Conformally flat 3-τ-a manifolds

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Abstract. In this paper, generalizing the results of [5] and [9], we investigate conformally flat 3-τ-a manifolds. We find a new class of contact metric manifolds whose non-compact examples have already been constructed by D. E. Blair ([3]) and we find out compact examples. We also give a new example of 3-dimensional contact metric manifold in paragraph 8.

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1 Introduction

It is well known that the tensors $\tau = L_\xi g$ and $\nabla_\xi \tau$ on a contact metric manifold (M, η, g, ϕ, ξ), introduced by S. S. Chern and R. S. Hamilton [4], play a fundamental role in the study of the geometry of (M, η, g, ϕ, ξ). The condition

$$\nabla_\xi \tau = 2a\tau\phi, \quad a = \text{constant} \tag{1}$$

where the composition $(\tau\phi)(X, Y)$ has to be interpreted as $\tau(\phi X, Y)$, is necessary condition for a contact metric 3-manifold to be homogeneous. If $a = 0$ then this condition is equivalent to the condition that the sectional curvatures of all planes, at a given point, perpendicular to the contact subbundle, are equal [8]. These manifolds are said to be 3-τ manifolds [8]. Known examples of 3-τ manifolds are the ones satisfying the stronger condition that their Ricci operator Q commutes with ϕ (η-Einstein). D. E. Blair [1] gave the first example of contact metric 3-τ manifolds with $Q\phi \neq \phi Q$, where the eigenvalue λ of the operator $h(:= \frac{1}{2} L_\xi \phi)$ is a non-constant function. G. Calvaruso and D. Perrone, also, gave examples with λ=constant [6].

The classification of conformally flat contact metric manifolds is an interesting problem which has been investigated by many authors. G. Calvaruso [5] proved that a conformally flat contact metric 3-manifold satisfying (1) has constant sectional curvature 0 or 1.

We call 3-τ-a manifold a 3-dimensional contact metric manifold satisfying $\nabla_\xi \tau = 2a\tau\phi$, where a is an arbitrary smooth function.

A nowhere zero, smooth vector field X on a 3-dimensional manifold M is called *Beltrami field* if it satisfies the equation curl$X = fX$, where f is a smooth function. If f is nowhere zero, then X is called *rotational* Beltrami field. Beltrami fields

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arise in several contexts: They are steady (i.e. time-independent) solutions to the Euler's equations of motion for an inviscid (i.e. without viscocity or dissipation) incompressible (i.e. volume preserving) fluid. They yield steady solutions to the ideal magnetohydrodynamics equations [7]. They are extrema of the L^2 energy functional $\|x\|_2 := \frac{1}{2}\int_M \|x\|^2 d\mu$ under the action of the volume - preserving diffeomorphism group of M. Eigenfields of curl having the smallest nonzero eigenvalue globally minimize the energy. They also play a role in the analysis of the stability of matter and in the formation of dynamos [7]. Etnyre and Ghrist in [7] proved that given a Reeb field ξ on a 3-dimensional contact manifold M^3, every vector field $\sigma\xi$, σ being a smooth positive function, is a smooth rotational Beltrami field on M^3. Blair in [3] proved that a solution of the equation curl$X = |X|\, X$, X being a nowhere zero, smooth vector field on a 3-dimensional manifold, is equivalent to the existence of a conformally flat contact metric structure.

In the present paper we investigate conformally flat 3-τ-a manifolds, where a and the scalar curvature S, are smooth functions constant along the flow of ξ. We prove the following

Theorem 1 (MAIN THEOREM) *Let M^3 be a conformally flat 3-τ-a manifold with a and S smooth functions, constant along the flow of ξ. Then M^3 is either flat or Sasakian with constant curvature 1 or a semi-K contact manifold. Furthermore a close semi-K contact manifold is a non-trivial torus bundle over the circle.*

D. E. Blair in [3] constructed examples of non-compact conformally flat, contact metric 3-manifolds with non-constant curvature. The analysis involved in [3] is interesting in its own right and gives another solution to the standard "force free" model equations of solar physics. In our paper we prove (paragraph 7) that these examples are semi-K contact manifolds.

We also construct compact examples (based on Blair's ones) of the same class.

In view of the result of the main Theorem we may ask if the above Theorem is true without the hypothesis of conformal flatness. The answer is no, because in paragraph 8 we give an example which is 3-τ-a manifold and neither conformally flat nor semi-K contact.

2 Preliminaries

A differentiable $(2n+1)$-dimensional manifold M is called *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ (called the *characteristic vector field* or *Reeb* field) such that $(d\eta)(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle defined by $\eta = 0$, one obtains a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$(d\eta)(X,Y) = g(\phi X, Y), \quad \eta(X) = g(X,\xi), \quad \phi^2 = -I + \eta \otimes \xi.$$

g is called *associated metric* of η, and (ϕ, η, ξ, g) a *contact metric structure* ([3] p.17). A differentiable $(2n+1)$-dimensional manifold M equipped with a contact metric structure is called *contact metric* $(2n+1)$-manifold.

On a contact metric manifold we have

$$\nabla_X \xi = -\phi X - \phi h X, \quad \nabla_\xi \phi = 0,$$

where ∇ is the *Levi-Civita* connection of g.

In the theory of contact metric manifolds there are two tensor fields h and l which play a fundamental role. These tensor fields are defined by

$$h = \frac{1}{2} L_\xi \phi, \quad l = R(\cdot, \xi)\,\xi,$$

where R and L are the Riemannian curvature tensor and Lie derivative respectively. h is self-adjoint and satisfies:

$$\text{(1)} \qquad h\xi = 0, \quad Trh = 0, \quad Tr(h\phi) = 0, \quad h\phi = -\phi h, \quad \eta \circ h = 0,$$
$$\phi l \phi - l = 2\left(\phi^2 + h^2\right), \quad Trl = g\left(Q\xi, \xi\right) = 2n - Trh^2.$$

$h = 0$ if and only if ξ is Killing and M is called K-*contact*.

The Ricci tensor of type $(2,0)$, the corresponding endomorphism field and the scalar curvature are respectively denoted by ρ, Q and S. Moreover, we consider the tensor $\tau = L_\xi g$. The tensors h and τ are symmetric and satisfy:

$$\text{(2)} \qquad (\nabla_\xi h)\,\phi = -\phi(\nabla_\xi h), \quad \nabla_\xi h = \phi - \phi l - \phi h^2, \quad (\nabla_\xi h)\,\xi = 0$$
$$\tau = 2g\left(\phi., h.\right), \quad \nabla_\xi \tau = 2g\left(\phi, \nabla_\xi h\right).$$

If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta\left(X\right)\frac{d}{dt}\right)$$

is integrable, M is said to be *Sasakian*. A Sasakian manifold is K-contact and the converse is true for 3-dimensional spaces.

The sectional curvature of a plane section containing ξ is called ξ-*sectional curvature*. We denote the *contact subbundle* or *contact distribution* defined by the subspace $\{X \in T_pM : \eta(X) = 0\}$ by D. If $X \in D$ we denote the ξ-*sectional curvature* by $K(X, \xi)$. The sectional curvature $K(X, \phi X)$ of a plain section spanned by the vector fields X (orthogonal to ξ) and ϕX is called ϕ-*sectional curvature*.

In what follows, let $M^3 \equiv (M^3, \phi, \eta, \xi, g)$ be a 3-dimensional contact metric manifold (*contact metric* 3-*manifold*) and m a point of M^3. Let U_1 be the open subset of M^3 where $h \neq 0$ and U_0 the open subset of points $m \in M^3$ such that $h = 0$ in a neighborhood of $m \in M^3$. Then, $U_1 \cup U_0$ is an open and dense subset of M^3. For every point $m \in U_0 \cup U_1$ there exists a local orthonormal basis $\{e, \phi e, \xi\}$ (called ϕ-*basis*) of smooth eigenvectors of h in a neighborhood of m. On U_1 we put $h\,e = \lambda\,e$, where λ is a non-vanishing smooth function. We can, easily, prove that $h\,\phi e = -\lambda\,\phi e$.

