

The dynamics of Rabinovich system

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Abstract. The paper presents some dynamical aspects of Rabinovich type. For the system (1.1) we have presented some Hamilton-Poisson realizations, a metriplectic structure, the system with distributed delay, with fractional derivatives and some numerical applications using Moulton-Adams algorithm for differential systems with fractional derivatives.

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1 Introduction

The control of chaos lies at the interface of control theory and the theory of dynamical systems. It studies methods of controlling deterministic systems with chaotic behavior. Using computer aided modelling, it was easy to notice the possibility of substantial variation of the characteristics of chaotic systems by relatively small variations of their parameters and external actions. A method of transmitting information using chaotic signal was proposed by A.S.Pikovsky and M.I. Rabinovich [8] using the differential system

$$\begin{cases} \dot{x}_1 = -\nu_1 x_1 + h x_2 + x_2 x_3 \\ \dot{x}_2 = h x_1 - \nu_2 x_2 - x_1 x_3 \\ \dot{x}_3 = -\nu_3 x_3 + x_1 x_2. \end{cases}$$

In this paper we will consider the Rabinovich system:

$$(1.1) \quad \begin{cases} \dot{x}_1 = x_2 x_3 \\ \dot{x}_2 = -x_1 x_3 \\ \dot{x}_3 = x_1 x_2 \end{cases},$$

and we will analyze some global properties, the local study of stationary points, compatible Poisson structures and corresponding tri-Hamiltonian systems are also discussed.

We will consider a Poisson structure on \mathbb{R}^n as $\{\cdot, \cdot\} : \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}) \times \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R})$, see [10]. A Hamiltonian equation is called tri-Hamiltonian if it admits two Hamiltonian representations with compatible Poisson structures $\frac{dx}{dt} = J\nabla H =$

$\tilde{J}\nabla\tilde{H} = \bar{J}\nabla\bar{H}$, where J , \tilde{J} and \bar{J} are three Hamiltonian matrices (of the form $[\{x_i, x_j\}]$ and $\{\cdot, \cdot\}$ is the Poisson structure) and they are also compatible.

Revised dynamical system, with distributive delay, associated to system (1.1), allow the description of new crypting methods.

2 The analysis of classical Rabinovich differential equations

2.1 Geometrical properties of the system (1.1)

In this subsection we will present some dynamical and geometrical properties, from geometrical mechanical point of view.

Proposition 2.1. *The dynamics (1.1) have the following 3 Hamilton-Poisson realizations (\mathbb{R}^3, P^i, h_i) , $i = 1, 2, 3$ and their linear combinations where*

$$P^1 = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 0 & \frac{1}{2}x_2 \\ 0 & 0 & -\frac{1}{2}x_1 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & 0 \end{bmatrix}, P^3 = \begin{bmatrix} 0 & 0 & -\frac{1}{2}x_2 \\ 0 & 0 & \frac{1}{2}x_1 \\ \frac{1}{2}x_2 & -\frac{1}{2}x_1 & 0 \end{bmatrix}$$

$$h_1(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2), h_3(x_1, x_2, x_3) = x_1^2 - x_3^2, h_2(x_1, x_2, x_3) = x_2^2 + x_3^2. \quad \square$$

From direct computations, using the algebraic technique of Bermejo and Fairen [3], we get the following results.

Proposition 2.2. *There exists only one functionally independent Casimir of our Poisson configurations (\mathbb{R}^3, P^i) , $i = 1, 2, 3$ are given by:*

$$c_1(x_1, x_2, x_3) = \frac{1}{2}(x_2^2 + x_3^2), c_2(x_1, x_2, x_3) = x_1^2 + x_2^2, c_3(x_1, x_2, x_3) = x_1^2 + x_3^2. \quad \square$$

2.2 Stability problem

From the analysis of the stationary points, using [9], we get the following statements.

