Financial dynamical systems

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Abstract. We show that a correct definition of a state space allows to define dynamical systems representing the financial evolutions of standard economics. This goal is completely explained by means of several new results on a certain kind of actions defined in the paper. Moreover, our approach to the problems of financial evolutions not only reduces the heuristic methods of financial mathematics to a well founded and developed mathematical theory - the theory of Dynamical Systems - but, moreover, it induces a more deep and prolific understanding of the nature of economic evolutions themselves. In the paper, the general definitions of reversible and non-reversible dynamical systems (as algebraic structures) are used.

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1 Introduction: evolutions in financial mathematics

In financial mathematics the evolutions of a capital (or more precisely, of a financial event) is ruled by *capitalization factors*. A capitalization factor is a somehow heuristic concept and who manages it has to acquire a certain "practical language and spirit" to formulate the problems (in a way more or less univocal) and to obtain the correct results. Moreover, the resort to a capitalization factor is a very particular manner to produce an evolution, it is typical of financial calculus and it is not used in other scientific fields.

Firstly we recall the basic theoretic substratum on which we move. We want to conduct a rigorous discussion, thus we must establish the concepts one time for all; what follows is our choice of formalization (some definitions being already present in the literature, in a more or less formalized way).

Definition 1 (of financial event). The plane of financial events is the usual cartesian plane $\mathbb{R}^2$. It is interpreted as the cartesian product of a time-axis and of a capital-axis. Every pair (t, C) belonging to this plane is called a financial event of time t and capital C.

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Remark 1. The first and, indubitably, one of the principal aims of financial mathematics is to determine how a financial event (t_0, C_0) evolves in time. In order to do this it is necessary to specify the laws of evolutions in some way.

Nowadays, the standard manner to define the laws of evolution of a financial event is to give a *capitalization factor*.

Remark 2. A capitalization factor f is a continuous function from the non-negative real line to the real line, enjoying not universally established (and sometimes disputable) properties. Two properties are always present (although not in the following clear way) and we shall use solely them: the function f must be positive and such that $f(0) = 1$.

Let us formalize the concept of capitalization factor.

Definition 2 (of capitalization factor). A capitalization factor on the financial events plane is a continuous positive function from the non-negative real line into the real line such that $f(0) = 1$.

By means of capitalization factors we can rigorously formalize the first evolution concept.

Definition 3 (of capital-evolution of an event). The capital-evolution of the event (t_0, C_0) with respect to a capitalization factor f is the function M from the time interval $[t_0, +\infty[$ to the real line defined, for every real $t \geq t_0$, by $M(t) = C_0 f(t - t_0)$.

Remark 3. Regrettably, in practice, the function M is almost always said to be the evolution of the capital C_0, and not of the event (t_0, C_0). This draws away from the real state of the things.

Definition 4 (of value of an event at a time). Let t be a real number and let $E_0 = (t_0, C_0)$ be a financial event with time less or equal to t. The real value $C_0 f(t - t_0)$, denoted by $V_t(E_0)$, is called the value of the event E_0at time t. The real functional V_t, defined on the half-plane of all events $E_0 = (t_0, C_0)$ with time $t_0 \leq t$, associating to each event E_0 its value in t, is called the value functional at t induced by the capitalization factor f.

2 The setting of the problem

We are tempted to consider the multiplication by a capitalization factor as an evolution operator of a dynamical system. More precisely, we are tempted to define the operator e from the half-plane of $\mathbb{R}^2$ characterized by the inequality $\mathrm{pr}_1 \geq \mathrm{pr}_2$ into the function-space $\mathcal{F}(\mathbb{R}, \mathbb{R})$, set of all the functions from the real line into itself, defined by

$$e(t, t_0)(C_0) = C_0 f(t - t_0).$$

But to represent an evolution operator (see [1]) this function e must to satisfy two basic properties: 1) for every time t, $e(t, t) = \mathrm{id}_{\mathbb{R}}$; 2) for every triple of times (t_0, t_1, t_2) with $t_0 \leq t_1 \leq t_2$, $e(t_2, t_1) \circ e(t_1, t_0) = e(t_2, t_0)$. The first one is obviously satisfied; on the contrary the second one (the so called *Chapman-Kolmogorov law*), in general, is very far to be true. Indeed, the only evolutions of the type proposed, enjoying the Chapman-Kolmogorov law, are those generated by exponential capitalization factors, more precisely, factors defined by $f(h) = e^{wh}$, where w is the so called *instant* *interest intensity*; this is a simple consequence of the fact that the only continuous homomorphisms of the additive group of real numbers into the multiplicative group of the non-vanishing real numbers are the exponential functions.

Let us see with an example that there are capitalization factors whose operators e are not enjoying the Chapman-Kolmogorov law.

Example 1. Let i be a real non-negative number. The capital-evolution, in simple capitalization, of the event $E_0 = (t_0, C_0)$ is the function M from the real interval $[t_0, +\infty[$ to the real axis defined by

$$M(t) = C_0(1 + i(t - t_0)),$$

for every real $t \geq t_0$. It is clear that, if we consider two instants of time t_1 and t_2, the value of the event E_0 in t_2 is different from the value of the event $(t_1, M(t_1))$ in t_2, in fact

$$V_{t_2}(t_1, M(t_1)) = M(t_1)f(t_2 - t_1) \neq C_0(1 + i(t_2 - t_0)),$$

and this is equivalent to

$$e(t_2, t_1) \circ e(t_1, t_0)(C_0) \neq e(t_2, t_0)(C_0).$$

Problem 1. The preceding example proves that the state of the evolution of a financial event E_0 at a time t *cannot be* the simple capital $M(t)$: *we need additional information to define the state-space of the evolution.* This is the problem we shall solve in the following.

