Fractional dynamical systems and applications in economy

Mihai Boleantu

Abstract. Using Caputo fractional derivative of order α we build the fractional jet bundle of order α and its main geometrical structures. Defined on that bundle, some fractional dynamical systems with applications in economics are studied and a numerical simulation is done.

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1 Introduction

In this paper, using Caputo fractional derivative [1], the fractional jet fibre bundle is built on a differentiable manifold and its main geometric structures are emphasized. Some significant examples from economics are presented. The definitions, notations and the main properties of the Caputo fractional derivative used in this paper may be explicitly found in [1]. In section 2, the fractional osculator bundle of order k is described. In section 3, the fractional jet fibre bundle is defined and the fractional Euler-Lagrange equations together with the fractional Hamilton equations are established. In section 4 two fractional economic models are studied and a numerical simulation is done, using the method of Adams-Bashforth-Moulton (see [2]).

2 The fractional osculator bundle of order k on a differentiable manifold

Let $\alpha \in (0,1)$ be fixed and M a differentiable manifold of dimension n. Two curves $c_1, c_2 : I \to \mathbb{R}$ with $c_1(0) = c_2(0) = x_0 \in M$, $0 \in I$, have a fractional contact α of order $k \in \mathbb{N}^*$ in x_0, if for any $f \in \mathcal{F}(U)$, $x_0 \in U$, U a chart on M, it holds that

$$D_t^{\alpha a}(f \circ c_1)\,|_{t=0} = D_t^{\alpha a}(f \circ c_2)\,|_{t=0}\ , \tag{2.1}$$

where $a = \overline{1,k}$ and $D_t^{\alpha a} = {}_0D_t^{\alpha a}$ (Caputo left derivative [1]).

The set of equivalence classes defined by (2.1) is called the *k-osculator fractional space* at M in x_0 and it will be denoted by $Osc_{x_0}^{\alpha k}(M)$. If the curve $c : I \to M$ is

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given by $x^i = x^i(t)$, $t \in I$, $i = \overline{1,n}$, in the chart U, then the class $[c]^{\alpha k}_{x_0} \in Osc^{\alpha k}_{x_0}(M)$ is given by

$$x^i(t) = x^i + \sum_{a=1}^{k} t^{\alpha a} y^{i(\alpha a)}, \quad t \in (-\varepsilon, \varepsilon),$$
$$y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t) \mid_{t=0}, \tag{2.2}$$

where $i = \overline{1,n}$, $a = \overline{1,k}$ and $x^i = x^i(0)$. The *fractional osculator bundle of order* k is the bundle $(Osc^{\alpha k}(M),\ M)$ where $Osc^{\alpha k}(M) = \bigcup_{x\in M} Osc^{\alpha k}_x(M)$ and $\pi^{\alpha k}$: $Osc^{\alpha k}(M) \to M$ is defined by $\pi^{\alpha k}([c]^{\alpha k}_x) = x$, $\forall [c]^{\alpha k}_x \in Osc^{\alpha k}(M)$.

For $f \in \mathcal{F}(U)$, the fractional derivative of order α, $\alpha \in (0,1)$, with respect to the variable x^i is defined by

$$D^\alpha_{x^i} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{a^i}^{x^i} \frac{\partial f(x^1, ..., x^{i-1}, s, x^{i+1}, ..., x^n)}{\partial x^i} \frac{1}{(x^i - s)^\alpha} ds,$$

where $x \in U_{ab} = \{x \in U,\ a^i \le x^i \le b^i,\ i = \overline{1,n}\}$, x^i, $i = \overline{1,n}$, are the coordinate functions on U and $\left\{\frac{\partial}{\partial x^i}\right\}$, $i = \overline{1,n}$, is the canonical base of the vector fields on U.

Using the fractional exterior differential [1] $d^\alpha : \mathcal{F}(U) \to \mathcal{D}^1(U)$ given by

$$d^\alpha = d(x^j)^\alpha D^\alpha_{x^j}, \tag{2.3}$$

where $(x^j)^\alpha \in \mathcal{F}(U)$ and $\mathcal{D}^1(U)$ is the module of the differential 1-forms on U with the canonical base $\{dx^i\}$, $i = \overline{1,n}$, we get [1]

