Classification of lines in finite affine coordinate plane of odd order

Sunčica Blažev and Vlado Cigić

Abstract. For the plane $A(K)$ over the field $K = GF(q)$, of odd order q, classification of all lines in $A(K)$ is given. So, for $q \equiv 3 \pmod 4$, we have ordinary and specific lines in $A(K)$ exclusively, and for $q \equiv 1 \pmod 4$, besides ordinary and specific lines, we have singular lines also. An example of application of conclusions in this classification is determination of the group of all isometries of the plane $A(K)$.

M.S.C. 2000: 05B25.

Key words: lines in affine plane, finite field, distance, isometries.

§1. Introduction

In Blažev-Cigić [2], the classification of the affine coordinate planes $A(K)$, where $K = GF(q)$ (q is a power of the prime $p = charK \neq 2$) is given.
So, for $q \equiv 3 \pmod 4$, the ring $K[i] = \{x + yi \mid x, y \in K\}$ ($i = \sqrt{-1}$) is a field of the Galois complex numbers and, with the identification $x + yi \equiv (x, y)$, we have the points of the plane $A(K)$, which is the Galois complex plane. The lines of the plane $A(K)$ are the sets of the points (x, y), determinated with the equation $y = kx + l$, respectively $x = x_0$ (where $k, l, x_0 \in K$). If we add the "point at infinity" ∞ to the plane $A(K)$, we get the extended Galois plane $A(K) \cup \{\infty\}$ (with the note that every line goes through the point ∞).
One of the characterizations of the Galois complex plane (besides $q \equiv 3 \pmod 4$ and some others [2]) is: $x^2 + y^2 \neq 0$ for all the points $(x, y) \neq (0, 0)$ from $A(K)$. The remaining case, $q \equiv 1 \pmod 4$, is characterized by the property: $K[i]$ is not a field. In that case, in the plane $A(K) = K[i]$ we have the sets of the "singular points", i.e. the points $(x, y) \in A(K)$ for which $x^2 + y^2 = 0$. These are all the points of "singular (isotropic) lines" $y = \pm kx$ (where $k^2 + 1 = 0$; [2]). The *squared length* $d^{(2)}$ of the point (x, y) is defined ([2]) as: $d^{(2)}((x, y)) = x^2 + y^2$, respectively the *squared distance* $d^{(2)}$ of the points (x_1, y_1) and (x_2, y_2) from $A(K)$ is defined as: $d^{(2)}((x_1, y_1), (x_2, y_2)) = (x_2 - x_1)^2 + (y_2 - y_1)^2$. In the case when $q \equiv 1 \pmod 4$, the singular figure in $A(K)$ consists of all the lines (isotropic lines) parallel with the lines $y = \pm kx$ ($k^2 + 1 = 0$). The squared distance between two points belonging to any line of this figure is equal to zero.

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 58-61.

We define the length $d((x,y)) = \sqrt{x^2+y^2} = \sqrt{d^{(2)}((x,y))}$ of the point (x,y) from the plane $A(K)$. Also, we define the euclidean distance between the points (x_1,y_1) and (x_2,y_2) as $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and we give the classification of the lines in $A(K)$: ordinary lines, specific lines and isotropic (singular) lines.

§2. The classification

The distance between the points (x_1,y_1) and (x_2,y_2), which belong to the line $y = kx+l$ is $d = (x_2-x_1)\sqrt{1+k^2}$ (not $\pm(x_2-x_1)\sqrt{1+k^2}$; we take the above value, where, generally, we can not avoid the indeterminability of that sign). On the line $x = x_0$, we have $d\left((x_0,y_1),(x_0,y_2)\right) = y_2 - y_1$.
If $1+k^2$ is a square (i.e. $\sqrt{1+k^2} \in K$), we will say that the line $y = kx+l$ is an ordinary line in $A(K)$. So, the distance between the points of an ordinary line is an element from the field K. Thereto, every line $x = x_0$ is an ordinary line. If $1+k^2 = 0$, we will say that the line $y = kx+l$ is a singular line in $A(K)$. Let us emphasize that the distance between two points on a singular line is equal to the zero. Finally, if $1+k^2$ is not a square in K (i.e. $\sqrt{1+k^2} \notin K$), we will say that the line $y = kx+l$ is a specific line in $A(K)$. The value of the distance $d = (x_2-x_1)\sqrt{1+k^2}$, between any of two points (x_1,y_1) and (x_2,y_2) on such a line (with the equation $y = kx+l$) is not an element from the field K, i.e. $d \notin K$. However, we also consider these values of the distance d (they are elements in some algebraic closure of the field K).