3 Preliminaries and terminology on actions

In this section we recall some classic concepts of algebra that have great importance in the foundation of dynamical systems.

We call *action* of a set X on a set Y every function $f : X \to \mathcal{F}(Y, Y)$, that is every application of the set X into the set of functions from Y to Y. For every x in X, the function $f(x) : Y \to Y : y \mapsto f(x)(y)$, is called the action of the element x on Y by means of f. To each action f of a set X on a set Y is naturally associated an external binary operation on Y with domain of operators X, defined, for every pair (x, y) in $X \times Y$, by $x.y = f(x)(y)$. Viceversa, if . is an external binary operation on a set Y with domain of operators X, it is possible to define an action of X on Y, putting, for each x in X and each y in Y, $f(x)(y) = x.y$. The natural correspondence from the set of actions of X on Y − we shall denote it by $\mathcal{A}(X, Y)$ − into the set of external binary operations on Y, with operators domain X, associating to each action the relative external binary operation, is bijective. An action f of a set X on a set Y is said faithful if it is injective. We say that an action f of a set X on a set Y is strongly faithful if for each distinct x and x' belonging to X the functions $f(x)$ and $f(x')$ have disjoint graphs. Let f be an action of a set X on a set Y; the family $(f(x))_{x\in X}$ is said the family of actions of the elements of X associated to f. Viceversa, given a family of functions from Y to Y indexed by X, say $(f_x)_{x\in X}$, there is only an action admitting the family as associated family of actions. Let $(X, .)$ be an algebraic structure with an internal associative operation . (eventually not everywhere defined) and let f be an action of the set X on a set Y. We shall say that the action f is an action of the structure $(X, .)$ if, for every x and x' in X and for each y in Y, we have $f(x.x') = f(x) \circ f(x')$, that is, if f is a homomorphism of the structure $(X, .)$ in the structure given by the set of function $\mathcal{F}(Y, Y)$ endowed with the composition law of functions. In other form, if we denote the action of $x \in X$ on $y \in Y$ (by means of f) by the multiplicative notation xy, the preceding relation can be written in the expressive form $(x.x')y = x(x'y)$. In other terms, f is an action of the structure $(X, .)$ on Y if the corresponding external binary operation is associative with respect to the internal operation of the associative magma $(X, .)$. Let us consider an associative unitary magma $(X, .)$ and an action f of X on a set Y we say that f is an action of the associative unitary magma on Y if it is an action of the associative magma on Y and we have $f(1_X) = \mathrm{id}_Y$.

4 Reversible dynamical systems

Now we propose the definition of reversible dynamical system that we shall use in the paper. To give this definition we introduce a particular operation on the plane $\mathbb{R}^2$, when it is interpreted as the product of a time-axis by itself.

Definition 5 (of dynamical composition of pairs of times). We define dynamical composition in the plane $\mathbb{R}^2$ the not-everywhere defined internal binary operation . given by $(t_2, t_1).(t'_1, t_0) = (t_2, t_0)$, for every choice of pairs (t_2, t_1) and (t'_1, t_0) such that $t_1 = t'_1$.

Theorem 1. *The plane $\mathbb{R}^2$ endowed with the dynamical composition, i.e, the algebraic structure $(\mathbb{R}^2, .)$, is a groupoid.*

Proof. For the definition of grupoid see [2] and [3], the proof is a straightforword computation. □

Definition 6 (of reversible dynamical system). Let X be a non void set. A reversible dynamical system on X is an algebraic structure (X, θ) where:
1) θ is an external binary operation on X with operators-domain $\mathbb{R}^2$;
2) the external law θ is associative with respect to the dynamical law of the plane;
3) each diagonal element of $\mathbb{R}^2$ acts as the identity operator on the set X.

Remark 4. In other terms, the action associated with the external law θ is an action of the grupoid $(\mathbb{R}^2, .)$ on the set X.

An equivalent way to define a dynamical system is to give an *evolution operator.*

Definition 7 (of reversible evolution operator). Let X be a non void set, a function G from the plane $\mathbb{R}^2$ to the set of all functions from X to X, denoted by $\mathcal{F}(X, X)$, is called a reversible evolution operator if
1) for every real t, $G(t, t) = \mathrm{id}_X$;
2) for every triple of reals (t_0, t_1, t_2), $G(t_2, t_1) \circ G(t_1, t_0) = G(t_2, t_0)$.

Remark 5. Note that from the preceding axioms (the latter is sometimes called the Chapman-Kolmogorov law) it follows that each function $G(t, t')$ is invertible and its inverse is $G(t', t)$.

5 Non-reversible dynamical systems

We are chiefly interested in non-reversible dynamical systems. To introduce non-reversible dynamical systems we must consider the graph of the natural majoring order of the real line, that we shall denote by $\mathrm{gr}(\geq)$, that is the half-plane $\{(t', t) \in \mathbb{R}^2 : t' \geq t\}$.

Definition 8 (of non-reversible evolution operator). Let X be a non void set, a function G from the half-plane $\mathrm{gr}(\geq)$ to the set of all functions from X to X is called a non-reversible evolution operator on the set X if
1) for every real t, $G(t, t) = \mathrm{id}_X$;
2) for every triple of reals (t_0, t_1, t_2), $G(t_2, t_1) \circ G(t_1, t_0) = G(t_2, t_0)$.