It is well known that on every 3-dimensional Riemannian manifold the curvature $R(X,Y)\,Z$ ($\equiv [\nabla_X, \nabla_Y]\,Z - \nabla_{[X,Y]}Z$) is given by

$$\text{(3)} \qquad R(X,Y)\,Z = g(Y,Z)\,QX - g(X,Z)\,QY + g(QY,Z)\,X - g(QX,Z)\,Y$$
$$- \frac{S}{2}\left[g(Y,Z)\,X - g(X,Z)\,Y\right]$$

where X, Y and Z are arbitrary vector fields on M^3.

For a ϕ-basis $\{e, \phi e, \xi\}$ we have:

Lemma 2 *On a contact metric* 3*-manifold we have:*

$$\begin{aligned}
&\nabla_e e = b\,\phi e, \quad \nabla_e \phi e = -b\,e + (\lambda+1)\ \xi, \quad \nabla_e \xi = -\,(\lambda+1)\ \phi e \\
&\nabla_{\phi e} e = -c\,\phi e + (\lambda-1)\ \xi, \quad \nabla_{\phi e}\phi e = c\,e, \quad \nabla_{\phi e}\xi = (1-\lambda)\ e \\
&\nabla_\xi e = -d\,\phi e, \quad \nabla_\xi \phi e = d\,e
\end{aligned}$$

All manifolds are assumed to be connected and smooth.

3 3-τ-a manifolds

Using (1) and (2) we can prove: $l = -\phi^2 - h^2 - 2dh$. This relation and straightforward calculation of l for a ϕ-basis yield $d = -a$. So we have [5]:

Proposition 3 *Let M^3 be a contact metric* 3*-manifold. Then, on M^3 we have $\nabla_\xi \tau = 0$ if and only if $a = \xi(\lambda) = 0$, while $\nabla_\xi \tau = 2a\tau\phi$ if and only if $\xi(\lambda) = 0$.*

Lemma 2 can be restated as follows:

Lemma 4 *On a contact metric* 3*-τ-a manifold we have:*

$$\begin{aligned}
&\nabla_e e = b\phi e, \quad \nabla_e \phi e = -be + (\lambda+1)\,\xi, \quad \nabla_e \xi = -(\lambda+1)\,\phi e \\
&\nabla_{\phi e} e = -c\phi e + (\lambda-1)\xi, \quad \nabla_{\phi e}\phi e = ce, \quad \nabla_{\phi e}\xi = (1-\lambda)e \\
&\nabla_\xi e = a\phi e, \quad \nabla_\xi \phi e = -ae
\end{aligned}$$

From $\xi(\lambda) = 0$ we obtain that on a 3-τ-a manifold we have:

(1) $$\xi\left(e\left(\lambda\right)\right) = \left(a+\lambda+1\right)\ \phi e\left(\lambda\right), \quad \xi\left(\phi e\left(\lambda\right)\right) = \left(-a+\lambda-1\right) e\left(\lambda\right)$$

Relation (3) and a straightforward calculation of $R(X,Y)Z$ if X, Y and Z belong to a ϕ-basis yield

(2) $$\begin{aligned}
Qe &= \left(\frac{S}{2} + \lambda^2 - 2\lambda a - 1\right) e + \left[2\lambda b - \phi e\left(\lambda\right)\right]\ \xi \\
Q\phi e &= \left(\frac{S}{2} + \lambda^2 + 2\lambda a - 1\right) \phi e + \left[2\lambda c - e\left(\lambda\right)\right]\ \xi \\
Q\xi &= \left[2\lambda b - \phi e\left(\lambda\right)\right]\ e + \left[2\lambda c - e\left(\lambda\right)\right]\ \phi e + 2\left(1-\lambda^2\right)\ \xi
\end{aligned}$$

(3) $$e\left(c\right) + \phi e\left(b\right) = \frac{S}{2} + b^2 + c^2 - 2a + \lambda^2 - 1$$

From Lemma 4 we have:

(4) $$[e,\xi] = -\left(a+\lambda+1\right)\ \phi e$$

(5) $$[\phi e,\xi] = \left(a-\lambda+1\right)\ e$$

(6) $$[e,\phi e] = -b\,e + c\,\phi e + 2\,\xi$$

It is well known that on every contact metric 3-manifold we have:

$$(\nabla_e Q)\ e + (\nabla_{\phi e} Q)\ \phi e + (\nabla_\xi Q)\ \xi = \frac{1}{2} \operatorname{grad} S$$

The above condition using (2) implies:

$$\xi\,(b) - e\,(a) + e\,(\lambda) = (-a + \lambda - 1)\ c \tag{7}$$

$$\xi\,(c) + \phi e\,(a) + \phi e\,(\lambda) = (a + \lambda + 1)\ b \tag{8}$$

Definition 5 *Let M^3 be a 3-dimensional contact metric manifold and $h = \lambda h^+ - \lambda h^-$ the spectral decomposition of h on U_1. If*

$$\nabla_{h^- X} h^- X = \left[\xi, h^+ X\right] \tag{9}$$

for all vector fields X on M^3 and all points of an open subset W of U_1 and $h = 0$ on the points of M^3 which do not belong to W, then the manifold is said to be semi-K contact manifold.

Remark 6 *From Lemma 2 and relations (4), (5), (6) the condition (9) for $X = e$ leads to $[\xi, e] = 0$ while for $X = \phi e$ leads to $\nabla_{\phi e} \phi e = 0$. Hence on a semi-K contact manifold we have $a + \lambda + 1 = c = 0$. If we apply the deformation $e \to \phi e$, $\phi e \to e$, $\xi \to -\xi$, $\lambda \to -\lambda$, $b \to c$, $c \to b$, then the contact structure remains the same. So the condition for a three-dimensional contact metric manifold to be semi-K contact is equivalent to $a - \lambda + 1 = b = 0$. On the other hand, if on a 3-dimensional contact metric manifold the relation $\nabla_{h^+ X} h^+ X = [\xi, h^- X]$ holds, then applying Lemma 2 and relations (4), (5), (6) we have $a - \lambda + 1 = b = 0$.*

4 Some lemmas on conformally flat 3-τ-a manifolds

A 3-dimensional Riemannian manifold is conformally flat if and only if the Ricci operator satisfies:

$$(\nabla_X Q)\, Y - (\nabla_Y Q)\, X = \frac{1}{4}\left[X(S)Y - Y(S)X\right]$$

for all vector fields X and Y.

In what follows we assume that M^3 is a conformally flat 3-τ-a manifold whose a and S are smooth functions constant along the geodesic foliation generated by ξ. This class of manifolds is denoted by $\mathcal{B}$. If X and Y belong to a ϕ-basis, then the above equation yields:

$$\frac{e\,(S)}{4} = -2\,\lambda\, e\,(a) - (2a + 3\lambda + 3)\ e\,(\lambda) + 2\,\lambda\,(2a + \lambda + 3)\ c \tag{1}$$

$$\frac{\phi e\,(S)}{4} = 2\,\lambda\,\phi e\,(a) + (2a - 3\lambda + 3)\ \phi e\,(\lambda) + 2\,\lambda\,(-2a + \lambda - 3\,)\, b \tag{2}$$

$$e(\phi e(\lambda)) = 2\,\lambda\, e(b) + 3\, b\, e(\lambda) - 2\,\lambda\, b\, c \tag{3}$$

$$\phi e(e(\lambda)) = 2\,\lambda\, \phi e(c) + 3\, c\, \phi e(\lambda) - 2\,\lambda\, b\, c \tag{4}$$

$$\begin{aligned} e(e(\lambda)) = {} & 2\,\lambda\, e(c) + 2\, c\, e(\lambda) - b\, \phi e(\lambda) + \frac{\lambda+1}{2} S + \\ & + \lambda\,(4a^2 + 2b^2 + 2\lambda a + 2a + 3\lambda^2 + 3\lambda - 3) - 3 \end{aligned} \tag{5}$$

$$\begin{aligned} \phi e(\phi e(\lambda)) = {} & -2\,\lambda\, e(c) - c\, e(\lambda) + 2\, b\, \phi e(\lambda) + \frac{3\lambda-1}{2} S + \\ & + \lambda\,(4\,a^2 + 2\,b^2 + 4\,c^2 - 2\,\lambda\, a - 2\,a + 5\,\lambda^2 - 3\,\lambda - 5) - 3 \end{aligned} \tag{6}$$

Before proving the main theorem, we shall prove some essential Lemmas. The proofs of the Lemmas 7 and 8 being elementary, are omitted.