Proposition 1. (a) *With respect to the transformation of coordinates* $\bar{x}^i = \bar{x}^i(x^1, ..., x^n)$, $i = \overline{1,n}$, $\det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \ne 0$, *corresponding to the charts* U, U', $U \cap U' \ne \varnothing$, *we have the relations*

$$d(\bar{x}^i)^\alpha = \overset{\alpha}{J}{}^i_j(\bar{x}, x) d(x^j)^\alpha, \quad D^\alpha_{\bar{x}^i} = \overset{\alpha}{J}{}^j_i(x, \bar{x}) D^\alpha_{x^j}, \tag{2.4}$$

where $\overset{\alpha}{J}{}^i_j(x, \bar{x}) = \frac{1}{\Gamma(1+\alpha)} D^\alpha_{\bar{x}^j}(x^j)^\alpha$.

(b) *The transformation of coordinates on* $(\pi^{\alpha k})^{-1}(U \cap U') \subset Osc^{\alpha k}(M)$ *are given by*

$$\frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \bar{y}^{i(\alpha a)} = \Gamma(1+\alpha) \overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)}, x) y^{j(\alpha)}$$
$$+ \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \sum_{b=1}^{a-1} \overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)}, y^{\alpha b}) y^{j((b+1)\alpha)} + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha k)}, \tag{2.5}$$

where $a = \overline{2,k}$ *and* $(x^i, y^{i(\alpha)}, ..., y^{i(\alpha k)}) \in (\pi^{\alpha k})^{-1}(U)$.

3 The fractional jet bundle on a differentiable manifold. Geometrical objects

3.1 The fractional jet bundle

The fractional jet bundle of order α on the manifold M is the space $J^\alpha(\mathbb{R},\ M) = \mathbb{R}\times Osc^\alpha(M)$, where $Osc^\alpha(M)$ is the fractional osculator bundle. The triplet $(J^\alpha(\mathbb{R},\ M),\ \pi_0^\alpha,\ M)$ has a structure of differentiable fiber bundle, where $\pi_0^\alpha : J^\alpha(\mathbb{R},\ M) \to M$ is the canonical projection. If (x^i), $i=\overline{1,n}$, are the coordinate functions on the chart $U\subset M$, then the coordinate functions on $(\pi_0^\alpha)^{-1}(U)$ are given by $(t, x^i, y^{i(\alpha)})$ where $y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha x^i(t)\,|_{t=0}\,,\ i=\overline{1,n}$.

From the properties of the Caputo fractional derivative and from Proposition 1, it follows:

Proposition 2. *Let us consider the functions* $(t)^\alpha$, $(x^i)^\alpha$, $(y^{i(\alpha)})^\alpha$ *of* $\mathcal{F}((\pi_0^\alpha)^{-1}(U))$, *the 1-forms* $\frac{1}{\Gamma(1+\alpha)}d(t)^\alpha$, $\frac{1}{\Gamma(1+\alpha)}d(x^i)^\alpha$, $\frac{1}{\Gamma(1+\alpha)}d((y^{i(\alpha)})^\alpha)$ *of* $\mathcal{D}^1((\pi_0^\alpha)^{-1}(U))$ *and the operators* D_t^α, $D_{x^i}^\alpha$, $D_{y^{i(\alpha)}}^\alpha$, $i=\overline{1,n}$. *The following relations hold:*

$$D_t^\alpha\left(\frac{1}{\Gamma(1+\alpha)}t^\alpha\right)=1,\ D_{x^i}^\alpha\left(\frac{1}{\Gamma(1+\alpha)}(x^j)^\alpha\right)=\delta_i^j$$
$$D_{y^{i(\alpha)}}^\alpha\left(\frac{1}{\Gamma(1+\alpha)}y^{j(\alpha)}\right)=\delta_i^j,\ \frac{1}{\Gamma(1+\alpha)}d(t^\alpha)(D_t^\alpha)=1,$$
$$\frac{1}{\Gamma(1+\alpha)}d(x^i)^\alpha(D_{x^j}^\alpha)=\delta_j^i,\ \frac{1}{\Gamma(1+\alpha)}d(y^{i(\alpha)})^\alpha(D_{y^{j(\alpha)}}^\alpha)=\delta_j^i.$$

The module generated by the operators D_t^α, $D_{x^i}^\alpha$, $D_{y^{i(\alpha)}}^\alpha$, $i=\overline{1,n}$, will be denoted by $\mathfrak{X}^\alpha((\pi_0^\alpha)^{-1}(U))$. For $\alpha\to 1$ this module represents the module of the vector fields defined on $\pi_0^{-1}(U)$.

