$ and $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sinh\theta$, where θ is the hyperbolic angle between $\vec{u}$ and $\vec{v}$.

iv. If $\vec{u}$ and $\vec{v}$ are spacelike vectors satisfying the equality $|g(\vec{u},\vec{v})| = \|\vec{u}\|\,\|\vec{v}\|$, then $\vec{u}\times\vec{v}$ is null.

4 Screw operators in Minkowski 3-space

In this section, we give the main results of the paper, first, some necessary definitions about dual split quaternions and point lines are given, then the properties of point lines are examined.

Theorem 4.1. *[5] There exist one-to-one correspondence between directed timelike (resp. spacelike) lines of R_1^3 and an ordered pair of vectors $A = (\vec{a},\vec{a}^*)$ such that $g(\vec{a},\vec{a}) = -1$ (resp. $g(\vec{a},\vec{a}) = 1$) and $g(\vec{a},\vec{a}^*) = 0$.*

Definition 4.2. Let us consider a dual split vector

$$V_Q = \vec{a} + \varepsilon\vec{a}^* = (a_1 i + a_2 j + a_3 k) + \varepsilon\,(a_1^* i + a_2^* j + a_3^* k)\,.$$

The dual split vector is said to timelike if $g(\vec{a},\vec{a}) < 0$, spacelike if $g(\vec{a},\vec{a}) > 0$ or $\vec{a} = 0$, and null if $g(\vec{a},\vec{a}) = 0$ and $\vec{a} \neq 0$, where g is a Minkowskian scalar product with signature $(-,+,+)$.

Theorem 4.3. *[5] Let A, B be two future pointing (or past pointing) unit timelike dual split vectors. Then we have*

$$g(A,B) = -\cosh\tilde{\varphi}$$

where $\cosh\tilde{\varphi} = -\cosh\varphi - d\sinh\varphi$.

Theorem 4.4. *Let A, B be two unit spacelike dual vectors.*

i. If $\vec{a}\times\vec{b}$ is spacelike, then

$$g(A,B) = \pm\cosh\tilde{\varphi},$$

where $\cosh\tilde{\varphi} = \pm\cosh\varphi \pm d\sinh\varphi$.

ii. If $\vec{a}\times\vec{b}$ is timelike, then

$$g(A,B) = \cos\tilde{\theta},$$

where $\cos\tilde{\theta} = \cos\theta - d\sin\theta$.

Proof. Moment vectors $\vec{a}^*$ and $\vec{b}^*$ independent of choice of the points p and q on the directed spacelike lines l_1 and l_2 that correspond to A and B in R^3_1 space of lorentzian lines. Thus, p and q points can be thought of as feet of common perpendicular of l_1 and l_2.

If we show the shortest distance between l_1 and l_2 by d,

$$p - q = \mp\frac{\vec{a}\times\vec{b}}{\left\|\vec{a}\times\vec{b}\right\|}d.$$

Now we consider following equations

$$g(\vec{a},\vec{b}^*) = g(\vec{a},\vec{q}\times\vec{b}) = -g(\vec{q},\vec{a}\times\vec{b}),$$

$$g(\vec{a}^*,\vec{b}) = g(\vec{p}\times\vec{a},\vec{b}) = g(\vec{p},\vec{a}\times\vec{b}),$$

where $\vec{a}^* = \vec{p}\times\vec{a}$ and $\vec{b}^* = \vec{q}\times\vec{b}$.
i. From the third item of Lemma 3.4, we have

$$\begin{aligned} g(\vec{a},\vec{b}^*) + g(\vec{a}^*,\vec{b}) &= g(\overrightarrow{p-q},\vec{a}\times\vec{b}) \\ &= g\left(\pm\frac{\vec{a}\times\vec{b}}{\left\|\vec{a}\times\vec{b}\right\|}d, \vec{a}\times\vec{b}\right) \\ &= \pm d\left\|\vec{a}\times\vec{b}\right\| \\ &= \pm d\sinh\varphi. \end{aligned}$$