Notation 1. Sometimes we shall write $G_{t_0}^{t}$ for $G(t, t_0)$. In such a way the latter law of the preceding definition appears (using the multiplicative notation for the composition) as $G_{t_1}^{t_2} G_{t_0}^{t_1} = G_{t_0}^{t_2}$.

The non-reversible evolution operators can be viewed as actions of a semi-grupoid on a set X. We propose indeed the following definition of semi-grupoid.

Definition 9 (of semi-grupoid). An algebraic structure endowed with a not necessarily everywhere defined internal binary operation $(X, .)$ is said to be a semi-grupoid if it satisfies the following axioms:
(1) for every triple (x, y, z) of elements of X, for every x, y, z in X such that the compositions yz and $x(yz)$ or the compositions xy and $(xy)z$ are defined so are the other two and we have $x(yz) = (xy)z$;
(2_r) if the compositions xy and $x'y$ are defined and equal then $x = x'$;
(2_l) if the compositions yx and yx' are defined and equal then $x = x'$;
(3) for all x there are e_x and e'_x such that $e_x x = x e'_x = x$;
(4) for all indempotents e and e' there exists x such that $e_x x = x e'_x = x$.

Definition 10 (left and right units). With reference to the above definition, every element e of X such that there is an other element x for which $ex = x$ (resp. $xe = x$) is said a left (right) unit of the semi-grouppoid (by abuse of language).

Theorem 2. *In a semi-grupoid it is equivalent for an element e to be a right unit a left unit or an idempotent element.*

Proof. Each left unit of the semigrupoid is idempotent; indeed let e be a left unit of an element x, it follows $(ee)x = e(ex) = ex$, and by right cancellation $ee = e$ (note that the compositions ex and $e(ex)$ are defined and equal x, hence, by (1) the compositions ee and $(ee)x$ are defined). Analogously, we can proceed for the right units. The property (4) of the preceding definition says that (conversely) each idempotent element is both a rigth and left unit, so the claim is proved. □

Remark 6. If a semi-grupoid $(E, .)$ satisfies also the following: (5) for every x in E there is an x' in E such that $xx' = e_x$; then it is a grupoid in the sense of Bourbaki.

Definition 11 (of action of a semi-grupoid on a set). An action of a semi-grupoid $(X, .)$ on a set Y is an action f of the set X on the set Y such that the corresponding external binary operation is associative with respect to the internal operation of $(X, .)$ and sending every idempotent element into the identity operator on Y.

Theorem 3. *The algebraic structure* $(\mathrm{gr}(\geq), .)$*, where the not everywhere defined operation . is given, for every* $(t_2, t_1), (t_1', t_0)$ *in the half-plane* $\mathrm{gr}(\geq)$*, by* $(t_2, t_1).(t_1', t_0) = (t_2, t_0)$ *if and only if* $t_1' = t_1$ *, is a semi-grupoid.*

Proof. (1). Let (q, r, r', s, s', t) be a (weakly) increasing sequence of times. If the compositions $(t, s').(s, r')$ and $[(t, s').(s, r')].(r, q)$ are defined, then we have $s' = s$ and $(t, s').(s, r') = (t, r')$; so, the composition $(t, r').(r, q)$, since it is equal to the composition $[(t, s').(s, r')].(r, q)$, is defined and equal to (t, q). Consequently, we have $r' = r$, and the composition $(s, r').(r, q)$ is defined and equal to (s, q). Finally, we have

$$\begin{aligned}(t, s').[(s, r').(r, q)] &= (t, s').(s, q) = \\ &= (t, s).(s, q) = \\ &= (t, q).\end{aligned}$$

The symmetric part of (1) is analogous.

(2_l). Let (q, r, s, t) and (q', r', s, t) be two (weakly) increasing sequences of times. Suppose $(t, s).(r, q) = (t, s).(r', q')$, where the compositions are suppose to be defined; by definition of the operation ., we deduce $r' = s = r$; moreover the equality is equivalent to $(t, q) = (t, q')$, hence $q = q'$ and $(r, q) = (r', q')$. (2_r) is analogous.

(3). For (t, s) in $\mathrm{gr}(\geq)$ we have $(t, s).(s, s) = (t, t).(t, s) = (t, s)$, thus every diagonal element (s, s) is both a right and left unity of the structure.

(4). If the element (t, s) is idempotent, that is, if the composition $(t, s).(t, s)$ is defined and equal to (t, s), we have $t = s$ and $(t, s) = (s, s)$, so (t, s) (being diagonal) is both a right and left unit. We then proved that the structure $(\mathrm{gr}(\geq), .)$ is a semi-grupoid. □

Remark 7. If we consider the entire plane we have that $(t_2, t_1).(t_1, t_2) = (t_2, t_2)$ and hence we have a grupoid.

6 The dynamical system associated with a capitalization

Let us associate with a capitalization factor f from the semi-line $\mathbb{R}_\geq$ (i.e., the interval $[0, +\infty[$) to $\mathbb{R}$ a dynamical system.