Let us consider two charts U, U' on M with $U\cap U'\neq\varnothing$, $(\pi_0^\alpha)^{-1}(U)$, $(\pi_0^\alpha)^{-1}(U')\subset J^\alpha(\mathbb{R},\ M)$ the corresponding charts on $J^\alpha(\mathbb{R},\ M)$ and the coordinate functions (x^i), $(\bar{x}^i)$, respectively, $(t, x^i, y^{i(\alpha)})$, $(t,\bar{x}^i,\bar{y}^{i(\alpha)})$. From Proposition 1, we obtain the transformations of coordinates

$$\begin{aligned}\bar{x}^i &= \bar{x}^i(x^1,...,x^n)\\ \bar{y}^{i(\alpha)} &= \overset{\alpha}{J}{}_j^i(x,\bar{x})y^{j(\alpha)}.\end{aligned}\tag{3.1}$$

3.2 Geometrical objects on $J^\alpha(\mathbb{R},\ M)$

On the manifold $J^\alpha(\mathbb{R},\ M)$, the following canonical structures may be defined:

$$\begin{aligned}&\overset{\alpha}{\theta}_1 = d(t^\alpha)\otimes(D_t^\alpha + y^{i(\alpha)}D_{x^i}^\alpha),\ \overset{\alpha}{\theta}_2=\overset{\alpha}{\theta^i}\otimes D_{x^i}^\alpha,\\ &\overset{\alpha}{\theta^i}=\frac{1}{\Gamma(1+\alpha)}(d(x^i)^\alpha - y^{i(\alpha)}d(t^\alpha))\\ &\overset{\alpha}{S}=\overset{\alpha}{\theta^i}\otimes D_{y^{i(\alpha)}}^\alpha,\ \overset{\alpha}{V}_i = D_{y^{i(\alpha)}}^\alpha.\end{aligned}\tag{3.2}$$

From (3.1) and from Proposition 1, it follows that the structures (3.2) have geometrical character.

The vector field $\overset{\alpha}{\Gamma} \in \mathfrak{X}^\alpha((\pi_0^\alpha)^{-1}(U))$ is called a *fractional vector field* (*FVF*) iff

$$d(t)^\alpha(\overset{\alpha}{\Gamma}) = 1, \;\; \overset{\alpha}{\theta}{}^i(\overset{\alpha}{\Gamma}) = 0, \;\; i = \overline{1,n}. \tag{3.3}$$

In local coordinates, (*FVF*) is given by

$$\overset{\alpha}{\Gamma} = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + F^i D_{y^{i(\alpha)}}^\alpha, \tag{3.4}$$

where $F^i \in C^\infty((\pi_0^\alpha)^{-1}(U))$, $i = \overline{1,n}$. The integral curves of (*FVF*) are the solutions of the fractional differential equations (*FDE*)

$$D_t^{2\alpha} x^i(t) = F^i(t, x(t), D_t^\alpha x(t)), \;\; i = \overline{1,n}. \tag{3.5}$$

The system (3.5), with given initial conditions, admits solution [2].

Let $L \in C^\infty(J^\alpha(\mathbb{R}, \; M))$ be a *fractional Lagrange function.* By definition, the *Cartan fractional* 1-*form* is the 1-form $\overset{\alpha}{\theta}_L$ given by

$$\overset{\alpha}{\theta}_L = L d(t)^\alpha + \overset{\alpha}{S}(L). \tag{3.6}$$

The *Cartan* 2-*form* $\overset{\alpha}{\omega}_L$ is defined by

$$\overset{\alpha}{\omega}_L = d^\alpha \, \overset{\alpha}{\theta}_L, \tag{3.7}$$

where d^α is the fractional exterior differential

$$d^\alpha = d(t)^\alpha D_t^\alpha + d(x^i)^\alpha D_{x^i}^\alpha + d(y^{i(\alpha)})^\alpha D_{y^{i(\alpha)}}^\alpha. \tag{3.8}$$