If we choose the signal $-$(minus)[$+$(plus)], we get

$$g(A,B) = \pm\cosh\varphi \pm d\sinh\varphi.$$

Thus, from the Taylor formula

$$g(A,B) = \pm\cosh\tilde{\varphi} = \pm\cosh(\varphi+\varepsilon d).$$

ii. Also from the second item of Lemma 3.4, we get

$$\begin{aligned} g(\vec{a},\vec{b}^{*})+g(\vec{a}^{*},\vec{b}) &= \mp d\left\|\vec{a}\times\vec{b}\right\| \\ &= \mp d\sin\theta. \end{aligned}$$

If we choose signal $-$(minus) in above equation,

$$g(A,B)=\cos\theta-d\sin\theta.$$

Then, we obtain

$$g(A,B)=\cos(\theta+\varepsilon d)=\cos\tilde{\theta}.$$

This completes the proof of the theorem. □

Lemma 4.5. *Let A and B be two dual split quaternion. We have the following,*

$$g(A\times B,A\times B)=-g(A,A)g(B,B)+[g(A,B)]^{2}$$

Proof of the Lemma is obtained by [1, 3].

Let given two spacelike line be l_1 and l_2 represented with unit dual split vectors A and B, respectively.

Now $\vec{a}\times\vec{b}$ be timelike vector. Then we have $g(A,B)=\cos\tilde{\theta}$ and we can calculate $\|A\times B\|=\sin\tilde{\theta}$, where $\tilde{\theta}$ is the dual angle between A and B. Dual quaternionic multiplication of A and B is determined by

$$AB=g(A,B)+A\times B.$$

Then we find

$$AB=\cos\tilde{\theta}+C\sin\tilde{\theta} \tag{4.1}$$

where $C=\frac{A\times B}{\|A\times B\|}$.

From the equation (2.3), we obtain $B^{-1}=-B$, then we get

$$A=-\left(\cos\tilde{\theta}+C\sin\tilde{\theta}\right)B.$$

The screw operator on B is denoted as S_1, i.e.

$$S_1=-(\cos\tilde{\theta}+C\sin\tilde{\theta}). \tag{4.2}$$

And now $\vec{a}\times\vec{b}$ be spacelike split vector. Let we choose $g(A,B)=\cosh\tilde{\varphi}$, then we can find the screw operator in the form:

$$S_2=-(\cosh\tilde{\varphi}+C\sinh\tilde{\varphi}), \tag{4.3}$$

where $C=\frac{A\times B}{\|A\times B\|}$.

Surprisingly; if there are future pointing (or past pointing) two unit timelike dual vector instead of spacelike, the operator is $S_3=-S_2$.

Now we can give following theorem.

Theorem 4.6. *Let A and B be unit dual split vectors.*

i. If A and B are spacelike vectors and $a \times b$ is a timelike vector, then the screw operator is $S_1 = -\cos\tilde{\theta} - C\sin\tilde{\theta}$, where $\tilde{\theta}$ is the dual angle between A and B.

ii. If A and B are spacelike vectors satisfying the inequality $g(\vec{a},\vec{b}) > 0$ and $a \times b$ is a spacelike vector, then the screw operator is $S_2 = -\cosh\tilde{\varphi} - C\sinh\tilde{\varphi}$, where $\tilde{\varphi}$ is the dual hyperbolic angle between A and B.

iii. If A and B are spacelike vectors satisfying the inequality $g(\vec{a},\vec{b}) < 0$ and $a \times b$ is a spacelike vector, then the screw operator is defined by $S_3 = \cosh\tilde{\varphi} + C\sinh\tilde{\varphi}$, where $\tilde{\varphi}$ is the dual hyperbolic angle between $-A$ and B.

iv. If A and B are timelike vectors satisfying the inequality $g(\vec{a},\vec{b}) < 0$, then $a \times b$ is a spacelike vector and the screw operator is $S_4 = \cosh\tilde{\varphi} + C\sinh\tilde{\varphi}$, where $\tilde{\varphi}$ is the dual hyperbolic angle between A and B.