Theorem 4. *Let* $f : \mathbb{R}_\geq \to \mathbb{R}$ *be a function never-vanishing and let* G *be the application from the graph* $\mathrm{gr}(\geq)$ *into the space of functions of the half-plane* $\mathbb{R}_\geq \times \mathbb{R}$ *into itself defined by*

$$G(t_1, t_0)(h_0, C_0) = \left(h_0 + (t_1 - t_0), C_0 \frac{f(h_0 + (t_1 - t_0))}{f(h_0)} \right).$$

Then G *is a non-reversible evolution operator.*

Proof. We have obviously the first property of the evolution operators. Moreover, putting $h_1 = t_1 - t_0$, and $h_2 = t_2 - t_1$, we have

$$\begin{aligned} G(t_2,t_1)\circ G(t_1,t_0)(h_0,C_0) &= G(t_2,t_1)\left(h_0+h_1, C_0\frac{f(h_0+h_1)}{f(h_0)}\right) = \\ &= \left(h_0+h_1+h_2, C_0\frac{f(h_0+h_1)}{f(h_0)}\frac{f(h_0+h_1+h_2)}{f(h_0+h_1)}\right) = \\ &= \left(h_0+(t_2-t_0), C_0\frac{f(h_0+(t_2-t_0))}{f(h_0)}\right) = \\ &= G(t_2,t_0)(h_0,C_0), \end{aligned}$$

and so the Chapman-Kolmogorov law is also verified. □

Definition 12 (of non-reversible evolution operator associated to a capitalization factor). Let $f : \mathbb{R}_{\geq} \to \mathbb{R}$ be a capitalization factor and let G be the application from the graph $\mathrm{gr}(\geq)$ into the space $\mathcal{F}(\mathbb{R}_{\geq}\times\mathbb{R}, \mathbb{R}_{\geq}\times\mathbb{R})$ (set of functions of the half-plane $\mathbb{R}_{\geq}\times\mathbb{R}$ into itself) defined by

$$G(t_1,t_0)(h_0,C_0) = \left(h_0+(t_1-t_0), C_0\frac{f(h_0+(t_1-t_0))}{f(h_0)}\right),$$

for every pair of times (t_1,t_0) in $\mathrm{gr}(\geq)$ and for every pair (h_0,C_0) in the half-plane $\mathbb{R}_{\geq}\times\mathbb{R}$. The evolution operator G is called the non-reversible evolution operator associated with the factor f.

Interpretation. So we have a non-reversible evolution operator whose space of states is the half-plane $[0,+\infty[\times\mathbb{R}$. We interpret, from an economic point of view, this half-plane as the product of the set of positive time-lengths times the space of capitals. A pair (h,C) belonging to this state-space is said a *financial evolution state*, h is called the *capitalization length of the state* and C the *capital of the state*.

Remark 8. Note that, for every pair of times (t_1,t_0) in the half-plane $\mathrm{gr}(\geq)$, the application $G(t_1,t_0)$ has two components, the first is the translation on the convex $[0,+\infty[$ of vector $h_1 = t_1 - t_0$, the second is the function

$$(h_0,C_0) \mapsto C_0\frac{f(h_0+h_1)}{f(h_0)},$$

that is the function

$$\mathrm{pr}_2 \cdot \frac{f\circ\tau_{h_1}\circ\mathrm{pr}_1}{f\circ\mathrm{pr}_1}.$$

It is evident that if two pairs (t_1,t_0) and (t_1',t_0') have the same time-length, i.e., if $t_1 - t_0 = t_1' - t_0'$, then the corresponding operators $G(t_1,t_0)$ and $G(t_1',t_0')$ are equals, consequently the action G is not faithful (i.e., injective), nevertheless we can say to this purpose something, as show the following result.

Theorem 5. *Let the application* $a : \mathbb{R}_{\geq} \to \mathcal{F}(\mathbb{R}_{\geq}\times\mathbb{R}, \mathbb{R}_{\geq}\times\mathbb{R})$ *be the action of the semi-line* $\mathbb{R}_{\geq}$ *on the half-plane* $\mathbb{R}_{\geq}\times\mathbb{R}$ *defined by*

$$a(h)(h_0,C_0) = \left(h_0+h, C_0\frac{f(h_0+h)}{f(h_0)}\right),$$

for every pair (h_0, C_0)*. Then, it is also an action of the monoid* $(\mathbb{R}_{\geq}, +)$ *on the set* $\mathbb{R}^2$*; moreover, the action is strongly faithful and free.*

Proof. The fact that the action a is an homomorphism of the monoid $(\mathbb{R}_{\geq}, +)$ in the monoid $(\mathcal{F}(\mathbb{R}_{\geq} \times \mathbb{R}, \mathbb{R}_{\geq} \times \mathbb{R}), \circ)$ can be proved as in the preceding theorem. Concerning the injectivity of a, the first component of the operator $a(h)$ is the translation τ_h and then for every distinct time-lenghts h_1 and h_2 and for every state $s \in \mathbb{R}_{\geq} \times \mathbb{R}$ we have $a(h_1)(s) \neq a(h_2)(s)$. Concerning the freedom of a, we have to prove that, fixed a state (h_0, c_0), the orbital mapping starting from (h_0, c_0), that is the mapping $h \mapsto a(h)(h_0, c_0)$, is injective, and this follows immeaditly from the injectivity of the translation $\tau_{h_0} : h \mapsto h + h_0$. □

Remark 9. Note that, if G is the action of the preceding theorem, for every real t_0 and for every real $t \geq t_0$, the following relation holds $G^t_{t_0} = a(t - t_0)$.