2.1 The case $q \equiv 3 \pmod 4$

On the base of [2], there are no singular lines in $A(K)$. It holds:

Proposition 1. *For $q \equiv 3 \pmod 4$ we have exactly $\frac{q+1}{2}$ parallel-classes of ordinary lines in $A(K)$ (so thereby exactly $q+1-\frac{q+1}{2} = \frac{q+1}{2}$ classes of specific lines).*

Proof. Let us consider the equation $1+k^2 = x^2$ (for $x \in K$). Obviously $x \neq 0$ (otherwise, for $x = 0$ it holds $1+k^2 \neq 0$). We write $\frac{1}{x^2} + (\frac{k}{x})^2 = 1$. In fact, we consider the equation $u^2+(ku)^2 = 1$, respectively the system $\begin{cases} u^2+v^2=1 \\ v = ku. \end{cases}$
Each point $(u,v) \in A(K)$ belongs to some line $v = ku$, so each point (u,v) of the circle $\mathcal{K} : u^2+v^2 = 1$ belongs to some of those lines also. If the point (u,v) of that circle is on the line $v = ku$, we emphasize that the point $(-u,-v)$ of the circle $\mathcal{K}$ is also on that line. Thereto, the points $(0,1)$ and $(0,-1)$ from the circle $\mathcal{K}$ are on the ordinary line $x = 0$. Let s be the number of the above mentioned lines. Then exactly $2s$ points of the circle $\mathcal{K}$ lie on those lines. Therefore: $2s+2 = q+1$, so $s = \frac{q-1}{2}$ and we have exactly $s+1 = \frac{q-1}{2}+1 = \frac{q+1}{2}$ of ordinary lines through the origin. ■

Remark 1. Each line through the origin $0 = (0,0)$ is a representative of exactly one parallel-class. Each ordinary line through 0 passes through diametrical points of the circle $\mathcal{K} : x^2+y^2 = 1$ (i.e. besides the point (x,y) it goes through the point $(-x,-y)$ of the circle $\mathcal{K}$). How many rotations around 0 do we have? There are as many rotations around 0 as the points of the circle $\mathcal{K}$ which can be images of the point $(1,0)$ (from $x-$axis) under any such rotation. Rotations

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix}, \text{ respectively } \begin{bmatrix} -x & -y \\ y & -x \end{bmatrix}; \ (x^2+y^2=1)$$

map the point $(1,0) \in \mathcal{K}$ to diametrical points: $(1,0) \mapsto (1,0)\begin{bmatrix} x & y \\ -y & x \end{bmatrix} = (x,y)$, respectively $(1,0) \mapsto (1,0)\begin{bmatrix} -x & -y \\ y & -x \end{bmatrix} = (-x,-y)$ of the circle $\mathcal{K}$. According to Proposition 1, we have exactly $\frac{q+1}{2}$ ordinary lines through 0 and thereby exactly $2 \cdot \frac{q+1}{2} = q+1$ rotations around 0.
There are exactly $q+1$ reflections in lines through 0 (as many as the lines through 0). It can be verified (!) that the reflection in line $y = kx$ is given with the matrix $\frac{1}{1+k^2}\begin{bmatrix} 1-k^2 & 2k \\ 2k & k^2-1 \end{bmatrix}$ (when we put $-k$ instead k, we get reflection in line $y = -kx$).
Therewith we have determined all the elements of the subgroup $\mathcal{I}(A(K))_0$ of the group $\mathcal{I}(A(K))$ of all isometries of the plane $A(K)$. All translations $(x,y) \mapsto (x+r, y+s)$ form the normal subgroup $\mathcal{T}$ of the order q^2 in $\mathcal{I}(A(K))$. In such a manner, the group $\mathcal{I}(A(K))$ is a semidirect product: $\mathcal{I}(A(K)) = \mathcal{I}(A(K))_0 \cdot \mathcal{T}$ with the order $\mid \mathcal{I}(A(K)) \mid = 2q^2(q+1)$.