Remark 4.7. *The screw operator is a unit dual split quaternion.*

Corollary 4.8. *S_1, S_2 and S_3 are called screw operator in R_1^3 and describes a screw motion about orthogonal line axis C, through a dual angle $\tilde{\theta}$ (or a dual hyperbolic angle $\tilde{\varphi}$)*

Example 4.9. Let us consider two spacelike lines represented with unit dual vectors $A = (-\varepsilon\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2})$ and $B = (1, -\sqrt{2}, -\varepsilon)$. For these vectors $a \times b = (\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{\sqrt{3}}{2})$ is spacelike and $g(\vec{a},\vec{b}) = -\frac{\sqrt{3}}{\sqrt{2}} < 0$. From the equation (2.2) we obtain

$$AB = \left(-\frac{\sqrt{3}}{\sqrt{2}} + \varepsilon\frac{\sqrt{3}-1}{2}\right) + \left(-\frac{1}{\sqrt{2}} + \varepsilon\frac{\sqrt{3}}{2}\right)i + \left(\frac{1}{2}\right)j + \left(-\frac{\sqrt{3}}{2} + \varepsilon\frac{\sqrt{3}}{\sqrt{2}}\right)k.$$

From the equation (2.3) we obtain $B^{-1} = -B$, then we get

$$A = \left[\left(\frac{\sqrt{3}}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}-1}{2}\right) + \left(\frac{1}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}}{2}\right)i\left(-\frac{1}{2}\right)j + \left(\frac{\sqrt{3}}{2} - \varepsilon\frac{\sqrt{3}}{\sqrt{2}}\right)k\right]B$$

Here the unit dual quaternion $S_3 = (\frac{\sqrt{3}}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}-1}{2}) + (\frac{1}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}}{2})i + (-\frac{1}{2})j + (\frac{\sqrt{3}}{2} - \varepsilon\frac{\sqrt{3}}{\sqrt{2}})k$ represents a screw motion throught a dual hyperbolic angle φ such that $\cosh\tilde{\varphi} = \frac{\sqrt{3}}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}-1}{2}$ and $\sinh\tilde{\varphi} = \frac{1}{\sqrt{2}} - \varepsilon\frac{3-\sqrt{3}}{2}$.

5 Point-line operator in Minkowski 3-space

A point-line contains a directed line, which is a concept of oriented projective geometry, and the location of an endpoint along the line. We know that from the Theorem 4.1, a directed line which is not null can be represented with a unit dual split vector

Thus, the directed line passing through points E and H (see Fig. 1) can be represented by a unit dual split vector

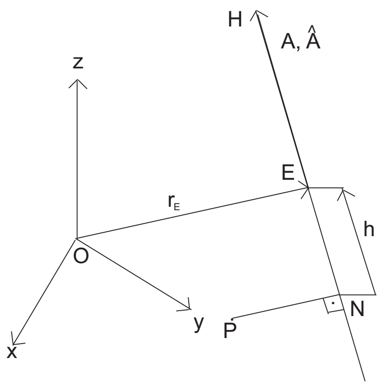

Figure 1: Pointline representation.

$$A = (\vec{a}, \vec{a}^*) \quad (g(\vec{a}, \vec{a}) = \mp 1 \text{ and } g(\vec{a}, \vec{a}^*) = 0),$$

where $\vec{a}$ represent to unit vector along the oriented line and $\vec{a}^*$ is the moment vector of the line with respect to the origin of reference frame.

Let P be a reference point, N is the foot of the perpendicular from P to the line, and d is the distance from N to E. The distance, termed end point offset, can be either positive or negative, depending on the consistency of the direction from N to E and the directed line.