The above theorem can be generalized. To this aim we note that the relation $\mathcal{E}$ on the graph $\mathrm{gr}(\geq)$ for which $(q, r)\mathcal{E}(s, t)$ means $r - q = t - s$, is an equivalence relation, we call it the *duration-equivalence on* $\mathrm{gr}(\geq)$.

Theorem 6. *The duration-equivalence* $\mathcal{E}$ *on* $\mathrm{gr}(\geq)$ *is compatible with the action induced by a capitalization factor, that is, if two bi-times are equivalent then the corresponding operators are equal.*

Proof. In fact, if $(q, r)\mathcal{E}(s, t)$, putting h the common length of the two bi-times, we have

$$G(q, r)(h_0, C_0) = \left(h_0 + h, C_0 \frac{f(h_0 + h)}{f(h_0)}\right) = G(s, t)(h_0, C_0),$$

for every state (h_0, C_0), and then the operators $G(q, r)$ and $G(s, t)$ coincide. □

Theorem 7. *The non-reversible evolution associated with a capitalization factor* f *is free and faithful up to the equivalence relation* $\mathcal{E}$.

Theorem 8. *The application leght on* $\mathrm{gr}(\geq)$ *is a representation of the semi-grupoid* $(\mathrm{gr}(\geq), .)$ *onto the additive monoid of non-negative real numbers (that is a semi-grupoid since the cancellation law holds). Moreover, the quotient* $(\mathrm{gr}(\geq), .)/\mathcal{E}$ *is a monoid and it is isomorph to the the additive monoid of non-negative real numbers.*

Proof. Let $L : \mathrm{gr}(\geq) \to \mathbb{R}_{\geq}$ be the leght on $\mathrm{gr}(\geq)$, we have $L(x.y) = L(x) + L(y)$, in fact $x.y = (x_1, y_2)$ and so $L(x.y) = x_1 - y_2$, on the other hand $L(x) + L(y) = (x_1 - x_2) + (y_1 - y_2)$ and since $x_2 = y_1$ the claim is proved. The equivalence $\mathcal{E}$ is compatible with the operation of the semigrupoid $(\mathrm{gr}(\geq), .)$ then, if we put, for every pair of class of equivalence x and y in $\mathrm{gr}(\geq)/\mathcal{E}$, $x.y := [a.b]_{\mathcal{E}}$, where $a \in x$ and $b \in y$, when $a.b$ is defined, is a good definition for an operation in $\mathrm{gr}(\geq)/\mathcal{E}$. Let us prove that the structure $(\mathrm{gr}(\geq)/\mathcal{E}, .)$ is a magma (we have to prove that the operation is everywhere defined). We have

$$\begin{aligned} [(t, s)]_{\mathcal{E}} \cdot [(r, q)]_{\mathcal{E}} &= [(t - s + (r - q), r - q)]_{\mathcal{E}} \cdot [(r - q, 0)]_{\mathcal{E}} = \\ &= [(t - s + (r - q), 0)]_{\mathcal{E}}, \end{aligned}$$

and then is always possible to define the composition of two equivalence classes. Consider the application $f : \mathbb{R}_{\geq} \to \mathrm{gr}(\geq)/\mathcal{E}$ defined by $f(h) = [(h, 0)]_{\mathcal{E}}$. For every non-negative real numbers h and k, we have

$$\begin{aligned} f(h+k) &= [(h+k,0)]_{\mathcal{E}} = \\ &= [(h+k,h).(h,0)]_{\mathcal{E}} = \\ &= [(h+k,h)]_{\mathcal{E}}\,[(h,0)]_{\mathcal{E}} = \\ &= [(k,0)]_{\mathcal{E}}\,[(h,0)]_{\mathcal{E}} = \\ &= f(k).f(h), \end{aligned}$$

so f is a representation of the monoid in the magma $(\mathrm{gr}(\geq)/\mathcal{E}, .)$. Moreover, if $f(h) = f(k)$ then $(h,0)$ has lenght k, and so $h = k$, thus f is injective. Let $x \in \mathrm{gr}(\geq)/\mathcal{E}$ and let (t,s) in the class of equivalence x, hence $x = f(t-s)$, and so f is also surjective, and the theorem is proved. □

7 Capital evolution of a capitalized event

Recall that if G is a dynamical system, the pair $I_0 = (t_0, s_0)$ where s_0 is state of the system is said an initial condition. The set $o_G(I_0)$ of all the states s such that $s = G_{t_0}^{t_1} s_0$, for some t_1, is called the orbit of the system starting from the initial condition I_0. The function $t \mapsto G_{t_0}^{t} s_0$ is called the parametrized orbit of the system starting from I_0.

We propose the following definition.

Definition 13 (of capitalized event and its evolution). Let E_0 be a financial event and let h_0 be a non negative real number the pair (E_0, h_0) is said to be a capitalized event, h_0 is said the length of capitalization of the capitalized event. We say that the function M from the interval $[t_0, +\infty[$ to the real line defined by

$$M(t) = C_0 \frac{f(h_0 + (t - t_0))}{f(h_0)},$$

is the capital evolution of the capitalized event (E_0, h_0). In other terms, the evolution M is the second component of the parametrized orbit starting from the initial condition $(t_0, (h_0, C_0))$.

The definition of evolution of a capitalized event generalizes the evolution of an event. In fact, the following proposition holds.