Proposition 2.3. *The stationary points $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$ and $e_3^m(0, 0, m)$, $m \in \mathbb{R}$ are spectrally stable, unstable, respectively. \square*

Proposition 2.4. *The stationary points $e_1^m(m, 0, 0)$ and $e_3^m(0, 0, m)$ are nonlinear stable. \square*

3 The metriplectic structure associated to Rabinovich system

A Leibniz structure on a smooth manifold M is defined by a tensor field P of type $(2,0)$, [2]. The tensor field P and a smooth function h on M , called a Hamiltonian function, define a vector field X_h which generates a differential system, called Leibniz system. If P is skew-symmetric then we have an almost symplectic structure and if P is symmetric then we have an almost metric structure.

Let P be a skew-symmetric tensor field in \mathbb{R}^3 of type $(2,0)$, g a 2-symmetric tensor field and $h \in C^\infty(\mathbb{R}^3)$. If P is a Poisson tensor field and g is a nondegenerate tensor field, then (\mathbb{R}^3, P, g) is called a metriplectic manifold of the first kind ([4], [6], [7]). The differential system is given by

$$(3.1) \quad \dot{x}_i = \sum_{j=1}^3 P_{ij} \frac{\partial h}{\partial x_j} + \sum_{j=1}^3 g_{ij} \frac{\partial h}{\partial x_j}, \quad g_{ii} = - \sum_{k=1, k \neq i}^3 \frac{\partial h}{\partial x_k} \frac{\partial h}{\partial x_k}, \quad g_{ij} = \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j}, \quad i, j = 1, 2, 3.$$

If P is a (almost) Poisson differential system on \mathbb{R}^3 with Hamiltonian function h_1 and a Casimir function h_2 , there exists a tensor field g such that (\mathbb{R}^3, P, g) is a metriplectic manifold of second kind. The differential system associated with it is given by:

$$(3.2) \quad \dot{x}_i = \sum_{j=1}^3 P_{ij} \frac{\partial h_1}{\partial x_j} + \sum_{j=1}^3 g_{ij} \frac{\partial h_2}{\partial x_j}, \quad g_{ii} = - \sum_{k=1, k \neq i}^3 \frac{\partial h_1}{\partial x_k} \frac{\partial h_2}{\partial x_k}, \quad g_{ij} = \frac{\partial h_1}{\partial x_i} \frac{\partial h_2}{\partial x_j}, \quad i, j = 1, 2, 3.$$

Let (\mathbb{R}^3, P^α) , $\alpha = 1, 2, 3$ realizations of Rabinovich system of differential equations, with Hamiltonian functions h_α , $\alpha = 1, 2, 3$ and Casimir functions c_α , $\alpha = 1, 2, 3$, given in Proposition 2.1 and Proposition 2.2.

Using (3.1) and (3.2) we get the following results.

Proposition 3.1. (a) The metriplectic realization of the first kind of the first Hamilton-Poisson realization is given by (\mathbb{R}^3, P_1, g_1) where: $g_{11}^1 = -(x_2)^2$, $g_{22}^1 = -(x_1)^2$, $g_{33}^1 = 0$ and $g_{12}^1 = x_1 x_2$, $g_{21}^1 = x_1 x_2$, $g_{13}^1 = g_{31}^1 = 0$, $g_{23}^1 = g_{32}^1 = 0$.

(b) The associated differential system is given by:

$$(3.3) \quad \begin{cases} \dot{x}_1 = x_2 x_3 + x_1 x_2 (x_1 - x_2) \\ \dot{x}_2 = -x_1 x_3 + (x_1)^2 \\ \dot{x}_3 = x_1 x_2. \end{cases}$$

(c) The differential system (3.3) has the following stationary points: $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$, $e_3^m(0, 0, m)$.

(d) The matrix of the linear part of the system (3.3) in $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$, resp. in $e_3^m(0, 0, m)$ is given by:

$$A_1 = \begin{bmatrix} 0 & m^2 & 0 \\ 0 & 0 & m^2 + m \\ 0 & m & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -m^2 & 0 & m \\ 0 & 0 & 0 \\ m & 0 & 0 \end{bmatrix}, \quad \text{resp} \quad A_3 = \begin{bmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) The characteristic equation of A_1 for (3.3) in $e_1^m(m, 0, 0)$ is: $\lambda(-\lambda^2 + m^2(m + 1)) = 0$ and so, we have two cases:

(i) if $m > -1$, then $e_1^m(m, 0, 0)$ are unstable;

(ii) if $m < -1$, then we have a limit cycle.