Lemma 7 *Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth real function. Let U be an open subset of M and U_1, U_2 open subsets of U defined by*

$$U_1 = \{m \in U : f(m) = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in U : f(m) \neq 0 \text{ in a neighborhood of } m\}$$

Then $U_1 \cup U_2$ is open and dense in the closure of U.

Lemma 8 *Let M be a smooth manifold and $f, g : M \to R$ be smooth real functions. Let U be an open subset of M and U_1, U_2, U_3 open subsets of U defined by*

$$U_1 = \{m \in U : f(m) = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in U : g(m) = 0 \text{ in a neighborhood of } m\}$$

$$U_3 = \{m \in U : f(m)\, g(m) \neq 0 \text{ in a neighborhood of } m\}$$

Then $U_1 \cup U_2 \cup U_3$ is open and dense in the closure of U.

Lemma 9 *Let M be a 3-dimensional contact metric manifold. If M has constant sectional curvature 1 then M is Sasakian. Conversely if M is Sasakian and $S = 6$ then M has constant sectional curvature 1. In particular if M is Sasakian and conformally flat then $S = 6$.*

Proof. A calculation of the sectional curvature on a contact metric 3-dimensional manifold yields $K(e, \phi e) = 2\left(\lambda^2 - 1\right) + \frac{S}{2}$, $K(e, \xi) = 1 - 2a\lambda - \lambda^2$, $K(\phi e, \xi) = 1 + 2a\lambda - \lambda^2$. Suppose first that the sectional curvature of M is 1. Then $\lambda = 0$ hence M is Sasakian. Conversely if M is Sasakian then $\lambda = 0$. Hence $K(e, \xi) = K(\phi e, \xi) = 1$ and $K(e, \phi e) = -2 + \frac{S}{2}$. If $S = 6$ then $K(e, \phi e) = 1$. From (5) we have that if M is conformally flat, then $S = 6$. ☐

Lemma 10 *If $M^3 \in \mathcal{B}$ and $b = c = 0$ in an open subset U of M^3, then U is flat or Sasakian with constant curvature 1.*

Proof. From (7) and (8) we have

$$e\,(a) = e\,(\lambda)\,,\ \ \phi e\,(a) = -\phi e\,(\lambda) \tag{7}$$

Because of the above relations and (3), relations (1) and (2) imply:

$$(a + 2\lambda + 2)\,e\,(a) = 0, \qquad (a - 2\lambda + 2)\,\phi e\,(a) = 0 \tag{8}$$

Consider the following open subsets of U:

$$U_1 = \{m \in U : a + 2\lambda + 2 = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in U : a - 2\lambda + 2 = 0 \text{ in a neighborhood of } m\}$$

$$U_3 = \{m \in U : (a + 2\lambda + 2)\,(a - 2\lambda + 2) \neq 0 \text{ in a neighborhood of } m\}$$

From Lemma 8 we have that $U_1 \cup U_2 \cup U_3$ is open and dense in the closure of U. (7) implies that a and λ are constant on U_1 and on U_2. On U_3 because of (8) we have $e\,(a) = \phi e\,(a) = 0$, therefore a and λ are constant. Hence a and λ are constant on $U_1 \cup U_2 \cup U_3$ and hence on U. From [5] (Theorem 3), we know that U has constant curvature 0 or 1. From Lemma 9 we conclude that U is flat or Sasakian with constant curvature 1. □

Lemma 11 *If $M^3 \in \mathcal{B}$ and Trl =constant on an open subset U of M^3, then U is flat or Sasakian with constant curvature* 1.

Proof. The condition Trl =constant is equivalent to λ =constant. If $\lambda = 0$ then Lemma 9 implies that U is Sasakian with constant curvature 1. Thus we suppose that $\lambda \neq 0$ on U. We will prove that in this case U is flat and hence U is flat or Sasakian space form. The relations (1) and (2) and $\lambda = cons\tan t$ yield:

$$\frac{e(S)}{4} = 2\,\lambda\,\left[-e(a) + (2a + \lambda + 3)\ c\right] \tag{9}$$

$$\frac{\phi e(S)}{4} = 2\,\lambda\,\left[\phi e(a) - (2a - \lambda + 3)\,b\right] \tag{10}$$

The relations (4) and (5) imply:

$$\xi\,(e\,(a)) = (a + \lambda + 1)\,\phi e\,(a)\,, \quad \xi\,(\phi e\,(a)) = -\,(a - \lambda + 1)\,e\,(a)$$

Using the above equations, (7), (8), (1) and (2), the differentiations of (9) and (10) with respect to ξ respectively yield:

$$\begin{aligned}(4a + 3\lambda + 5)\,\phi e(a) &= 2(a + \lambda + 1)(2a + 3)b \\ (4a - 3\lambda + 5)\,e(a) &= 2(a - \lambda + 1)(2a + 3)c\end{aligned} \tag{11}$$

Consider the open sets

$$U_1 = \{m \in U : 4a + 3\lambda + 5 = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in U : 4a - 3\lambda + 5 = 0 \text{ in a neighborhood of } m\}$$

$$U_3 = \{m \in U : (4a + 3\lambda + 5)\,(4a - 3\lambda + 5) \neq 0 \text{ in a neighborhood of } m\}\,.$$

From Lemma 8 we obtain that $U_1 \cup U_2 \cup U_3$ is open and dense in the closure of U. In U_1 and in U_2 a is constant and from [5] we have that U_1 and U_2 have constant curvatures 0 or 1. Since $\lambda \neq 0$, U_1 and U_2 are flat. On U_3, the relations (7), (8), (9) and (10) because of (11) can be written:

$$\xi(b) = \frac{(3\lambda + 1)(a - \lambda + 1)}{4a - 3\lambda + 5} \quad , \quad \xi(c) = \frac{(3\lambda - 1)(a + \lambda + 1)}{4a + 3\lambda + 5}. \tag{12}$$

$$\frac{e(S)}{4} = \frac{2\lambda c}{4a - 3\lambda + 5}[(2a + \lambda + 3)(4a - 3\lambda + 5) - 2(2a + 3)(a - \lambda + 1)] \tag{13}$$

$$\frac{\phi e(S)}{4} = \frac{2\lambda b}{4a + 3\lambda + 5}[-(2a - \lambda + 3)(4a + 3\lambda + 5) + 2(2a + 3)(a + \lambda + 1)] \tag{14}$$

Differentiating (13) and (14) with respect to ξ and using (12) we obtain:

$$b = 0 \text{ or } P = 0 \text{ or } a + \lambda + 1 = 0 \tag{15}$$

and

$$c = 0 \text{ or } Q = 0 \text{ or } a - \lambda + 1 = 0 \tag{16}$$

where $P = 4a^3 - 2\lambda a^2 + 16a^2 + 21a - 5\lambda a - 3\lambda + 9$ and $Q = 4a^3 + 2\lambda a^2 + 16a^2 + 21a + 5\lambda a + 3\lambda + 9$. Consider the open sets

$$U_1' = \{m \in U_3 : b = c = 0 \text{ in a neighborhood of } m\}$$

$$U_2' = \{m \in U_3 : b = Q = 0 \text{ in a neighborhood of } m\}$$

$$U_3' = \{m \in U_3 : b = a - \lambda + 1 = 0 \text{ in a neighborhood of } m\}$$

$$U_4' = \{m \in U_3 : P = c = 0 \text{ in a neighborhood of } m\}$$

$$U_5' = \{m \in U_3 : P = Q = 0 \text{ in a neighborhood of } m\}$$

$$U_6' = \{m \in U_3 : P = a - \lambda + 1 = 0 \text{ in a neighborhood of } m\}$$

$$U_7' = \{m \in U_3 : a + \lambda + 1 = c = 0 \text{ in a neighborhood of } m\}$$

$$U_8' = \{m \in U_3 : Q = a + \lambda + 1 = 0 \text{ in a neighborhood of } m\}$$

$$U_9' = \{m \in U_3 : a + \lambda + 1 = a - \lambda + 1 = 0 \text{ in a neighborhood of } m\}$$