In the chart $(\pi_0^\alpha)^{-1}(U)$, $\overset{\alpha}{\theta}_L$ and $\overset{\alpha}{\omega}_L$ are given by

$$\begin{aligned} \overset{\alpha}{\theta}_L &= (L - \frac{1}{\Gamma(1+\alpha)} y^{i(\alpha)} D_{y^{i(\alpha)}}^\alpha(L)) d(t)^\alpha + \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(L) d(x^i)^\alpha \\ \overset{\alpha}{\omega}_L &= A_i d(t)^\alpha \wedge d(x^i)^\alpha + B_i d(t)^\alpha \wedge d(y^{i(\alpha)})^\alpha + A_{ij} d(x^i)^\alpha \wedge d(x^j)^\alpha + \\ &\qquad B_{ij} d(x^i)^\alpha \wedge d(y^{j(\alpha)})^\alpha, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} A_i &= \frac{1}{\Gamma(1+\alpha)} D_t^\alpha D_{y^{i(\alpha)}}^\alpha(L) + \frac{1}{\Gamma(1+\alpha)} y^{j(\alpha)} D_{x^i}^\alpha D_{y^{j(\alpha)}}^\alpha(L) - D_{x^i}^\alpha(L) \\ B_i &= \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(y^{j(\alpha)} D_{y^{j(\alpha)}}^\alpha(L)) \\ A_{ij} &= D_{x^i}^\alpha D_{y^{i(\alpha)}}^\alpha(L), \;\; B_{ij} = -D_{y^{j(\alpha)}}^\alpha D_{y^{i(\alpha)}}^\alpha(L). \end{aligned}$$

Proposition 3. *If the fractional Lagrange function is regular i.e.,* $\det\left(\overset{\alpha}{g}_{ij}\right) \neq 0$, $\overset{\alpha}{g}_{ij} = D^\alpha_{y^{i(\alpha)}} D^\alpha_{y^{j(\alpha)}} L$, *then there is a fractional vector field (*FVF*)* $\overset{\alpha}{\Gamma}_L$ *such that* $i_{\overset{\alpha}{\Gamma}_L}(\overset{\alpha}{\omega}_L) = 0$. *In the chart* $(\pi^\alpha_0)^{-1}(U)$, $\overset{\alpha}{\Gamma}_L$ *is given by*

$$\overset{\alpha}{\Gamma}_L = D^\alpha_t + y^{i(\alpha)} D^\alpha_{x^i} + \overset{\alpha}{M}{}^i \, D^\alpha_{y^{i(\alpha)}}, \tag{3.10}$$

where

$$\begin{gathered} \overset{\alpha}{M}{}^i = \overset{\alpha}{g}{}^{ik}(D^\alpha_{x^k} L - d^\alpha_t(D^\alpha_{y^{k(\alpha)}} L)) \\ \left(\overset{\alpha}{g}{}^{ik}\right) = \left(\overset{\alpha}{g}_{ik}\right)^{-1}, \quad d^\alpha_t = D^\alpha_t + y^{i(\alpha)} D^\alpha_{x^i}. \end{gathered} \tag{3.11}$$

3.3 The fractional Euler-Lagrange equations

Let $c : t \in [0,1] \to (x^i(t)) \in M$ be a parameterized curve such that $\mathrm{Im} c \subset U \subset M$. The extension of the curve c to $J^\alpha(\mathbb{R},\ M)$ is the curve $c^\alpha : t \in [0,1] \to (t, x^i(t), y^{i(\alpha)}(t)) \in J^\alpha(\mathbb{R},\ M)$ with $\mathrm{Im} c^\alpha \subset (\pi^\alpha_0)^{-1}(U) \subset J^\alpha(\mathbb{R},\ M)$. Let $L \in C^\infty(J^\alpha(\mathbb{R},\ M))$ be a fractional Lagrange function. The action of L along the curve c^α is

$$A(c^\alpha) = \int_0^1 L(t, x(t), y^{(\alpha)}(t))dt. \tag{3.12}$$

Let $c_\varepsilon : t \in [0,1] \to (x^i(t,\varepsilon)) \in M$ be a family of curves, with ε sufficiently small in absolute value so that $\mathrm{Im} c_\varepsilon \subset M$, $c_0(t) = c(t)$, $D^\alpha_\varepsilon c_\varepsilon(0) = D^\alpha_\varepsilon c_\varepsilon(1) = 0$. The action of L on the curves c^α_ε is

$$A(c^\alpha_\varepsilon) = \int_0^1 L(t, x(t,\varepsilon), y^{(\alpha)}(t,\varepsilon))dt, \tag{3.13}$$

where $y^{i(\alpha)}(t,\varepsilon) = \frac{1}{\Gamma(1+\alpha)} D^\alpha_t x^i(t,\varepsilon)$, $i = \overline{1,n}$. The action (3.13) has a fractional extremal value if

$$D^\alpha_\varepsilon A(c^\alpha_\varepsilon) \mid_{\varepsilon=0} = 0. \tag{3.14}$$