A point-line expressed by multiplying a dual number $\exp(\varepsilon d)$, to A, namely

$$\hat{A} = A \exp(\varepsilon h) = \vec{a} + \varepsilon(\vec{a}^* + h\vec{a}). \tag{5.1}$$

If the reference point is changed from point P to P', the new representation of the point-line can be calculated easily. Let r represent to free vector from P to P'. π and $\pi^{'}$ are the two planes perpendicular to the point-line through P and $P^{'}$ respectively. The shortest distance between the planes π and $\pi^{'}$ is the variation of the endpoint offset and calculated easily, using the third and forth item of Theorem 3.3.
Thus, if $\vec{r}$ is spacelike, then

$$\hat{A}_{p'} = \hat{A}_p - \varepsilon g(\vec{r}, \vec{a})\vec{a}. \tag{5.2}$$

If $\vec{r}$ is timelike, then we have

$$\hat{A}_{p'} = \hat{A}_p + \varepsilon g(\vec{r}, \vec{a})\vec{a}. \tag{5.3}$$

Else we get following relationship

$$\hat{A}_{p'} = \hat{A}_p.$$

Let, we consider two given point-lines $\hat{A}$ and $\hat{B}$ in the following form:

$$\hat{A} = \exp(\varepsilon h_A)A,$$

$$\hat{B} = \exp(\varepsilon h_B)B.$$

In this case, the inner, cross , and geometric (quaternionic) products of $\hat{A}$ and $\hat{B}$ are as follows;

$$g(\hat{A}, \hat{B}) = \exp[\varepsilon(h_A + h_B)]g(A, B), \tag{5.4}$$

$$\hat{A} \times \hat{B} = \exp[\varepsilon(h_A + h_B)]A \times B, \tag{5.5}$$

$$\hat{A}\hat{B} = \exp[\varepsilon(h_A + h_B)]AB. \tag{5.6}$$

As split dual vectors, we can easily write following relationship:

$$\hat{A}\hat{B} = g(\hat{A}, \hat{B}) + \hat{A} \times \hat{B}, \tag{5.7}$$

Comparing equations (5.1) and (5.2) with equation (5.4) yields

$$\hat{A}\hat{B} = \exp[\varepsilon(h_A + h_B)]\{g(A, B) + A \times B\}. \tag{5.8}$$

Definition 5.1. A point-line $\hat{A} = A\exp(dh_A)$ is called timelike (spacelike) if A is timelike(spacelike).

Let now we choose two spacelike point-lines $\hat{A}$ and $\hat{B}$. If $\vec{a} \times \vec{b}$ is timelike, then we have $g(A, B) = \cos\tilde{\theta}$ and $\|A \times B\| = \sin\tilde{\theta}$. In this case, equation (5.8) can be written as:

$$\hat{A}\hat{B} = \exp[\varepsilon(h_A + h_B)]\{\cos\tilde{\theta} + C\sin\tilde{\theta}\}, \tag{5.9}$$

where $C = \frac{A\times B}{\|A\times B\|}$ (see Fig. 2).
Using the equation (2.3) , the conjugation of $\hat{A}\hat{B}$ can be obtained as

$$\overline{(\hat{A}\hat{B})} = \exp[\varepsilon(h_A + h_B)]\{\cos\tilde{\theta} - \gamma\sin\tilde{\theta}\}. \tag{5.10}$$

Since the inverse of $\hat{A}$ is defined as $\hat{A}^{-1} = \frac{\overline{\hat{A}}}{\hat{A}\overline{\hat{A}}}$ then

$$\overline{\hat{A}} = (\hat{A}\overline{\hat{A}})\hat{A}^{-1} = \exp(2\varepsilon d_A)\hat{A}^{-1}. \tag{5.11}$$

Using the above equation and the fact that $\hat{B}^* = -\hat{B}$, we have

$$\overline{(\hat{A}\hat{B})} = \overline{\hat{B}\hat{A}} = -\exp(2\varepsilon h_A)\hat{B}\hat{A}^{-1}. \tag{5.12}$$

Comparing equation (5.10) with equation (5.12) yields

$$\hat{B}\hat{A}^{-1} = \exp[\varepsilon(h_B - h_A)]\{-\cos\tilde{\theta} - C\sin\tilde{\theta}\} \tag{5.13}$$