From Lemma 8 we have that $\bigcup_{i=1}^{9} U_i'$ is open and dense in the closure of U_3. Since we have already assumed that $\lambda \neq 0$ we have that $U_9' = \varnothing$. In U_i', $i = 2, \ldots, 8$ we can easily obtain that a is constant, since λ is constant. Hence from [5] we conclude that U_i', $i = 2, \ldots 8$ are flat. In U_1' we can apply Lemma 10 and thus we conclude that U_1' is flat too. Hence U_3 is flat. Since $U_1 \cup U_2 \cup U_3$ is open and dense in the closure of U it follows that U is flat. □

Lemma 12 *If $M^3 \in \mathcal{B}$ and $(2a+3\lambda+3)(2a-3\lambda+3)=0$ on an open subset U of M^3, then U is flat or Sasakian with constant curvature 1.*

Proof. Consider the following subsets of U:

$$G_1 = \{m \in U : \lambda = 0 \text{ in a neighborhood of } m\}$$

$$G_2 = \{m \in U : \lambda \neq 0 \text{ in a neighborhood of } m\}$$

Since $G_1 \in \mathcal{B}$ from Lemma 9 it follows that G_1 is Sasakian with constant curvature 1. On G_2 consider the following subsets:

$$U_1 = \{m \in G_2 : 2a+3\lambda+3 = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in G_2 : 2a-3\lambda+3 = 0 \text{ in a neighborhood of } m\}$$

It is easy to see that $U_1 \cup U_2$ is open and dense in the closure of G_2. We will consider only the case of U_1 because the case of U_2 is similar. The equations (7) and (8) take the forms:

$$(17) \qquad 2\xi(b) = -5e(\lambda) + (5\lambda+1)c, \quad 2\xi(c) = \phi e(\lambda) - (\lambda+1)b$$

The relations (1) and (2) can be written:

$$(18) \qquad e(S) = -4\lambda\,(4c\lambda - 3e(\lambda)), \quad \phi e(S) = 4\lambda\,(8b\lambda - 9\phi e(\lambda))$$

Differentiating the above equations with respect to ξ and using (1), (4) and (5) we obtain:

$$(19) \qquad \lambda\,(3+4\lambda)\,\phi e(\lambda) = 3b\lambda^2\,(1+\lambda), \quad \lambda\,(3+25\lambda)\,e(\lambda) = 3c\lambda^2\,(1+5\lambda)$$

Consider the subsets of U_1:

$$V_1 = \{m \in U_1 : 4\lambda+3 = 0 \text{ in a neighborhood of } m\}$$

$$V_2 = \{m \in U_1 : 4\lambda+3 \neq 0 \text{ in a neighborhood of } m\}$$

$$V_3 = \{m \in U_1 : 25\lambda+3 = 0 \text{ in a neighborhood of } m\}$$

$$V_4 = \{m \in U_1 : 25\lambda+3 \neq 0 \text{ in a neighborhood of } m\}$$

In V_1 and in V_3 λ is constant. Hence from Lemma 11 we obtain that V_1 and V_3 have constant curvature 0 or 1. Now, in $V_2 \cup V_4$ we differentiate (19) with respect to ξ and make use of (4), (5) and (17). Then we obtain

$$c\lambda^2\,(5\lambda+3)\,(5\lambda+1) = 0, \quad b\lambda^2\,(1+\lambda)\,(5\lambda+3) = 0$$

Taking account the above, in $V_2 \cup V_4$ we can apply a similar argument to that of the proof of Lemma 11, especially from equations (15), (16) and beyond. Then we can see that $V_2 \cup V_4$ has constant curvature 0 or 1. Since $V_1 \cup V_2 \cup V_3 \cup V_4$ is open and dense in the closure of G_2 it follows that G_2 has constant curvature 0 or 1. Now, $G_1 \cup G_2$ is open and dense in the closure of U. Hence U has constant curvature 0 or 1 and thus Lemma 9 implies that U is flat or Sasakian with constant curvature 1. □

Lemma 13 *If $M^3 \in \mathcal{B}$ and one of the following conditions holds on an open subset U of M^3, then U has constant curvature 0 or 1.*
(i) $\Phi := 4a^2 + 2(6-7\lambda)a - 11\lambda^2 - 18\lambda + 9 = 0$,
(ii) $\Gamma := 4a^2 + 2(6+7\lambda)a - 11\lambda^2 + 18\lambda + 9 = 0$,
(iii) $\Lambda := 2a^2 + (5+2\lambda)a + 3(\lambda^2+\lambda+1) = 0$,
(iv) $\Psi := 2a^2 + (5-2\lambda)a + 3(\lambda^2-\lambda+1) = 0$.

Proof. We shall prove the first of the above cases. The proofs of the other ones are similar. Consider the following subsets of U:

$$G_1 = \{m \in U : \lambda = 0 \text{ in a neighborhood of } m\}$$

$$G_2 = \{m \in U : \lambda \neq 0 \text{ in a neighborhood of } m\}$$

Since $G_1 \in \mathcal{B}$, from Lemma 9 it follows that G_1 is Sasakian with constant curvature 1. On G_2 we differentiate the relation *(i)* with respect to e and ϕe respectively:

$$\begin{aligned} &(4a-7\lambda+6)e(a) - (7a+11\lambda+9)e(\lambda) = 0 \\ &(4a-7\lambda+6)\phi e(a) - (7a+11\lambda+9)\phi e(\lambda) = 0 \end{aligned} \tag{20}$$

Consider the following subsets of G_2:

$$U_1 = \{m \in G_2 : 4a - 7\lambda + 6 = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in G_2 : 4a - 7\lambda + 6 \neq 0 \text{ in a neighborhood of } m\}$$

From Lemma 7 we can see that $U_1 \cup U_2$ is open and dense in the closure of G_2. On U_1 a combination of *(i)* and $4a - 7\lambda + 6 = 0$ yields $\lambda(4 - 31\lambda) = 0$. Hence λ is constant and thus a is constant too. Therefore Lemma 10 implies that U_1 is flat because U_1 is subset of G_2. Now we study the case of U_2. The equations (7), (8), (1) and (2), because of (20) take the forms

$$\xi(b) = \frac{3(a+6\lambda+1)}{4a-7\lambda+6}e(\lambda) - (a-\lambda+1)c \tag{21}$$

$$\xi(c) = -\frac{11a+4\lambda+15}{4a-7\lambda+6}\phi e(\lambda) + (a+\lambda+1)b \tag{22}$$

$$\frac{e(S)}{4} = -\frac{2(3+2a)^2 + 3(5+4a)\lambda + \lambda^2}{6+4a-7\lambda}e(\lambda) + 2\lambda c(2a+\lambda+3)$$

$$\frac{\phi e(S)}{4} = \frac{2(3+2a)^2 - 3(7+4a)\lambda + 43\lambda^2}{6+4a-7\lambda}\phi e(\lambda) + 2\lambda b(-2a+\lambda-3)$$

Differentiations of the above two relations with respect to ξ, using (1), (21), (22) and *(i)* imply:

$$\begin{aligned} &(4a-7\lambda+6)\xi(e(S))/4 \\ &= (-18 - 42a - 32a^2 - 8a^3 - 123\lambda - 177a\lambda - 64a^2\lambda - 70\lambda^2 - 51a\lambda^2 - 9\lambda^3)\phi e(\lambda) \\ &+ 2b\lambda(4a-7\lambda+6)(1+a+\lambda)(3+2a+\lambda) \end{aligned}$$

$$(4a-7\lambda+6)\xi(\phi e(S))/4$$
$$=(-18-42a-32a^2-8a^3+21\lambda+27a\lambda+8a^2\lambda-166\lambda^2-121a\lambda^2+79\lambda^3)e\,(\lambda)$$
$$+\,2c\lambda\,(4a-7\lambda+6)\,(1+a-\lambda)\,(3+2a-\lambda)$$