Using the properties of the Caputo fractional derivative, it results

Proposition 4. (a) *If the action* (3.13) *reaches a fractional extremal value then a necessary condition is that* $c(t)$ *satisfies* the fractional Euler-Lagrange equations

$$\begin{gathered} D^\alpha_{x^i} L - d^{2\alpha}_t(D^\alpha_{y^{i(\alpha)}} L) = 0, \quad i = \overline{1,n} \\ d^{2\alpha}_t = D^\alpha_t + y^{i(\alpha)} D^\alpha_{x^i} + y^{i(2\alpha)} D^\alpha_{y^{i(\alpha)}}. \end{gathered} \tag{3.15}$$

(b) *If the fractional Lagrange function is nondegenerated, then the equations* (3.15) *are the fractional differential equations associated to the fractional vector field* $\overset{\alpha}{\Gamma}_L$ *given by* (3.10).

(c) *If the fractional Lagrange function is nondegenerated, then the system* (3.15) *may be written in the form of* the fractional Hamilton equations

$$D_t^\alpha p_i^{(\alpha)} = -D_{x^i}^\alpha H, \;\; D_t^\alpha x^i = D_{p_i^{(\alpha)}}^\alpha H, \tag{3.16}$$

where

$$\begin{aligned} H &= p_i^{(\alpha)} D_t^\alpha x^i - L(t, x(t), y^{(\alpha)}(t)) \\ p_i^{(\alpha)} &= D_{y^{i(\alpha)}}^\alpha L(t, x, y^{(\alpha)}), \;\; i = \overline{1, n}. \end{aligned} \tag{3.17}$$

(d) *If for* $f, h : J^1(\mathbb{R},\ M)^* \to \mathbb{R}$ the fractional Poisson bracket *is defined by*

$$\{f, h\}^\alpha = D_{p_i^{(\alpha)}}^\alpha f D_{x^i}^\alpha g - D_{x^i}^\alpha f D_{p_i^{(\alpha)}}^\alpha g, \tag{3.18}$$

where the local coordinates on $J^1(\mathbb{R},\ M)^*$ *are* $(x, p^{(\alpha)})$, *then*

$$\{H, p_i^{(\alpha)}\}^\alpha = D_t^\alpha p_i^{(\alpha)}, \quad \{H, x^i\}^\alpha = D_t^\alpha x^i, \;\; i = \overline{1, n}. \tag{3.19}$$

4 Economic models described by fractional differential equations

4.1 The fractional model of Liviatan-Samuelson

Let us consider the fractional Lagrange function $L \in \mathcal{F}(J^\alpha(\mathbb{R},\ M))$ given by

$$L(t, x, y^{(\alpha)}) = L_1(x, y^{(\alpha)}) E_\alpha(-\rho t^\alpha), \tag{4.1}$$

where E_α is the Mittag-Leffler function, $E_\alpha(t) = \sum_{k=0}^\infty \frac{t^{\alpha k}}{\Gamma(1+\alpha k)}$ and $\rho > 0$ is the discount rate. Using the relation $D_t^\alpha E_\alpha(-\rho t^\alpha) = -\rho E_\alpha(-\rho t^\alpha)$ and Proposition 4 we obtain

Proposition 5. (a) *The fractional Euler-Lagrange equations* (3.15) *for* (4.1) *are*

$$\begin{aligned} y^{j(2\alpha)} D_{y^{i(\alpha)}}^\alpha D_{y^{j(\alpha)}}^\alpha L_1 + y^{j(\alpha)} D_{x^j}^\alpha D_{y^{i(\alpha)}}^\alpha L_1 - \rho D_{y^{i(\alpha)}}^\alpha L_1 \\ - D_{x^i}^\alpha L_1 = 0, \;\; i = \overline{1, n}. \end{aligned} \tag{4.2}$$