(f) The characteristic equation of A_2 for (3.3) in $e_2^m(0, m, 0)$ is: $-\lambda(\lambda^2 + m^2\lambda - m^2) = 0$ and so, it can be easily seen that $e_2^m(0, m, 0)$ are unstable.

(g) The characteristic equation of A_3 for (3.3) in $e_3^m(0, 0, m)$ is: $\lambda(\lambda^2 + m^2) = 0$.

(h) In a neighborhood of $e_3^m(0, 0, m)$, $m > 0$ there exists a limit cycle. \square

Proposition 3.2. (a) The metriplectic realization of the second kind of the first Hamilton-Poisson realization is given by (\mathbb{R}^3, P_1, g_1) where: $g_{11}^1 = -(x_2)^2$, $g_{22}^1 = 0$, $g_{33}^1 = 0$ and $g_{12}^1 = x_1x_2$, $g_{21}^1 = 0$, $g_{13}^1 = x_1x_3$, $g_{31}^1 = 0$, $g_{23}^1 = x_2x_3$, $g_{32}^1 = 0$.

(b) The associated differential system is given by:

$$(3.4) \quad \begin{cases} \dot{x}_1 = x_2x_3 + x_1((x_2)^2 + (x_3)^2) \\ \dot{x}_2 = -x_1x_3 + x_2x_3 \\ \dot{x}_3 = x_1x_2. \end{cases}$$

(c) The differential system (3.4) has the following stationary points: $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$, $e_3^m(0, 0, m)$.

(d) The matrix of the linear part of the system (3.3) in $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$, resp. in $e_3^m(0, 0, m)$ is given by:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -m \\ 0 & m & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} m^2 & 0 & m \\ 0 & 0 & m \\ m & 0 & 0 \end{bmatrix}, \quad \text{resp.} \quad A_3 = \begin{bmatrix} m^2 & m & 0 \\ -m & m & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) The characteristic equation of A_1 for (3.4) in $e_1^m(m, 0, 0)$ is: $\lambda(\lambda^2 + m^2) = 0$.

(f) The characteristic equation of A_2 for (3.4) in $e_2^m(0, m, 0)$ is: $\lambda(\lambda^2 - \lambda m^2 - m^2) = 0$ and so, it can be easily seen that $e_2^m(0, m, 0)$ are unstable.

(g) The characteristic equation of A_3 for (3.4) in $e_3^m(0, 0, m)$ is: $\lambda(\lambda^2 - \lambda(m + m^2) + m^2) = 0$. \square

Remark 3.3. In an analogous way we can discuss the metriplectic realization of first kind and second kind of the other two Hamilton-Poisson realization.

4 The differential systems with distributed delay

Let us consider the product $\mathbb{R}^3 \times \mathbb{R}^3 = \{(\tilde{x}, x) \mid \tilde{x} \in \mathbb{R}^3, x \in \mathbb{R}^3\}$ and the canonical projections $\pi_i : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $i = 1, 2$. A vector field $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$, satisfying the condition $X(\pi_1^* f) = 0$, for any $f \in \mathbf{C}^\infty(\mathbb{R}^3)$, is given by:

$$(4.1) \quad X(\tilde{x}, x) = \sum_{i=1}^n X_i(\tilde{x}, x) \frac{\partial}{\partial x_i}.$$

The differential system associated to X is given by: $\dot{x}_i(t) = X(\tilde{x}, x)$, $i = 1, 2, 3$.

