Semi-symmetric non-metric connections on a Kähler manifold

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Abstract. In the present paper, we have study Kähler manifolds equipped with a semi-symmetric non-metric connection. It has been shown that a contravariant almost analytic vector field with respect to a Riemannian connection D will be contravariant almost analytic with respect to the semi-symmetric non-metric connection ∇^*. We have also shown that the Nijenhuis tensor on a Kähler manifold with respect to semi-symmetric non-metric connection identically vanishes.

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Key words: semi-symmetric non-metric connection, Kähler manifold, Nijenhuis tensor, contravariant almost analytic vector field.

1 Introduction

The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O.C. Andonie [2], M.C. Chaki and A. Konar [3], U.C.De [4] etc., while a special type of semi-symmetric metric connection on a Riemannian manifold has been studied by U.C. De and S.C. Biswas [5]. P.N. Pandey and S.K. Dubey [7] discussed an almost Grayan manifold admitting a semi-symmetric metric connection while a Kähler manifold equipped with semi-symmetric metric connection and an almost Hermitian manifold with semi-symmetric recurrent connection have been studied by P.N. Pandey and B.B. Chaturvedi [6, 8]. Nirmala S. Agashe and Mangala R. Chafle [1] have studied semi-symmetric non-metric connection on a Riemannian manifold in 1992.

Let *Mn* be an even dimensional differentiable manifold of differentiability class C^{r+1}. If there exists a vector valued linear function F of differentiability class C^r such that for any vector field X

$$\overline{\overline{X}} + X = 0\,, \tag{1.1}$$

$$g\,(\overline{X},\overline{Y}) = g(X,Y), \tag{1.2}$$

$$(D_X F)Y = 0\,, \tag{1.3}$$

where $\overline{X} = FX$, g is non-singular metric tensor and D is Riemannian connection, then M_n is called a Kähler manifold.

A linear connection ∇^* on $\{M_n, g\}$ is said to be a semi-symmetric non-metric connection if the torsion tensor T^* of the connection ∇^* and the metric g of the manifold satisfy the following conditions [1]:

$$(\nabla^*_X g)(Y,Z) = -u(Y)g(X,Z) - u(Z)g(X,Y)\,, \tag{1.4}$$

$$T^*(X,Y) = u(Y)X - u(X)Y\,, \tag{1.5}$$

for arbitrary vector fields X and Y. The symbol u stands for a 1-form associated with the vector field U on M by

$$u(X) = g(U,X)\,.$$

The relation between the semi-symmetric non-metric connection ∇^* and Riemannian connection D is given by [1]:

$$\nabla^*_X Y = D_X Y + u(Y)X\,. \tag{1.6}$$

2 Some theorems on a Kähler manifold with connection ∇^*

Let us consider a Kähler manifold M_n equipped with a semi-symmetric non-metric connection ∇^* and define

$$\left\{\begin{array}{ll} (i) & {}'F(Y,Z) = g(\overline{Y}, Z)\,, \\ (ii) & {}'T^*(X,Y,Z) = g(T^*(X,Y),Z)\,, \\ (iii) & H(X,Y) = u(Y)X\,, \\ (iv) & {}'H(X,Y,Z) = g(H(X,Y),Z)\,. \end{array}\right. \tag{2.1}$$

In view of (2.1)iii, the equation (1.6) becomes

$$\nabla^*_X Y = D_X Y + H(X,Y)\,.$$

For any vector field $\overline{Y}$, the equation (1.6) becomes

$$\nabla^*_X \overline{Y} = D_X \overline{Y} + u(\overline{Y})X\,, \tag{2.2}$$

which implies

$$(\nabla^*_X F)Y = (D_X F)Y + \overline{D_X Y} + u(\overline{Y})X - \overline{\nabla^*_X Y}\,. \tag{2.3}$$

Operating both sides of the equation (1.6) by F, we have

$$\overline{\nabla^*_X Y} = \overline{D_X Y} + u(Y)\overline{X}\,. \tag{2.4}$$

Using (1.3) and (2.4) in (2.3), we get

$$(\nabla^*_X F)Y = u(\overline{Y})X - u(Y)\overline{X}\,. \tag{2.5}$$

Barring X and Y in (2.5), we get

$$(\nabla^*_{\overline{X}} F)\overline{Y} = u(\overline{Y})X - u(Y)\overline{X}\,. \tag{2.6}$$

Thus, we conclude:

Theorem 2.1. *A Kähler manifold equipped with a semi-symmetric non-metric connection* ∇^* *satisfies the following*

$$\begin{cases} (i) \quad (\nabla^*_{\overline{X}} F)\overline{Y} = (\nabla^*_X F)Y\,, \\ (ii) \quad (\nabla^*_X F)Y = 0 \quad \textit{if and only if} \quad u(\overline{Y})X = u(Y)\overline{X}\,. \end{cases} \tag{2.7}$$