(b) *If* $L_1 \in \mathcal{F}(J^\alpha(\mathbb{R},\ \mathbb{R}))$ *is of the form*

$$L_1(x, y^{(\alpha)}) = U(g(x) - y^{(\alpha)}), \tag{4.3}$$

where U *is the utility (welfare) function and* $c = g(x) - y^{(\alpha)}$ *is the consumption function, then the fractional Euler-Lagrange equation is*

$$\begin{aligned} \Gamma(1+\alpha)^2 U''(g(x) - y^{(\alpha)}) y^{(2\alpha)} - \Gamma(1+\alpha) D_x^\alpha g(x) U''(g(x) - y^{(\alpha)}) y^{(\alpha)} \\ + \rho U'(g(x) - y^{(\alpha)}) \Gamma(1+\alpha) - U'(g(x) - y^{(\alpha)}) D_x^\alpha g(x) = 0. \end{aligned} \tag{4.4}$$

For $U(x) = e^{-\rho_1 x}$ and $g(x) = x^\gamma$, from (4.4) we get

$$y^{(2\alpha)} = \frac{x^{\gamma-\alpha}\Gamma(1+\gamma)}{\rho_1\Gamma(1+\gamma-\alpha)\Gamma(1+\alpha)}y^{(\alpha)} - \frac{\rho}{\rho_1\Gamma(1+\alpha)} - \frac{x^{\gamma-\alpha}\Gamma(1+\gamma)}{\rho_1\Gamma(1+\gamma-\alpha)\Gamma(1+\alpha)^2}. \tag{4.5}$$

Proposition 6. *If $L_1 \in \mathcal{F}(J^\alpha(\mathbb{R},\ \mathbb{R}))$ is given by*

$$L_1(x, y^{(\alpha)}) = -a_1(y^{(\alpha)})^{2\alpha} - a_2(y^{(\alpha)})^\alpha x^\alpha - a_3 x^{2\alpha},\ \ a_1, a_2, a_3 \in \mathbb{R}, \tag{4.6}$$

then the fractional Euler-Lagrange equation is

$$\begin{aligned} &a_1\Gamma(1+\alpha)\Gamma(1+2\alpha)y^{(2\alpha)} - (a_2\Gamma(1+\alpha)^2 + \rho a_1\Gamma(1+2\alpha))(y^{(\alpha)})^\alpha \\ &\quad + a_2\Gamma(1+\alpha)^3 y^{(\alpha)} - (a_3\Gamma(1+2\alpha) + \rho a_2\Gamma(1+\alpha)^2)x^\alpha = 0. \end{aligned} \tag{4.7}$$

If $a_1 = a_3 = \frac{1}{2}$, $a_2 = a$ and $\alpha \to 1$, then the function L_1 and the corresponding Euler-Lagrange equation for $L = e^{-\rho t}L_1$ become, respectively

$$\begin{aligned} L_1(x, \dot{x}) &= -\frac{1}{2}\dot{x}^2 - ax\dot{x} - \frac{1}{2}x^2 \\ -\ddot{x} + (\rho a + 1)x + \rho\dot{x} &= 0, \end{aligned} \tag{4.8}$$

and we obtain the classic model of Samuelson [3].

4.2 Fractional economic models with restrictions

Let us consider the Lagrange function $L \in C^\infty(J^\alpha(\mathbb{R},\ M))$ and the function $F \in C^\infty(J^\alpha(\mathbb{R},\ M))$. The fractional Euler-Lagrange equations of L on the restriction $F(x, y^{(\alpha)}) = 0$, $(x, y^{(\alpha)}) \in (\pi_0^\alpha)^{-1}(U)$, are given by the fractional Euler-Lagrange equations of the fractional Lagrange function

$$L_2(t, \lambda, x, y^{(\alpha)}) = L(t, x, y^{(\alpha)}) + \lambda F(x, y^{(\alpha)}), \tag{4.9}$$

where $\lambda(t)$ is a Lagrange multiplier. From Proposition 4, we obtain

Proposition 7. (a) *The fractional Euler-Lagrange equations of* (4.9) *are:*

$$\begin{aligned} &D^\alpha_{x^i}L + \lambda D^\alpha_{x^i}F - d^{2\alpha}_t(D^\alpha_{y^{i(\alpha)}}L) - \lambda y^{j(\alpha)}D^\alpha_{x^j}(D^\alpha_{y^{i(\alpha)}}F) \\ &-\lambda y^{j(\alpha)}D^\alpha_{y^{j(\alpha)}}(D^\alpha_{y^{i(\alpha)}}F) - D^\alpha_t\lambda D^\alpha_{y^{i(\alpha)}}F = 0,\ \ i = \overline{1,n}. \end{aligned} \tag{4.10}$$