A differential system with distributed delay is a differential system associated to a vector field $X \in \mathcal{X}(\mathbb{R}^3 \times \mathbb{R}^3)$ for which $X(\pi_1^* f) = 0$, $\forall f \in \mathbf{C}^\infty(\mathbb{R}^3)$, and it is given by (4.1) where $\tilde{x}(t)$ is: $\tilde{x}(t) = \int_0^\infty k(s)x(t-s)ds$

$k(s)$ is a distribution density. In the following we will consider the following densities:

$$1. \text{ uniform: } k_\tau^N(s) = \begin{cases} 0, & 0 \leq s \leq a \\ \frac{1}{\tau}, & a \leq s \leq a + \tau \\ 0, & s > a + \tau. \end{cases} \quad \text{where } a > 0, \tau > 0 \text{ are fixed numbers.}$$

2. exponential: $k_\alpha(s) = \alpha e^{-\alpha s}$, $\alpha > 0$;
3. Erlang: $k_\alpha(s) = \alpha^2 s e^{-\alpha s}$, $\alpha > 0$;
4. Dirac: $k_\alpha(s) = \delta(s - \tau)$, $\tau > 0$;

The differential equations with distributed delay for Rabinovich system are generated by an antisymmetric tensor field P on $\mathbb{R}^3 \times \mathbb{R}^3$ that satisfies the following relations: $P(\pi_1^* f_1, \pi_1^* f_2) = 0$, $P(\pi_2^* f_1, \pi_2^* f_2) = 0$ for all $f_1, f_2 \in \mathbf{C}^\infty(\mathbb{R}^3)$.

The differential equation with distributed delay is given by:

$$(4.2) \quad \dot{x}(t) = P(x(t), x(t)) \nabla_x h(\tilde{x}(t), x(t)),$$

where $\tilde{x}(t) = \int_0^\infty k(s)x(t-s)ds$, and $h \in \mathbf{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

$$\text{Let } P_0(x) = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}, P_1(\tilde{x}, x) = \begin{bmatrix} 0 & x_3 & -\tilde{x}_2 \\ -x_3 & 0 & 0 \\ \tilde{x}_2 & 0 & 0 \end{bmatrix},$$

$$P_2(\tilde{x}, x) = \begin{bmatrix} 0 & \tilde{x}_3 & -x_2 \\ -\tilde{x}_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}, P_3(\tilde{x}, x) = \begin{bmatrix} 0 & \tilde{x}_3 & -\tilde{x}_2 \\ -\tilde{x}_3 & 0 & 0 \\ \tilde{x}_2 & 0 & 0 \end{bmatrix} \text{ and we define}$$

$P(\tilde{x}, x) = \sum_{i=0}^3 \varepsilon_i P_i$, with $\varepsilon_i \geq 0$, $\sum_{i=0}^3 \varepsilon_i = 1$. Let $h_0(\tilde{x}, x) = \frac{1}{2}((x_1)^2 + (x_2)^2)$, $h_1(\tilde{x}, x) = \tilde{x}_1 x_1 + \frac{1}{2}(x_2)^2$, $h_2(\tilde{x}, x) = \frac{1}{2}(x_1)^2 + \tilde{x}_2 x_2$, $h_3(\tilde{x}, x) = \tilde{x}_1 x_1 + \tilde{x}_2 x_2$. We define $h(\tilde{x}, x) = \sum_{i=0}^3 \delta_i h_i$, with $\delta_i \geq 0$, $\sum_{i=0}^3 \delta_i = 1$. The Rabinovich differential equation with distributed delay is given by (4.2) with P and h given above, with initial value $x(s) = \phi(s)$, $s \in (-\infty, 0]$ where $\phi : (-\infty, 0] \rightarrow \mathbb{R}^3$, $\phi \in \mathbf{C}^\infty(\mathbb{R}^3)$.

In what follows we consider the functions $l \in \mathbf{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ given by: $l_0(\tilde{x}, x) = \frac{1}{2}((x_2)^2 + (x_3)^2)$, $l_1(\tilde{x}, x) = \tilde{x}_2 x_2 + \frac{1}{2}(x_3)^2$, $l_2(\tilde{x}, x) = \frac{1}{2}(x_2)^2 + \tilde{x}_3 x_3$, $l_3(\tilde{x}, x) = \tilde{x}_2 x_2 + \tilde{x}_3 x_3$.

We define $l(\tilde{x}, x) = \sum_{i=0}^3 \varepsilon_i l_i$, with $\varepsilon_i \geq 0$, $\sum_{i=0}^3 \varepsilon_i = 1$.