Differentiating (2.1)i with respect to connection ∇^* and using (1.4), (1.6) and $g(\overline{X}, Y) = -g(X, \overline{Y})$, we have

$$(\nabla^*_X {}'F)(Y, Z) = u(Y)g(\overline{Z}, X) + u(Z)g(\overline{Y}, Z)\,. \tag{2.8}$$

$d^*{}'F(X, Y, Z)$ is defined as

$$d^*{}'F(X, Y, Z) = (\nabla^*_X {}'F)(Y, Z) + (\nabla^*_Y {}'F)(Z, X) + (\nabla^*_Z {}'F)(X, Y)\,. \tag{2.9}$$

Using (2.8) in (2.9), we get

$$d^*{}'F(X, Y, Z) = 2[u(X)g(\overline{Y}, Z) + u(Y)g(\overline{Z}, X) + u(Z)g(\overline{X}, Y)]\,.$$

Thus, we conclude:

Theorem 2.2. *On a Kähler manifold equipped with semi-symmetric non-metric connection* ∇^**, we have*

$$d^*{}'F(X, Y, Z) = 2[u(X)g(\overline{Y}, Z) + u(Y)g(\overline{Z}, X) + u(Z)g(\overline{X}, Y)]\,.$$

3 Nijenhuis tensor with connection ∇^*

The Nijenhuis tensor with respect to semi-symmetric non-metric connection ∇^* is given by

$$N^*(X, Y) = (\nabla^*_{\overline{X}} F)Y - (\nabla^*_{\overline{Y}} F)X - \overline{(\nabla^*_X F)Y} + \overline{(\nabla^*_Y F)X}\,. \tag{3.1}$$

From equations (2.5) and (1.1), we have

$$(\nabla^*_{\overline{X}} F)Y = u(\overline{Y})\overline{X} + u(Y)X\,. \tag{3.2}$$

Interchanging X and Y in (3.2), we get

$$(\nabla^*_{\overline{Y}} F)X = u(\overline{X})\overline{Y} + u(X)Y\,. \tag{3.3}$$

Operating F on both sides of the equation (2.5) and using (1.1), we get

$$\overline{(\nabla^*_X F)Y} = u(\overline{Y})\overline{X} + u(Y)X\,. \tag{3.4}$$

Interchanging X and Y in (3.4), we have

$$\overline{(\nabla^*_Y F)X} = u(\overline{X})\overline{Y} + u(X)Y\,. \tag{3.5}$$

From the equations (3.2), (3.3),(3.4) and(3.5),we get

$$(\nabla^*_{\overline{X}} F)Y - (\nabla^*_{\overline{Y}} F)X - \overline{(\nabla^*_X F)Y} + \overline{(\nabla^*_Y F)X} = 0\,. \tag{3.6}$$

From equations (3.1) and (3.6), we may conclude

Theorem 3.1. *On a Kähler manifold, Nijenhuis tensor with respect to Semi-symmetric non-metric connection vanishes, i.e.* $N^*(X,Y) = 0$.

4 Contravariant almost analytic vector fields on a Kähler manifold

A vector field V is said to be contravariant almost analytic if the Lie-derivative of F with respect to V vanishes identically, i.e.

$$(L_V F)X = 0\,, \tag{4.1}$$

for all X. The equation (4.1) is equivalent to the equation

$$[V, \overline{X}] = \overline{[V, X]}\,. \tag{4.2}$$

In a Kähler manifold, the equation (4.2) becomes

$$D_{\overline{X}} V - \overline{D_X V} = 0 \quad \text{if and only if} \quad \overline{D_{\overline{X}} V} + D_X V = 0\,. \tag{4.3}$$

Now, we propose:

Theorem 4.1. *On a Kähler manifold, a contravariant almost analytic vector* V *with respect to Riemannian connection* D *is also contravariant almost analytic with respect to semi-symmetric non- metric connection* ∇^* .

Proof. For any vector field V, the equation (1.5) gives

$$\nabla^*_X V = D_X V + u(V)X\,. \tag{4.4}$$

For the vector field $\overline{X}$ the equation (4.4) becomes

$$\nabla^*_{\overline{X}} V = D_{\overline{X}} V + u(V)\overline{X}\,. \tag{4.5}$$

Operating both sides of the equation (4.4) by F, we have

$$\overline{\nabla^*_X V} = \overline{D_X V} + u(V)\overline{X}\,. \tag{4.6}$$

Subtracting the equation (4.6) from (4.5), we get

$$\nabla^*_{\overline{X}} V - \overline{\nabla^*_X V} = D_{\overline{X}} V - \overline{D_X V}\,.$$

Since V is contravariant almost analytic with respect to connection D, we have $D_{\overline{X}} V - \overline{D_X V} = 0$, and then $\nabla^*_{\overline{X}} V - \overline{\nabla^*_X V} = 0$. Therefore V is contravariant almost analytic with respect to connection ∇^*. □

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