(b) *If the Lagrange function is given by* (4.1) *then the fractional Euler-Lagrange equations* (4.10) *become*

$$\begin{aligned} &E_\alpha(-\rho t^\alpha)(D^\alpha_{x^i}L_1 + \rho D^\alpha_{y^{i(\alpha)}}L_1 - y^{j(\alpha)}D^\alpha_{x^j}(D^\alpha_{y^{i(\alpha)}}L_1) \\ &-y^{j(2\alpha)}D^\alpha_{y^{j(\alpha)}}(D^\alpha_{y^{i(\alpha)}}L_1)) + \lambda(D^\alpha_{x^i}F - y^{j(\alpha)}D^\alpha_{x^j}(D^\alpha_{y^{i(\alpha)}}F) \\ &-y^{j(2\alpha)}D^\alpha_{y^{j(\alpha)}}(D^\alpha_{y^{i(\alpha)}}F)) - D^\alpha_t\lambda D^\alpha_{y^{i(\alpha)}}F = 0, \end{aligned} \tag{4.11}$$

for $i = \overline{1,n}$.

The fractional model of investments with restriction is described by the function $L_1(K, I, N)$ where $K(t) = x^1(t)$, $I(t) = x^2(t)$, $N(t) = x^3(t)$ represent the capital stock, the investment and the labor, respectively. The restriction is given by $F(K^{(\alpha)}, K, I, N) = \phi(K, I, N) - K^{(\alpha)} = 0$, $K^{(\alpha)} = D_t^\alpha K$. From (4.11) we obtain the fractional Euler-Lagrange equations

$$\begin{aligned} E_\alpha(-\rho t^\alpha) D_K^\alpha L_1 + \lambda D_K^\alpha \phi &= -D_t^\alpha \lambda \\ E_\alpha(-\rho t^\alpha) D_I^\alpha L_1 + \lambda D_K^\alpha \phi &= 0 \\ E_\alpha(-\rho t^\alpha) D_N^\alpha L_1 + \lambda D_N^\alpha \phi &= 0. \end{aligned} \tag{4.12}$$

If L_1 and ϕ satisfy the relations

$$\begin{aligned} K^\alpha D_K^\alpha L_1 + I^\alpha D_I^\alpha L_1 + N^\alpha D_N^\alpha L_1 &= \frac{1}{\Gamma(1+\alpha)} r L_1, \quad r \in \mathbb{R} \\ K^\alpha D_K^\alpha \phi + I^\alpha D_I^\alpha \phi + N^\alpha D_N^\alpha \phi &= \frac{1}{\Gamma(1+\alpha)} \phi, \end{aligned} \tag{4.13}$$

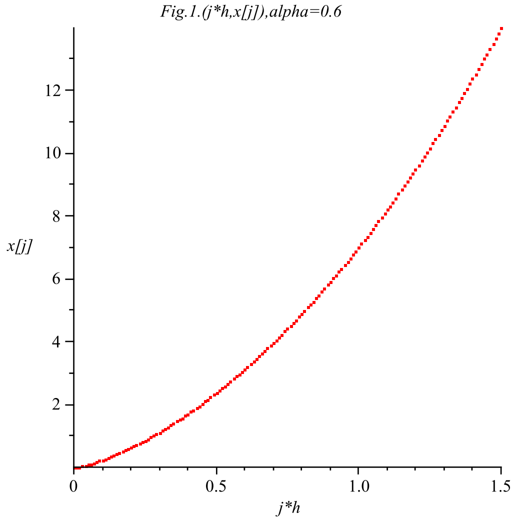
from (4.12) we get

$$r E_\alpha(-\rho t^\alpha) L_1 = -\lambda K^{(\alpha)} - \Gamma(1+\alpha) K^\alpha \lambda^{(\alpha)}. \tag{4.14}$$

For $\alpha \to 1$ the classic model of investments [3] is obtained.

The figure represents a numerical simulation of the equation (4.5), for $\rho = 0.1$, $\rho_1 = 0.2$, $\gamma = 0.5$, and $\alpha = 0.6$, using Adams-Bashforth-Moulton algorithm (see [2]), in Maple 11.

*Fig.1.(j*h,x[j]),alpha=0.6*



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Author's address:

Mihai Boleantu
Dept. of Economy, West University of Timisoara,
Pestalozzi 16, 300115 Timisoara, Romania.
E-mail: mihai.boleantu@fse.uvt.ro