2.2 The case $q \equiv 1 \pmod 4$

In this case, the equation $k^2+1=0$ has the roots $\pm\kappa$ in the field $K = GF(q)$. Two singular lines (0−lines or isotropic lines) $y = \pm\kappa x$ go through the point $0 = (0,0)$. The length of the point $(x, \kappa x)$ is $d((x,\kappa x)) = x\sqrt{1+\kappa^2} = 0$. Therewith, in $A(K)$ we have two parallel-classes of singular lines. (The distance $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ between two points (x_1,y_1) and (x_2,y_2), belonging to any of the line from these two classes, is equal to the zero.) There remain parallel-classes of ordinary, respectively specific lines. Analogously to the proof of Proposition 1 (this time the "circle" u^2+v^2 is affine hyperbola with exactly $q-1$ points) we get

Proposition 2. *For $q \equiv 1 \pmod 4$ we have exactly $\frac{q-1}{2}$ parallel-classes of ordinary lines, exactly two classes of singular lines (so thereby exactly $q+1-(\frac{q-1}{2}+2) = \frac{q-1}{2}$ classes of specific lines) in $A(K)$.*

Remark 2. By the rotation around 0, ordinary line $y=0$ goes to ordinary line exactly. So, we have exactly $2 \cdot \frac{q-1}{2} = q-1$ rotations around 0. About the reflections in lines $y = kx$ (through 0), we can see that the matrix $\frac{1}{1+k^2}\begin{bmatrix} 1-k^2 & 2k \\ 2k & k^2-1 \end{bmatrix}$ (where instead k, it can stand $-k$) has no sense for $1+k^2=0$, i.e. for the singular lines $y = \pm\kappa x$. Therefore, *there do not exist reflections in singular lines*! So, we have exactly $(q+1)-2 = q-1$ reflections in lines through the origin 0. Thereby we have determined the subgroup $\mathcal{I}(A(K))_0$ which (this time!) has the order $2(q-1)$. In such a manner, the group $\mathcal{I}(A(K))$ is a semidirect product of the subgroup $\mathcal{I}(A(K))_0$ and the normal subgroup $\mathcal{T}$ of all translations in $A(K)$. Finally, it holds $\mid \mathcal{I}(A(K)) \mid = 2q^2(q-1)$.

Final remark. In this article, we have classified all the lines in the affine coordinate plane $A(K)$ over the finite field $K = GF(q)$ of odd order q. So, for $q \equiv 3 \pmod 4$, in the Galois plane $A(K)$ ([2]) we have ordinary and specific lines (Proposition 1) exclusively. For $q \equiv 1 \pmod 4$, except ordinary and specific lines, we have singular lines (Proposition 2) also.

About the description of the group of all isometries $\mathcal{I}(A(K))$ (in both cases; $q \equiv 3 \pmod 4$ respectively $q \equiv 1 \pmod 4$), we know that the results are familiar (for example [3]), but our intention was to derive that by the use of mentioned classification of the lines. So, for example, isometry maps ordinary line to ordinary line. Furthermore, if the line is singular then it can not be the axis of reflection. Finally, these specificities are related to the fact that the plane $A(K)$ has finite (and odd) order. (Why would not we notice: The plane $A(\mathbb{R})$ (real euclidean plane) has ordinary lines only; finite plane $A(K)$, $charK = 2$ has no specific lines.)

References

[1] S. Blažev and V. Cigić, *About induced orthogonality and generalized reflections of affine coordinate plane $A(K)$ of odd order*, Balkan Journal of Geometry and Its Applications 12 (2007), 9-15.

[2] S. Blažev and V. Cigić, *Classification of finite affine coordinate planes of odd order and corresponding circle geometries*, sent to publisher.

[3] E. Snapper and R.J. Troyer, *Metric affine geometry*, Academic Press, New York, 1971.

Authors' addresses:

Sunčica Blažev
Hrvatskih žrtava 85, 21210 Solin, Croatia
email: susa7777@yahoo.com

Vlado Cigić
Fakultet strojarstva i računarstva, University of Mostar,
Matice hrvatske bb., 88000 Mostar, BiH