As $e(\xi(S))=0$ and $\phi e(\xi(S))=0$ the above two relations because of (1), (4) and (5) yield:

$$K_1\phi e(\lambda)=2\lambda bL_1 \tag{23}$$

$$K_2 e(\lambda)=2\lambda cL_2 \tag{24}$$

where

$$K_1=2(1+a)(3+2a)^2+6(10+a(14+5a))\lambda+(46+41a)\lambda^2+26\lambda^3$$
$$L_1=(3+2a)\,(6+4a-7\lambda)\,(1+a+\lambda)$$
$$K_2=2(1+a)(3+2a)^2-2(6+a(6+a))\lambda+(76+55a)\lambda^2-40\lambda^3$$
$$L_2=(3+2a)\,(6+4a-7\lambda)\,(1+a-\lambda)$$

Differentiations of (23) and (24) with respect to ξ yield:

$$P_1\phi e(\lambda)=-2Ab \tag{25}$$

$$P_2 e(\lambda)=2Ac \tag{26}$$

where $P_1=2(1+a)^2(3+2\,a)^2+2(1+a)(48+a(69+25\,a))\lambda+[-2+a(33+25\,a)]\lambda^2-(-12+a)\lambda^3-40\,\lambda^4$

$P_2=-2(1+a)^2\,(3+2\,a)^2-2(1+a)[30+a(45+17\,a)]\,\lambda-(7+5\,a)(16+19\,a)\lambda^2-(88+57a)\lambda^3+26\lambda^4$

and $A=2(3+2a)(6+4a-7\lambda)(1+a+\lambda)(1+a-\lambda)$.

Consider the following subsets of U_2

$$V_1=\{m\in U_2 : P_1=0 \text{ in a neighborhood of } m\}$$
$$V_2=\{m\in U_2 : P_2=0 \text{ in a neighborhood of } m\}$$
$$V_3=\{m\in U_2 : P_1P_2\neq 0 \text{ in a neighborhood of } m\}$$

Lemma 8 implies that $V_1\cup V_2\cup V_3$ is open and dense in the closure of U_2. On V_1 and on V_2, a and λ are constant due to *(i)*. Therefore from Lemma 10 we obtain that V_1 and V_2 are flat. On V_3, solving (25), (26) with respect to $e\,(\lambda)$and $\phi e\,(\lambda)$ and substituting to (23) and (24) we obtain

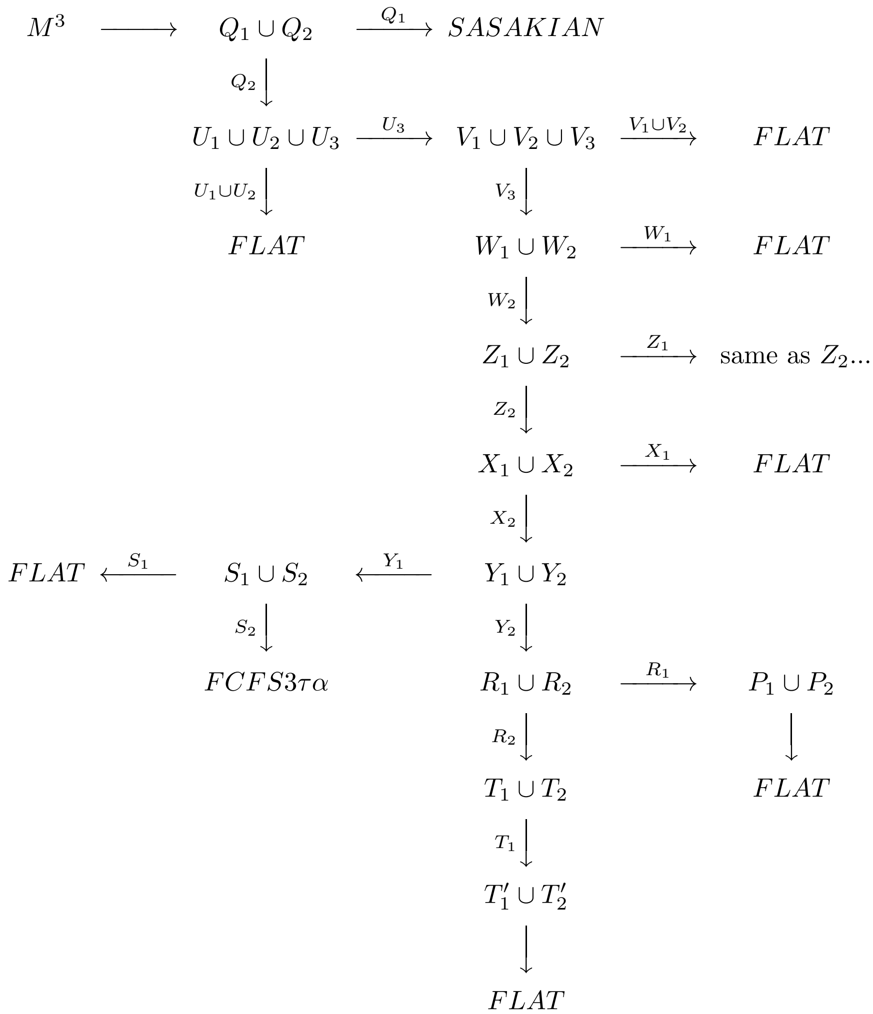
$$b\left(\lambda L_1+A\frac{K_1}{P_1}\right)=0,\quad c\left(\lambda L_2-A\frac{K_2}{P_2}\right)=0$$

Using the above relations, on V_3 we can apply an argument similar to that of the proof of Lemma 11. We conclude that V_3 is flat and thus $V_1\cup V_2\cup V_3$ is flat. Since this last set is open and dense in U_2 it follows that U_2 is flat. Hence $U_1\cup U_2$ and G_2 are flat also. Now, $G_1\cup G_2$ is open and dense in the closure of U. In G_1 we have proved that the sectional curvature equals to 1 and in G_2 the sectional curvature equals to 0. Since the curvature is continuous we conclude that U has constant curvature 0 or 1 and hence U is flat or Sasakian with constant curvature 1. □

5 Proof of main theorem

5.1 Plan of the proof

Before proving the main Theorem, one can see a diagram of the proof in the following figure. The symbol $FCFS3\tau a$ means flat or conformally flat semi-K, 3-τ-a manifold. The arrows denote a decomposition of an open subset of M^3 into open subsets such that their union is open and dense in the closure of this set. For example M^3 is decomposed into Q_1 and Q_2 where $Q_1 \cup Q_2$ is open and dense in M^3. Q_1 will be proved to be of constant curvature 1 and Q_2 will be decomposed into U_1, U_2, U_3 such that $U_1 \cup U_2 \cup U_3$ is open and dense in the closure of Q_2 and so on.



As we shall prove in the last subsection S_2 is not flat. These are the cases where $b = 0$ ($c = 0$) and $a + \lambda + 1 = 0$ ($a - \lambda + 1 = 0$) on S_2. In particular Blair's examples [3] fall in this case, i.e. in the case where S_2 is not flat. Hence M^3 is either flat or Sasakian or a semi-K contact manifold.