Theorem 9. *The evolution of an event E_0, with respect to a certain capitalization factor f coincides with the evolution of the capitalized event $(E_0, 0)$.*

Remark 10. The concept of evolution of a capitalized event gives an explanation of the fact that given an event E_0 it is possible to have different evolutions of this event depending on the state of its capitalization. For instance, often financial mathematics say "if we interrupt the capitalization and we restart newly the capitalization (with the same capitalization factor) we obtain a different evolution". The rigorous counterpart of this assertion is the evolutions of the capitalized events $((t,C),h)$ and $((t,C),0)$ are different: this is not more a problem since they are different capitalized events.

8 The extension of the model

In our definition of dynamical system associated with a capitalization factor we can define the dynamical system on the entire plane and not only on the half-plane $\mathrm{gr}(\geq)$, in fact we can give the following definition.

Definition 14 (of canonical extension of a capitalization factor). Let f be a capitalization factor. We define canonical extension of f the function ${}^e f$ from the entire real line to the real line defined by ${}^e f(h) = f(|h|)^{\mathrm{sgn}(h)}$.

Remark 11. Note that the extension ${}^e f$ is continuous and that ${}^e f(-h)^e f(h) = 1$. This obviously not implies that ${}^e f$ be an homomorphism of the additive group of reals into the multiplicative group of non-zero reals.

Theorem 10. *Let $f : \mathbb{R}_{\geq} \to \mathbb{R}$ be a function never vanishing and let G be the application from the plane $\mathbb{R}^2$ into the space of functions of the plane $\mathbb{R}^2$ into itself defined by*

$$G(t_1, t_0)(h_0, C_0) = \left(h_0 + (t_1 - t_0), C_0 \frac{{}^e f(h_0 + (t_1 - t_0))}{{}^e f(h_0)} \right)$$

Then G is a reversible evolution operator.

Proof. It is analogous to the case of the non-reversible evolution operators. Putting $h_1 = t_1 - t_0$, and $h_2 = t_2 - t_1$, we have

$$\begin{aligned} G(t_2, t_1) \circ G(t_1, t_0)(h_0, C_0) &= G(t_2, t_1) \left(h_0 + h_1, C_0 \frac{{}^e f(h_0 + h_1)}{f(h_0)} \right) = \\ &= \left(h_0 + h_1 + h_2, C_0 \frac{{}^e f(h_0 + h_1)}{{}^e f(h_0)} \frac{{}^e f(h_0 + h_1 + h_2)}{{}^e f(h_0 + h_1)} \right) = \\ &= \left(h_0 + (t_2 - t_0), C_0 \frac{{}^e f(h_0 + (t_2 - t_0))}{{}^e f(h_0)} \right) = \\ &= G(t_2, t_0)(h_0, C_0), \end{aligned}$$

and the Chapman-Kolmogorov law is verified also in this case. □

Definition 15 (of reversible evolution operator associated to a capitalization factor). Let $f : \mathbb{R}_{\geq} \to \mathbb{R}$ be a capitalization factor and let G be the application from the plane into the space $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$ (set of functions of the plane $\mathbb{R}^2$ into itself) defined by

$$G(t_1, t_0)(h_0, C_0) = \left(h_0 + (t_1 - t_0), C_0 \frac{{}^e f(h_0 + (t_1 - t_0))}{{}^e f(h_0)} \right),$$

for every pair of times (t_1, t_0) and for every pair (h_0, C_0) in the plane. The evolution operator G is called the reversible evolution operator associated with the factor f, or the dynamical action of the times-plane on the financial states plane induced by the factor f.

Definition 16 (of evolution factor). An evolution factor on the financial events plane is a continuous positive function from the real line into the real line such that $f(h)f(-h) = 1$ for every real h.

Remark 12. The value at 0 of an evolution factor is 1, since f is positive and since $f(0)^2 = 1$.

Remark 13. The canonical extension of a capitalization factor is an evolution factor on the financial events plane.

9 The case of separable capitalization factors

The following definition is classic, we simply recall it.

Definition 17 (of separable capitalization factor). A capitalization factor f is said to be separable if it is an homomorphism of the additive monoid of non-negative real numbers into the multiplicative monoid of positive real numbers, in other terms if, for every non-negative h_0 and h, we have $f(h_0 + h) = f(h_0)f(h)$.

In this particular case it is possible to simplify the space of states of the financial evolutions. Thanks to the following result.

Theorem 11. *With respect to a separable capitalization factor, the second component of the orbital parametric curve starting at a certain time from a state* (h_0, C_0) *does not depend on the capitalization time displacement* h_0.

Proof. In fact, say G the evolution operator associated with f, we have

$$\begin{aligned} G(t_1, t_0)(h_0, C_0) &= \left(h_0 + (t_1 - t_0), C_0 \frac{f(h_0 + (t_1 - t_0))}{f(h_0)}\right) = \\ &= \left(h_0 + (t_1 - t_0), C_0 \frac{f(h_0)f(t_1 - t_0)}{f(h_0)}\right) = \\ &= (h_0 + (t_1 - t_0), C_0 f(t_1 - t_0)), \end{aligned}$$

then the result is proved. □

We can define another simplified non-reversible evolution operator W associated with a separable capitalization factor.

Definition 18 (of action induced by a separable capitalization factor on the capital line). We call dynamical action induced on the capital line by a separable capitalization factor f the function from the half-plane $\mathrm{gr}(\geq)$ (graph of the usual majoring order $\geq$ of the real line) into the space of functions of the real axis into itself, defined by $W_{t_0}^{t_1}(C_0) = C_0 f(t_1 - t_0)$.