Proposition 4.1. • The function $l(\tilde{x}, x)$ satisfies the following relation:

$$(4.3) \quad \nabla_x l(\tilde{x}, x) P(\tilde{x}, x) \nabla_x f(\tilde{x}, x) = 0, \quad f \in \mathbf{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3);$$

- The revised differential equations with distributed delay satisfies the following relation:

$$(4.4) \quad \dot{x}(t) = P(\tilde{x}, x) \nabla_x h(\tilde{x}, x) + g(\tilde{x}, x) \nabla_{\tilde{x}} l(\tilde{x}, x)$$

where $\tilde{x}(t) = \int_0^\infty k(s)x(t-s)ds$ and $g(\tilde{x}, x)$ is a 2-tensor field given by: $g(\tilde{x}, x) =$

$$(g_{ij}(\tilde{x}, x)), g_{ij}(\tilde{x}, x) = \frac{\partial h(\tilde{x}, x)}{\partial x_i} \frac{\partial h(\tilde{x}, x)}{\partial x_j}, i \neq j, g_{ij}(\tilde{x}, x) = -\sum_{k=i, k \neq j} \left(\frac{\partial h(\tilde{x}, x)}{\partial x_k} \right)^2. \quad \square$$

The revised Rabinovich system with distributed delay has the following form:

$$(4.5) \quad \begin{cases} \dot{x}_1 = (\alpha_1 x_3 + \alpha_4 \tilde{x}_3)(\beta_1 x_2 + \beta_4 \tilde{x}_2) + (\beta_2 x_1 + \beta_3 \tilde{x}_1)(\beta_1 x_2 + \beta_4 \tilde{x}_2) \alpha_2 x_3 \\ \dot{x}_2 = -(\alpha_1 x_3 + \alpha_4 \tilde{x}_3)(\beta_2 x_1 + \beta_3 \tilde{x}_1) - (\beta_2 x_1 + \beta_3 \tilde{x}_1) \alpha_3 x_3 \\ \dot{x}_3 = \alpha_2 x_2 + \alpha_3 \tilde{x}_2)(\beta_2 x_1 + \beta_3 \tilde{x}_1) - (\beta_2 x_1 + \beta_3 \tilde{x}_1) \alpha_4 x_3 - (\beta_1 x_2 + \beta_4 \tilde{x}_2)^2 \alpha_3 x_3 \end{cases}$$

where we considered the following notations: $\alpha_1 = \varepsilon_0 + \varepsilon_1$, $\alpha_2 = \varepsilon_0 + \varepsilon_2$, $\alpha_3 = \varepsilon_1 + \varepsilon_2$, $\alpha_4 = \varepsilon_1 + \varepsilon_3$, $\alpha_5 = \varepsilon_3 + \varepsilon_2$, $\beta_1 = \delta_0 + \delta_1$, $\beta_2 = \delta_0 + \delta_2$, $\beta_3 = \delta_1 + \delta_3$, $\beta_4 = \delta_2 + \delta_3$.

Remark 4.2. *The analysis of stationary points of the system (4.5) is quite difficult, that is why we will present the main results for fractional Rabinovich differential system.*

5 Fractional Rabinovich differential systems

Generally speaking, there are three mostly used definitions for fractional derivatives, i.e. Grünwald-Latnikov fractional derivatives, Riemann-Liouville fractional derivatives and Caputo's fractional derivatives, ([1],[5]). Here we discuss Caputo derivative: $D_t^\alpha x(t) = I^{m-\alpha} \left(\frac{d}{dt} \right)^m x(t)$, $\alpha > 0$ where $m-1 < \alpha \leq m$, $m \geq 1$, $\left(\frac{d}{dt} \right)^m = \frac{d}{dt} \circ \dots \circ \frac{d}{dt}$, I^β is the β^{th} order Riemann-Liouville integral operator, which is expressed in the following manner: $I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) ds$, $\beta > 0$.

In this paper we consider that $\alpha \in (0, 1)$. A fractional system of differential equations with distributed delay in \mathbb{R}^3 is given by:

$$(5.1) \quad D_t^\alpha x(t) = X(x(t), \tilde{x}(t)), \quad \alpha \in (0, 1)$$

where $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$.