5.2 Proof

Consider the following open sets of M^3:

$$Q_1 = \{m \in M^3 : \lambda = 0 \text{ in a neighborhood of } m\}$$

$$Q_2 = \{m \in M^3 : \lambda \neq 0 \text{ in a neighborhood of } m\}$$

By Lemma 9 we obtain that Q_1 has constant sectional curvature 1 and Lemma 7 implies that $Q_1 \cup Q_2$ is open and dense in M^3. From now on we will work in Q_2. This means that whenever an open subset of Q_2 is proved to have constant curvature 0 or 1, the value 1 will be excluded due to Lemma 9 and hence the subset will be flat. Consider the following open sets of Q_2:

$$U_1 = \{m \in Q_2 : 2a - 3\lambda + 3 = 0 \text{ in a neighborhood of } m\}$$

$$U_2 = \{m \in Q_2 : 2a + 3\lambda + 3 = 0 \text{ in a neighborhood of } m\}$$

$$U_3 = \{m \in Q_2 : (2a + 3\lambda + 3)(2a - 3\lambda + 3) \neq 0 \text{ in a neighborhood of } m\}$$

From (1) and (2) we obtain that $U_1 \cup U_2$ has constant sectional curvature 0 or 1 (Lemma 12), hence $U_1 \cup U_2$ is flat. In U_3 we have $(2a + 3\lambda + 3)(2a - 3\lambda + 3) \neq 0$. Differentiations of (1) and (2) with respect to ξ and use of (1), (4), (5), (7), (8) lead us to:

$$\begin{aligned} &\lambda(4a + 3\lambda + 5)\phi e(a) + (3 + 5a + 2a^2 + 6\lambda + 4\lambda a + \lambda^2)\phi e(\lambda) \\ &= 2\lambda(a + \lambda + 1)(2a + 3)b \end{aligned}$$

(1)
$$\begin{aligned} &\lambda(4a - 3\lambda + 5)e(a) + (3 + 5a + 2a^2 - 6\lambda - 4\lambda a + \lambda^2)e(\lambda) \\ &= 2\lambda(a - \lambda + 1)(2a + 3)c \end{aligned}$$

Relations (1), (2) and (1) imply:

$$(3+5a+2a^2-6\lambda-4\lambda a+\lambda^2)e(S) = 4\lambda\Gamma e(a)+8\lambda^2 c(-15-22a-8a^2+6\lambda+4\lambda a+\lambda^2)$$

(2)
$$(3+5a+2a^2+6\lambda+4\lambda a+\lambda^2)\phi e(S) = -4\lambda\Phi\phi e(a)-8\lambda^2 b(15+22a+8a^2+6\lambda+4\lambda a-\lambda^2)$$

where Γ and Φ are those of Lemma 13. From (4) and (5) we obtain $\xi(e(a)) = (a + \lambda + 1)\phi e(a)$ and $\xi(\phi e(a)) = -(a - \lambda + 1)e(a)$.These relations, (1) and (2) yield:

$$\Psi e(S) = 8\lambda^2 c(-15 - 22a - 8a^2 + 3\lambda^2),$$

(3)
$$\Lambda\phi e(S) = 8\lambda^2 b(-15 - 22a - 8a^2 + 3\lambda^2)$$

where Ψ and Λ are those of Lemma 13. Consider the following open sets of U_3:

$$V_1 = \{m \in U_3 : \Psi = 0 \text{ in a neighborhood of } m\}$$

$$V_2 = \{m \in U_3 : \Lambda = 0 \text{ in a neighborhood of } m\}$$

$$V_3 = \{m \in U_3 : \Psi\Lambda \neq 0 \text{ in a neighborhood of } m\}$$

The V_1 and V_2 have constant sectional curvature 0 or 1, (Lemma 13, (iii), (iv)) hence V_1 and V_2 are flat. In V_3, from (5) and (6) we have: $[\phi e, \xi](b) = (a - \lambda + 1)e(b)$, $[e, \phi e](a) = -be(a) + c\phi e(a)$. These equations, (7), (8), (3), (4), (1) and (2) imply:

$$(4) \qquad \begin{aligned} (3+2a)\,\lambda^2 \left(\Psi^2\Lambda e(b) - \Psi\Lambda^2\phi e(c)\right) &= (3+2a)\,\lambda^2 P_3, \\ \lambda^3 \left(\Psi^2\Lambda e(b) + \Psi\Lambda^2\phi e(c)\right) &= \lambda^3 Q_3 \end{aligned}$$

where

$$P_3 = 2bc\lambda\,(3+2a)\left[(1+a)^2\,(3+2a)^2 - (3+2a)\,(15+14a)\,\lambda^2 + 3\lambda^4\right]$$

$$Q_3 = -2bc\left[5\,(1+a)^3\,(3+2a)^3 + (a-4)\,(a+1)\,(3+2a)^2\,\lambda^2 - (3+2a)\,(18+19a)\,\lambda^4 - 27\lambda^6\right]$$

Consider the following open sets of V_3:

$$W_1 = \{m \in V_3 : 3 + 2a = 0 \text{ in a neighborhood of } m\}$$

$$W_2 = \{m \in V_3 : 3 + 2a \neq 0 \text{ in a neighborhood of } m\}$$

W_1 has constant sectional curvature 0 or 1, hence W_1 is flat [5]. In W_2 solving the system (4) with respect to $e(b)$ and $\phi e(c)$ we get:

$$(5) \qquad e\,(b) = \frac{Q_3 + P_3}{2\Psi^2\Lambda}, \quad \phi e\,(b) = \frac{Q_3 - P_3}{2\Psi\Lambda^2}$$

Substituting $e(b)$, $\phi e(c)$ from (5) in (3) we obtain

$$(3 + 2a)bc\lambda^4 = 0$$

Consider the following open sets of W_2:

$$Z_1 = \{m \in W_2 : b = 0 \text{ in a neighborhood of } m\}$$

$$Z_2 = \{m \in W_2 : c = 0 \text{ in a neighborhood of } m\}$$

We shall study only the case of Z_2 because Z_1 is similar. Equations (3), (7) and (8) can be written

$$(6) \qquad \begin{aligned} 2\phi e(b) &= S + 2b^2 - 4a + 2(\lambda^2 - 1) \\ \phi e(a) &= (a + \lambda + 1)b - \phi e(\lambda) \\ \xi(b) &= e(a) - e(\lambda) \end{aligned}$$

Differentiation of the second of (6) with respect to ξ and use of (1) and (5) yield

$$(7) \qquad (a + 1)e(a) = \lambda e(\lambda)$$

Consider the following open sets of Z_2:

$$X_1 = \{m \in Z_2 : a + 1 = 0 \text{ in a neighborhood of } m\}$$

$$X_2 = \{m \in Z_2 : a+1 \neq 0 \text{ in a neighborhood of } m\}$$

From [5] we can see that X_1 has constant sectional curvature 0 or 1 hence X_1 is flat. In X_2, differentiating (7) with respect to ξ and using (1), (4) and the second of (6) we obtain

$$(a+\lambda+1)^2[(a+1)b - \phi e(\lambda)] = 0$$

CASE I. Consider the following open sets of X_2:

$$Y_1 = \{m \in X_2 : a+\lambda+1 = 0 \text{ in a neighborhood of } m\}$$

$$Y_2 = \{m \in X_2 : a+\lambda+1 \neq 0 \text{ in a neighborhood of } m\}$$

In Y_2 we have

$$\phi e(\lambda) = (a+1)b$$

From (5) we have:

$$[\phi e, \xi](b) = (a-\lambda+1)e(b)$$

Using (6) and (6) the above equation yields:

$$\phi e(e(a)) = \frac{-a+3\lambda+1}{a+1} be(\lambda) + (2a-\lambda+2)e(b) \tag{8}$$