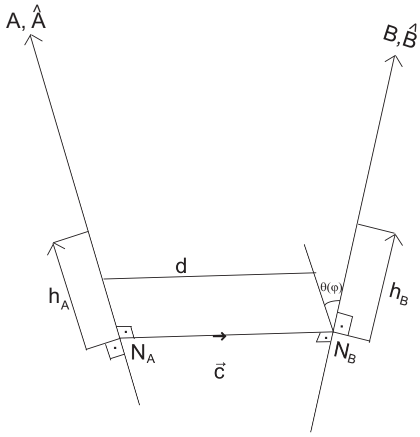

Figure 2: Pointline displacement.

or

$$\hat{B} = \exp[\varepsilon(h_B - h_A)]\{-\cos\tilde{\theta} - C\sin\tilde{\theta}\}\hat{A}. \tag{5.14}$$

The dual quaternion operator on $\hat{A}$ is referred to as *the point-line operator*, and is denoted as

$$\Omega = \exp[\varepsilon(h_B - h_A)]\{-\cos\tilde{\theta} - C\sin\tilde{\theta}\}. \tag{5.15}$$

From the equations (5.15) and (4.2), we get

$$\Omega = \exp(\varepsilon\Delta d)S_1 \tag{5.16}$$

where $\Delta d = h_B - h_A$ and $Q = -\cos\tilde{\theta} + C\sin\tilde{\theta}$.

Following theorem can be proved easily, using the similar way.

Theorem 5.2. *If S_i is given a screw operator on dual split vector A, then a point-line operator on dual split vector A can be expressed by the following equation*

$$\Omega_i = \exp(\varepsilon\Delta h)S_i$$

and the inverse of Ω_i can be calculated as the inverse of a dual quaternion

$$\Omega_i^{-1} = \exp(-\varepsilon\Delta h)S_i^{-1},$$

where $i = 1, 2, 3, 4$.

Example 5.3. For the two position $\hat{A}$ and $\hat{B}$ of the point-line shown in fig. 2, the direction-cosine vectors $\vec{a}$ and $\vec{b}$ of the directed line and the position vectors $\vec{E}$ and $\vec{F}$ of the end points E and F are respectively as follows:

$$\begin{aligned} \vec{a} &= \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), \ \vec{E} = (0,0,1) \\ \vec{b} &= \left(-1, \sqrt{2}, 0\right), \ \vec{F} = (0,1,0). \end{aligned}$$

The unit dual split vectors corresponding to initial and final points are

$$A = \vec{a} + \varepsilon \vec{a}^* = \vec{a} + \varepsilon \vec{a} \times \vec{E} = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) + \varepsilon \left(-\frac{\sqrt{3}}{2}, 0, 0\right)$$

$$B = \vec{b} + \vec{b}^* = \vec{b} + \varepsilon \vec{b} \times \vec{F} = \left(-1, \sqrt{2}, 0\right) + \varepsilon \left(0, 0, 1\right).$$

The endpoint offsets corresponding to the initial and final positions, with respect to the reference point at the origin O, are

$$h_{\hat{A}} = \sqrt{|g(E-K),(E-K)|} = -1,$$

and

$$h_{\hat{B}} = \sqrt{|g(F-L),(F-L)|} = 1.$$

Thus, the point-line operator is

$$\Omega_2 = \exp(2\varepsilon)\left[\left(\frac{\sqrt{3}}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}-1}{2}\right) + \left(\frac{1}{\sqrt{2}} - \varepsilon\frac{\sqrt{3}}{2}\right)i + \left(-\frac{1}{2}\right)j + \left(\frac{\sqrt{3}}{2} - \varepsilon\frac{\sqrt{3}}{\sqrt{2}}\right)k\right].$$

6 Conclusion

The representation and displacement of a point-line is a single rigid element. A point-line displacement is described by means of a point-line operator, which is composed of a screw operator and a translation operator.

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