The matrix associated to the linear part of the system (5.1) in the stationary point x_0 is given by the linear fractional differential system:

$$(5.2) \quad D_t^\alpha u(t) = Au(t) + B\tilde{u}(t),$$

where $A = \left(\frac{\partial X}{\partial x} \right) \Big|_{x=x_0}$ and $B = \left(\frac{\partial X}{\partial \tilde{x}} \right) \Big|_{x=x_0}$.

The characteristic equation of (5.2) is:

$$(5.3) \quad \Delta(\lambda) = \det(\lambda^\alpha - A - k^1(\lambda)B)$$

where $k^1(\lambda) = \int_0^\infty k(s)e^{-\lambda s} ds$ and k is given by 1-4 in the previous section.

Let us consider a fractional 2-tensor field $P^\alpha \in \mathcal{X}^\alpha(\mathbb{R}^3) \times \mathcal{X}^\alpha(\mathbb{R}^3)$ and $d^\alpha f, d^\alpha g \in \mathcal{D}(\mathbb{R}^3)$. The bilinear map $[\cdot, \cdot]^\alpha : \mathbf{C}^\infty(\mathbb{R}^3) \times \mathbf{C}^\infty(\mathbb{R}^3) \longrightarrow \mathbf{C}^\infty(\mathbb{R}^3)$ defined by: $[f, g]^\alpha = B^\alpha(d^\alpha f, d^\alpha g)$, $f, g \in \mathbf{C}^\infty(\mathbb{R}^3)$ is called the fractional Leibniz bracket. If P^α is skew-symmetric, we say that $(\mathbb{R}^3, [\cdot, \cdot]^\alpha)$ is a fractional almost Poisson manifold. For $h \in \mathbf{C}^\infty(\mathbb{R}^3)$ the fractional almost Poisson dynamical system is given by:

$$(5.4) \quad D_t^\alpha x_i(t) = [x_i(t), h(t)]^\alpha, \quad [x_i, h]^\alpha = \sum_{i,j=1}^3 P_{ij}^\alpha D_{x_j}^\alpha.$$

Let P^α be a skew-symmetric fractional 2-tensor field and a symmetric fractional 2-tensor field g^α on \mathbb{R}^3 . We define the bracket $[\cdot, \cdot]^\alpha : \mathbf{C}^\infty(\mathbb{R}^3) \times \mathbf{C}^\infty(\mathbb{R}^3) \longrightarrow \mathbf{C}^\infty(\mathbb{R}^3)$ by: $[f, g]^\alpha = P^\alpha(d^\alpha f, d^\alpha h) + g^\alpha(d^\alpha f, d^\alpha h)$, $f, h \in \mathbf{C}^\infty(\mathbb{R}^3)$.

The 4-tuple $(\mathbb{R}^3, P^\alpha, g^\alpha, [\cdot, \cdot]^\alpha)$ is called fractional almost metric manifold. The fractional dynamical system associated to $h \in \mathbf{C}^\infty(\mathbb{R}^3)$ and

$$(5.5) \quad D_t^\alpha x_i(t) = [x_i(t), h(t)]^\alpha, \quad [x_i, h]^\alpha = \sum_{i,j=1}^3 P_{ij}^\alpha D_{x_j}^\alpha + \sum_{i,j=1}^3 g_{ij}^\alpha D_{x_j}^\alpha.$$

Proposition 5.1. 1. The fractional dynamical system (5.4) is given by:

$$(5.6) \quad \begin{cases} D_t^\alpha x_1(t) = x_2(t)x_3(t) \\ D_t^\alpha x_2(t) = -x_1(t)x_3(t) \\ D_t^\alpha x_3(t) = x_1(t)x_2(t). \end{cases}$$