Differentiation of the second of (6) with respect to e implies:

$$e(\phi e(a)) = be(\lambda) + \lambda e(b)$$

Subtracting the last equation from (8) we obtain:

$$(a-\lambda+1)[(a+1)e\,(b) - be\,(\lambda)] = 0$$

Consider the following open subsets of Y_2:

$$R_1 = \{m \in Y_2 : a-\lambda+1 = 0 \text{ in a neighborhood of } m\}$$

$$R_2 = \{m \in Y_2 : a-\lambda+1 \neq 0 \text{ in a neighborhood of } m\}$$

In R_2 we have

$$e\,(b) = \frac{b}{a+1} e\,(\lambda)$$

Using this relation, (6), (3) and (4) we have:

$$\frac{2a+\lambda+2}{a+1} be\,(\lambda) = 0$$

Consider the following open sets of R_2:

$$T_1 = \{m \in R_2 : 2a+\lambda+2 = 0 \text{ in a neighborhood of } m\}$$

$$T_2 = \{m \in R_2 : be\,(\lambda) = 0 \text{ in a neighborhood of } m\}$$

and the following open sets of T_2:

$$T_1' = \{m \in T_2 : b = 0 \text{ in a neighborhood of } m\}$$

$$T_2' = \{m \in T_2 : e(\lambda) = 0 \text{ in a neighborhood of } m\}$$

Suppose that at a point p of T_1 we have $be(\lambda) \neq 0$. Then $be(\lambda) \neq 0$ in a neighborhood $A_p \subset T_1$ of p. From (7) we have a contradiction. Therefore at p we must have $be(\lambda) = 0$. Suppose that on a point $q \in A_p$ we have $be(\lambda) \neq 0$. Then at q we have a contradiction because of (7). Hence $be(\lambda) = 0$ in a neighborhood of p. Thus $p \in T_2$ and therefore $T_1 \subseteq T_2$. T_1 has constant sectional curvature 0 or 1, (Lemma 10) hence T_1 is flat. In $T_1' \cup T_2'$, (4) can be written

$$[e, \xi](\lambda) = -(a + \lambda + 1)\phi e(\lambda)$$

which implies $\phi e(\lambda) = 0$, therefore $\lambda =$ constant and $T_1' \cup T_2'$ has constant sectional curvature 0 or 1, (Lemma 11) hence $T_1' \cup T_2'$ is flat. Therefore T_2 is also flat.

Now we study the subset R_1, i.e. we assume that $a - \lambda + 1 = 0$, where a and λ are not constant. Then, (7) and (8) yield $\xi(b) = 0$ and $\phi e(\lambda) = \lambda b$. From these relations, (3) and (6) we get

$$2\phi e(b) = S + 2b^2 + 2(\lambda - 1)^2, \; (2\lambda - 1)S = 2(-2\lambda b^2 - 6\lambda^3 + 9\lambda^2 + 3)$$

Differentiation of the last equation with respect to ϕe and comparison of the result with (2) give

$$\lambda b[3b^2 + 10\lambda(\lambda - 2) - 6] = 0$$

Consider the following open sets of R_1:

$$P_1 = \{m \in R_1 : b = 0 \text{ in a neighborhood of } m\}$$

$$P_2 = \left\{m \in R_1 : 3b^2 + 10\lambda(\lambda - 2) - 6 = 0 \text{ in a neighborhood of } m\right\}$$

The P_1 has constant sectional curvature 0 or 1, (Lemma 10) hence P_1 is flat. In P_2 differentiation of $3b^2 + 10\lambda(\lambda - 2) - 6 = 0$ with respect to ϕe implies $\lambda =$ constant, therefore P_2 has constant sectional curvature 0 or 1, (Lemma 11) hence P_2 is flat.

Therefore $P_1 \cup P_2$, R_1 and Y_2 are flat.

CASE II

In Y_1 we have $a + \lambda + 1 = 0$. In this case the equations (1) and (2) can be written:

$$e(S) = 4(\lambda - 1)e(\lambda) \tag{9}$$

$$\phi e(S) = 4[(1 - 7\lambda)\phi e(\lambda) + 2\lambda(3\lambda - 1)b \tag{10}$$

Differentiating (10) with respect to ξ and using (1) and (6) we have

$$\xi(\phi e(S)) = 8\lambda(3 - 13\lambda)e(\lambda)$$

Since $\phi e(\xi(S)) = 0$, the above relation because of (5) implies:

$$e(S) = 4(3 - 13\lambda)e(\lambda)$$

The above relation and (9) yield:

$$(7\lambda - 2)e(\lambda) = 0$$

Consider the following open sets of Y_1:

$$S_1 = \{m \in Y_1 : 7\lambda - 2 = 0 \text{ in a neighborhood of } m\}$$

$$S_2 = \{m \in Y_1 : e\,(\lambda) = 0 \text{ in a neighborhood of } m\} \tag{11}$$

Since $a + \lambda + 1 = 0$ it follows from [5] that S_1 has constant sectional curvature 0 or 1, therefore S_1 is flat. In S_2 we have:

$$e\,(\lambda) = 0 \tag{12}$$

A simple calculation yields the sectional curvatures of S_2: $K(e, \phi e) = \frac{1}{2}(-4+S+4\,\lambda^2)$, $K(e, \xi) = (1 + \lambda)^2$, $K(\phi e, \xi) = (1 + \lambda)(1 - 3\lambda)$. There are two cases:

(A) S_2 has constant curvature, say K,

(B) S_2 has not constant curvature.

In case (A), the relation $(1 + \lambda)^2 = K$ yields λ=constant. Since $a + \lambda + 1 = 0$ it follows that a is constant too and hence [5] implies that S_2 has constant curvature 0 or 1. But the value 1 is excluded, hence S_2 must be flat.

In case (B), relations (10), (6), (7) and (9) yield:

$$c = \xi\,(b) = e\,(b) = e\,(S) = \xi(S) = e\,(a) = \xi(a) = e(\lambda) = \xi(\lambda) = 0$$

$$2b\phi e\,(\lambda) = (\lambda + 1)S + 2\lambda(2b^2 + 5\lambda^2 + 7\lambda - 1) - 6$$

$$\phi e\,(b) = \frac{S}{2} + b^2 + (\lambda + 1)^2 \tag{13}$$

$$b\phi e\,(S) = 2[-16\lambda^2 b^2 - (7\lambda^2 + 6\lambda - 1)S - 70\lambda^4 - 88\lambda^3 + 28\lambda^2 + 40\lambda - 6]$$

$$\phi e\,(a) = -\phi e(\lambda)$$

Taking account the result of [10] which states that every closed semi-K-contact manifold is a non-trivial torus bundle over the circle, the main Theorem is proved.

Remark 14 *The manifolds of case* (*B*) *given by (11) are exactly the conformally flat, 3-τ-a, semi-K contact manifolds.*

Concerning the structure of a conformally flat, semi-K contact, 3-τ-a manifold we state the following result:

Proposition 15 *Let M^3 be a conformally flat, semi-K contact manifold. If $S \geq 0$ then M^3 cannot be compact.*

Proof. Suppose that M^3 is compact and $S \geq 0$. Then $\overline{S}_2$ (the closure of S_2) is compact. In S_2 we have $\text{grad}b = e\,(b)\,e + \phi e\,(b)\,\phi e + \xi\,(b)\,\xi = \phi e\,(b)\,\phi e$. Hence for every $p \in S_2$ we have $\text{grad}b\,(p) = 0$ iff $\phi e\,(b)\,(p) = 0$. But $\phi e\,(b)\,(p) = 0$ iff $S\,(p) = b\,(p) = \lambda\,(p) + 1 = 0$. There are two cases:

(1) $\overline{S}_2 = M^3$.

(2) $\overline{S}_2 \subset M^3$.

In the first case, at every local maximum m and every local minimum μ of b we must have grad$b\,(m)$ =grad$b\,(\mu) = 0$, i.e., $\phi e\,(b)\,(m) = \phi e\,(b)\,(\mu) = 0$. Then $0 = b\,(m) \geq b\,(p) \geq b\,(\mu) = 0$ for all $p \in S_2$. Hence b must be zero everywhere on S_2 hence on M^3 too. Thus M^3 is flat. This is a contradiction because S_2, a subset of M^3, is non-flat.

In the second case we have already proved that the set $\Omega = S_1 \cup Y_2 \cup X_1 \cup Z_1 \cup V_1 \cup V_2 \cup U_1 \cup U_2$ is flat hence $b = 0$ on $\overline{\Omega}$. The set $\Omega \cup S_2$ is open and dense in M^3. Moreover Ω and S_2 have the same boundary points. Applying the proof of case (1) for the local minima and maxima of b in S_2 we conclude that $b = 0$ in S_2. Hence $b = 0$ on $S_2 \cup \Omega$ and thus in M^3 too. Therefore M^3 is flat. This is a contradiction because S_2, a subset of M^3, is non-flat.