Remark 14. It is clear that W is a non-reversible evolution operator on the real axis.

Remark 15. We can also define an action of the additive group of real numbers $(\mathbb{R}, +)$ on the plane of financial events and on the axis of capitals by (in multiplicative notation)

$$h.(t, C) = (t + h, Cf(h)), \qquad h.C = Cf(h).$$

To the first action is associated the dynamical system $(\mathbb{R}^2, .)$, where . is the external operation associated with the preceding action, and the corresponding one-parameter group $(\tau_h)_{h\in\mathbb{R}}$, where the translation τ_h is the application of the plane into itself defined by $\tau_h(E) = h.E$.

Theorem 12. *The action on the capital axis induced by a capitalization factor is distributive with respect to the addition on the capital axis, in multiplicative notation we have* $h.(C + C') = h.C + h.C'$.

The situation is more complicated for the action defined on the plane: it is obviously not distributive with respect to the standard addition of the euclidean plane, but this is not a natural addition among financial events, since it has no sense to sum financial events with different times. Nevertheless, just this simple financial consideration allow us to solve the problem.

We give the following definition of standard addition on the financial events plan.

Definition 19 (of standard addition in the financial events plane). In the plane of financial events we call standard addition the not-everywhere defined binary internal operation defined by $(t, c) + (t', c') = (t, c + c')$, when $t = t'$ and only in this case. We define, moreover, the everywhere defined external binary operation of multiplication by real scalars as follows $a(t, c) = (t, ac)$, we call it the standard multiplication by scalars of the financial plane.

Theorem 13. *The action induced on the events plane by a separable capitalization factor is distributive with respect to the standard addition of financial events and it is permutable with the action associated to the standard multiplication by real numbers of the financial events.*

Proof. It is left to the reader. □

Remark 16. That of separable capitalization factors is the only case in which we can define the dynamical actions directly on the plane of financial events.

10 The case of two-arguments capitalization factors

We have to generalize the setting since in the applications to finance are used also the two-arguments capitalization factors. We propose the following definition.

Definition 20 (of two-arguments capitalization factor). We define two-arguments capitalization factor each continuous positive function f from the half-plane $\mathrm{gr}(\leq)$, endowed with the standard topology induced on it by the usual topology of the plane, to the real line such that f is unitary on the diagonal of the plane, in other words, f associates with each diagonal element (t, t) of the plane the value 1.

Theorem 14. *Let* $f : \mathrm{gr}(\leq) \to \mathbb{R}$ *be a two-arguments capitalization factor and let* G *be the application from the graph of the usual majoring order of* $\mathbb{R}$ *into the space of functions of the half-plane* $[0, +\infty[\times \mathbb{R}$ *into itself defined by*

$$G(t, t_0)(h_0, C_0) = \left(h_0 + (t - t_0), C_0 \frac{f(t_0 - h_0, t)}{f(t_0 - h_0, t_0)} \right).$$

Then G *is a non-reversible evolution operator.*

Proof. We have obviously 1. Moreover, putting $h_1 = t_1 - t_0$, and $h_2 = t_2 - t_1$, we have (using the multiplicative notation for the action G, or, that it is the same, using the associated external binary operation)

$$
\begin{aligned}
(t_2,t_1).(t_1,t_0)(h_0,C_0) &= (t_2,t_1)\left(h_0+h_1, C_0\frac{f(t_0-h_0,t_1)}{f(t_0-h_0,t_0)}\right) = \\
&= \left(h_0+h_1+h_2, C_0\frac{f(t_0-h_0,t_1)}{f(t_0-h_0,t_0)}\frac{f(t_1-(h_0+h_1),t_2)}{f(t_0-h_0,t_1)}\right) = \\
&= \left(h_0+(t_2-t_0), C_0\frac{f(t_0-h_0,t_2)}{f(t_0-h_0,t_0)}\right) = \\
&= (t_2,t_0)(h_0,C_0). \ \square
\end{aligned}
$$

Definition 21 (of non-reversible evolution operator associated with a two-arguments capitalization factor). Let $f : \mathrm{gr}(\leq) \to \mathbb{R}$ be a two-arguments capitalization factor and let G be the application from the graph $\mathrm{gr}(\geq)$ into the space $\mathcal{F}(\mathbb{R}_\geq \times \mathbb{R}, \mathbb{R}_\geq \times \mathbb{R})$ (set of functions of the half-plane $\mathbb{R}_\geq \times \mathbb{R}$ into itself) defined by

$$
G(t,t_0)(h_0,C_0) = \left(h_0+(t-t_0), C_0\frac{f(t_0-h_0,t)}{f(t_0-h_0,t_0)}\right),
$$

for every pair of times (t,t_0) in $\mathrm{gr}(\geq)$ and for every pair (h_0,C_0) in the half-plane $\mathbb{R}_\geq \times \mathbb{R}$. The evolution operator G is called the non-reversible evolution operator associated with the factor f.

Also in this case we can extend the space of states to the entire plane and not only on the half-plane $\mathbb{R}_\geq \times \mathbb{R}$ and the bi-times space from $\mathrm{gr}(\geq)$ to $\mathbb{R}^2$, in fact we can give the following definition.

Definition 22 (of canonical extension of a capitalization factor). Let f be a two-arguments capitalization factor. We define canonical extension of f the function ${}^e f$ of the real line into itself coinciding with f when this latter is defined and such that ${}^e f(t,t_0) = f(t_0,t)^{-1}$ elsewhere.