2. The fractional dynamical system (5.5) is given by:

$$(5.7) \quad \begin{cases} D_t^\alpha x_1(t) = ((\alpha_1 x_3(t) + \alpha_4 \tilde{x}_3(t))(\beta_1 x_2(t) + \beta_4 \tilde{x}_2(t)) \\ \quad + (\beta_2 x_1(t) + \beta_3 \tilde{x}_1(t))(\beta_1 x_2(t) + \beta_4 \tilde{x}_2(t))\alpha_2 x_3(t) \\ D_t^\alpha x_2(t) = -(\alpha_1 x_3(t) + \alpha_4 \tilde{x}_3(t))(\beta_2 x_1(t) + \beta_3 \tilde{x}_1(t)) - (\beta_2 x_1(t) + \beta_3 \tilde{x}_1(t))\alpha_3 x_3(t) \\ D_t^\alpha x_3(t) = (\alpha_2 x_2(t) + \alpha_3 \tilde{x}_2(t))(\beta_2 x_1(t) + \beta_3 \tilde{x}_1(t)) \\ \quad - (\beta_2 x_1(t) + \beta_3 \tilde{x}_1(t))\alpha_4 x_3(t) - (\beta_1 x_2(t) + \beta_4 \tilde{x}_2(t))^2 \alpha_3 x_3(t). \end{cases}$$

3. The fractional dynamical systems (5.6) and (5.7) have the stationary points $e_1^m(m, 0, 0)$, $e_2^m(0, m, 0)$ and $e_3^m(0, 0, m)$, $m \in \mathbb{R}$.

4. The characteristic equations for (5.6), resp (5.7) are given by:

$$\begin{aligned} (i) \quad & e_1^m(m, 0, 0) : \lambda^\alpha(-\lambda^{2\alpha} + m^2(m+1)) = 0, \text{ resp} \\ & \lambda^\alpha(\lambda^{2\alpha} + a\lambda^\alpha + be^{-\lambda^\alpha\tau} + c) = 0, \quad a, b, c \in \mathbb{R}; \\ (ii) \quad & e_2^m(0, m, 0) : \lambda^\alpha(\lambda^{2\alpha} + m^2\lambda^2 - m^2) = 0, \\ & \text{resp } \lambda^\alpha(\lambda^{2\alpha} + a_1\lambda^\alpha + b_1e^{-\lambda^\alpha\tau} + c_1e^{-2\lambda^\alpha\tau}) = 0, \quad a_1, b_1, c_1 \in \mathbb{R}; \\ (iii) \quad & e_3^m(0, 0, m) : \lambda^\alpha(\lambda^{2\alpha} + m^2) = 0, \text{ resp} \\ & \lambda^\alpha(\lambda^{2\alpha} + a_2\lambda^\alpha + b_2e^{-\lambda^\alpha\tau} + c_2e^{-2\lambda^\alpha\tau} + d_2) = 0, \quad a_2, b_2, c_2, d_2 \in \mathbb{R}. \square \end{aligned}$$

We can integrate numerically the set of differential equations (5.6) with the Adams-Bashforth-Moulton method. We consider a graphic representation of Moulton method, for our system (1.1), for the following two cases: $\alpha = 0.8$ (Figure 1) and $\alpha = 1$ (Figure 2) for $x_1(0) = 0.001$, $x_2(0) = 0.001$, $x_3(0) = 6$.

Figure 1 alpha=0.8

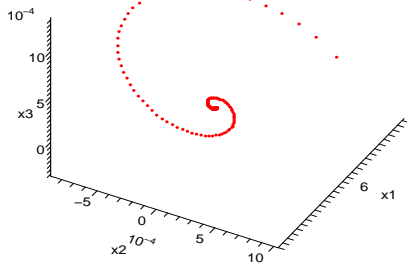
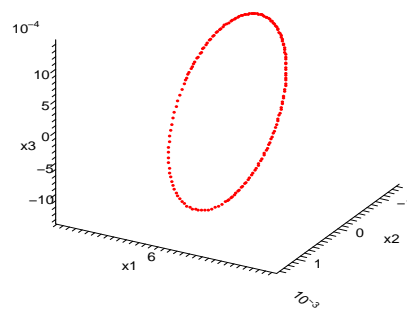


Figure 2 alpha=1



6 Conclusions

Until now we have an approach and also some solutions for metriplectic manifolds of the first and second kind, and also differential systems with distributed delay (for the first case presented here), and Rabinovich fractional differential system, with Dirac distribution ($\tilde{x}(t) = x(t - \tau)$). What we want to continue is to apply all other distributions for our three cases.

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