Remark 17. Note that the extension ${}^e f$ is continuous and that ${}^e f(t,t')\,{}^e f(t',t) = 1$. This obviously not implies that ${}^e f$ is a separable law of capitalization.

Theorem 15. *Let $f : \mathrm{gr}(\leq) \to \mathbb{R}$ be a two-arguments capitalization factor and let G be the application from the plane $\mathbb{R}^2$ into the space of functions of the plane $\mathbb{R}^2$ into the plane defined by*

$$
G(t,t_0)(h_0,C_0) = \left(h_0+(t-t_0), C_0\frac{{}^e f(t_0-h_0,t)}{{}^e f(t_0-h_0,t_0)}\right).
$$

Then G is a reversible evolution operator.

11 The case of hyperbolic capitalization factors

We first begin with the classic definition of hyperbolic capitalization factor.

Definition 23 (of hyperbolic capitalization factor). An hyperbolic capitalization factor is a function $f : [0,1/d[\to \mathbb{R}$, defined by $f(h) = (1-dh)^{-1}$, where d is a positive number called the discount rate of the capitalization factor. The reciprocal of the discount rate d is said to be the characteristic time of the factor f.

In this case the definition of non-reversible evolution operator associated with a capitalization factor f is more complicated. We give the following definition.

Definition 24 (of non-reversible evolution-operator associated with an hyperbolic factor). Let f be an hyperbolic capitalization factor of discount rate d. Let t^* be the characteristic time of the factor f. Let T be the set of all real pairs (t, t_0) such that the difference $h := t - t_0$ belongs to the interval $[0, t^*[$. We define non-reversible evolution operator associated with the factor f the mapping

$$G : T \to \bigcup_{t \in [0,t^*[} \mathcal{F}([0, t[\times \mathbb{R}, [0, t^*[\times \mathbb{R}),$$

sending every pair of times (t_0, t), with difference $h := t - t_0$, to the function

$$G^t_{t_0} : [0, t^* - h[\times \mathbb{R} \to [0, t^*[\times \mathbb{R},$$

defined, for every pair (h_0, C_0) belonging to the strip $[0, t^* - h[\times \mathbb{R}$, by

$$G^t_{t_0}(h_0, C_0) = \left(h_0 + h, C_0 \frac{f(h_0 + h)}{f(h_0)} \right).$$

Remark 18. The above set T is a strip, precisely the strip delimited by the vector-line generated by the vector $(1, 1)$ and the parallel affine-line containing the point $(0, -t^*)$.

Remark 19. The above name "non-reversible evolution" is not justified by the definitions of evolutions given in the paper. In this moment it is an abuse of language.

Remark 20. Every function $G^t_{t_0}$ is injective and smooth, it is then a local bord-chart on the closed half-plane $[0, t^*[\times \mathbb{R}$, moreover, the family $(G^t_{t_0})_{(t,t_0) \in T}$ is an atlas on this half-plane, we call it *the non-reversible atlas induced on the state space* $[0, t^*[\times \mathbb{R}$ *by the hyperbolic capitalization factor* f.

Remark 21 (the local Chapman-Kolmogorov law). To justifies the name non-reversible evolution operator we have to consider the definition of composition in the extended sense of differential geometry. In fact, to prove that the charts of the atlas induced by f on the state space are pairwise compatible we used this extended composition. Now it is clear that the composition (in this extended sense) is defined, with non-void domain, and it is clear that the composition of the local chart $G^{t_2}_{t_1}$ with the local chart $G^{t_1}_{t_0}$ is the mapping $G^{t_2}_{t_0}$: in this sense we say that *the mapping* G *satisfies the Chapman-Kolmogorov law and in this sense we say that it is a non-reversible evolution operator.*

Also in this case we can define a reversible dynamical system, in fact we can give the following definition.

Definition 25 (of canonical extension of an hyperbolic factor). Let f be an hyperbolic capitalization factor of discount rate d and characteristic time t^*. We define canonical extension of f the function ${}^e f$ from the interval $]-t^*, t^*[$ to the real line defined by ${}^e f(h) = f(|h|)^{\mathrm{sgn}(h)}$.

Remark 22. Note that the extension ${}^e f$ is continuous and that ${}^e f(-h) {}^e f(h) = 1$.

Definition 26 (of reversible evolution-operator associated with an hyperbolic factor). Let f be an hyperbolic capitalization factor of discount rate d. The reciprocal t^* of the discount rate d is said to be the characteristic time of the factor f. Let T be the set of all real pairs (t, t_0) such that the difference $h := t - t_0$ belongs to the interval $]-t^*, t^*[$. We define reversible evolution operator associated with the factor f the mapping

$$G : T \to \bigcup_{t \in [0,t^*[} \mathcal{F}(]-t^*, t[\times \mathbb{R},]-t^*, t^*[\times \mathbb{R}),$$

sending every pair of times (t_0, t), with difference $h := t - t_0$, to the function

$$G_{t_0}^t :]-t^*, t^* - h[\times \mathbb{R} \to]-t^*, t^*[\times \mathbb{R},$$

defined, for every pair (h_0, C_0) belonging to the strip $]-t^*, t^* - h[\times \mathbb{R}$, by

$$G_{t_0}^t(h_0, C_0) = \left(h_0 + h, C_0 \frac{{}^e f(h_0 + h)}{{}^e f(h_0)} \right).$$

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