In both cases we have a contradiction, namely that M^3 is flat. Hence M^3 cannot be compact and therefore it has $S \geq 0$. □

6 Blair's non-compact examples

In the present paragraph we prove that Blair's examples ([3] p. 108) are conformally flat, semi-K contact, 3-τ-a manifolds.

Proposition 16 *In all conformally flat, semi-K contact,* 3-τ-a *manifolds, the quantities* a, b, λ, S *vary only along* ϕe *or* e *(if* $c = 0$ *or* $b = 0$ *respectively).*

Proof. Let M^3 be a conformally flat, semi-K contact, 3-τ-a manifold. Using $a = -\lambda - 1$, equations (2) and (1) take the forms

$$2\,b\,\lambda\,(-1 + 3\,\lambda) - \frac{\phi e\,(S)}{4} + (1 - 7\,\lambda)\,\phi e\,(\lambda) = 0 \tag{1}$$

and

$$\frac{e\,(S)}{4} + e\,(\lambda) - \lambda\,e\,(\lambda) = 0 \tag{2}$$

Consider the following subsets of M^3:

$$E_1 = \left\{ m \in M^3 : \lambda = \frac{1}{7} \text{ in a neighborhood of } m \right\}$$

$$E_2 = \left\{ m \in M^3 : \lambda \neq \frac{1}{7} \text{ in a neighborhood of } m \right\}$$

Since $a = -\lambda - 1$, on E_1, a is a constant and [5] implies that E_1 is flat. Consider the following subsets of E_2:

$$E_3 = \left\{ m \in E_2 : \lambda = \frac{2}{7} \text{ in a neighborhood of } m \right\}$$

$$E_4 = \left\{ m \in E_2 : \lambda \neq \frac{2}{7} \text{ in a neighborhood of } m \right\}$$

Using the same argument as above we conclude that E_3 is flat. Consider the following subsets of E_4:

$$E_5 = \{m \in E_4 : \lambda = 1 \text{ in a neighborhood of } m\}$$

$$E_6 = \{m \in E_4 : \lambda \neq 1 \text{ in a neighborhood of } m\}$$

Using the above mentioned method we obtain that E_3 is flat. In E_6 (1) and (2) imply:

$$\phi e\,(\lambda) = \frac{8\,b\,\lambda - 24\,b\,\lambda^2 + \phi e\,(S)}{4\,(1 - 7\,\lambda)}$$

and

$$e\,(\lambda) = \frac{e\,(S)}{4\,(\lambda - 1)} \tag{3}$$

Also (5) acting on λ yields

$$\frac{\lambda\,(-2 + 7\,\lambda)\,e(S)}{(-1 + \lambda)\,(-1 + 7\,\lambda)} = 0$$

Since $\lambda \neq \frac{2}{7}$, the above implies $e(S) = 0$. Consider the following subsets of E_6:

$$E_7 = \{m \in E_6 : \lambda = 0 \text{ in a neighborhood of } m\}$$

$$E_8 = \{m \in E_6 : \lambda \neq 0 \text{ in a neighborhood of } m\}$$

and also the following subsets of E_8:

$$E_9 = \{m \in E_8 : b = 0 \text{ in a neighborhood of } m\}$$

$$E_{10} = \{m \in E_8 : b \neq 0 \text{ in a neighborhood of } m\}$$

From previous considerations we have that $E_8 \neq \varnothing$ and E_9 is flat. The subset E_{10} is neither flat nor Sasakian. In E_{10}, (3) implies that $e(\lambda) = 0$. The fact that $a = -\lambda - 1$ gives $e(a) = 0$ and (7) gives $\xi(b) = 0$. Thus on E_{10} the functions a, b, λ, S vary only along ϕe.

On the other hand, on E_i, i = 1,3,5,7,9 the functions a, b, λ, S are constant. Hence their derivatives along e and ξ vanish. Since $E_{i-1} \cup E_i$ is open and dense in the closure of E_{i-2}, i = 2, 4, 6, 8, 10, ($E_0 := M^3$), it follows that the derivatives of a, b, λ, S vanish along e and ξ. From hypothesis M^3 is conformally flat, semi-K contact, 3-τ-a manifold and hence it is not flat. Thus a, b, λ, S are not constant on M^3. Hence they vary only along ϕe.

For completeness of the proof we shall give the derivatives of a, b, λ, and S along ϕe. From (5) we get

$$\phi e(\lambda) = \frac{-6 + S + \left(-2 + 4\,b^2 + S\right)\,\lambda + 14\,\lambda^2 + 10\,\lambda^3}{2\,b} = -\phi e(a)$$

and from previous equations we get

$$\phi e(S) = \frac{-2\,\left(6 - S - 40\,\lambda + 6\,S\,\lambda - 28\,\lambda^2 + 16\,b^2\,\lambda^2 + 7\,S\,\lambda^2 + 88\,\lambda^3 + 70\,\lambda^4\right)}{b}$$

$$\phi e(b) = b^2 + \frac{S}{2} + (1+\lambda)^2$$

Also, simple calculation yields the sectional curvatures: $K(e, \phi e) = \frac{1}{2}(-4 + S + 4\lambda^2)$, $K(e, \xi) = (1+\lambda)^2$, $K(\phi e, \xi) = (1+\lambda)(1-3\lambda)$. □

Proposition 17 *Blair's examples are conformally flat, semi-K contact, 3-τ-a manifolds.*

Proof. As in [3] p. 108, we introduce cylindrical coordinates in an open subset U of R^3, namely r, θ, z. The vectors $\frac{\partial}{\partial r}$, $\frac{1}{r}\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$ constitute an orthonormal basis for the metric

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2$$

The most general form of a conformally flat metric in U is given in cylindrical coordinates by

$$ds^2 = \frac{1}{4} e^{2\sigma}\left(dr^2 + r^2\, d\theta^2 + dz^2\right) \tag{4}$$

where σ is some smooth function. Let

$$\eta = \frac{1}{2}\left(\alpha dr + \beta r d\theta + \gamma dz\right)$$

be a 1-form, where α, β, γ are smooth functions. η is a contact form if $\eta \wedge d\eta \neq 0$ everywhere on U. Provided that the metric (4), is an associated contact metric, the characteristic vector field ξ can be calculated using $g(\xi, X) = \eta(X)$ for every vector field X. It is found to be

$$\xi = 2e^{-2\sigma}\left(\alpha\frac{\partial}{\partial r} + \frac{\beta}{r}\frac{\partial}{\partial \theta} + \gamma\frac{\partial}{\partial z}\right) \tag{5}$$

The condition $\eta(\xi) = 1$ leads to $e^{2\sigma} = \alpha^2 + \beta^2 + \gamma^2$. From equation $g(\phi X, Y) = d\eta(X, Y)$ we get the following expression of the $(1,1)$-tensor ϕ in cylindrical coordinates

$$\phi = e^{-\sigma}\begin{pmatrix} 0 & r\gamma & -\beta \\ -\frac{\gamma}{r} & 0 & \frac{\alpha}{r} \\ \beta & -r\alpha & 0 \end{pmatrix} \tag{6}$$

We will also need the Christoffel symbols. These are

$\Gamma^1_{11} = \sigma_r$, $\Gamma^1_{12} = \sigma_\theta$, $\Gamma^1_{13} = \sigma_z$, $\Gamma^1_{22} = -r(1 + r\sigma_r)$, $\Gamma^1_{23} = 0$, $\Gamma^1_{33} = -\sigma_r$, $\Gamma^2_{11} = -\frac{\sigma_\theta}{r^2}$, $\Gamma^2_{12} = \frac{1}{r} + \sigma_r$, $\Gamma^2_{13} = 0$, $\Gamma^2_{22} = \sigma_\theta$, $\Gamma^2_{23} = \sigma_z$, $\Gamma^2_{33} = -\frac{\sigma_\theta}{r^2}$, $\Gamma^3_{11} = -\sigma_z$, $\Gamma^3_{12} = 0$, $\Gamma^3_{13} = \sigma_r$, $\Gamma^3_{22} = -r^2\sigma_z$, $\Gamma^3_{23} = \sigma_\theta$, $\Gamma^3_{33} = \sigma_z$.

From the formula $\nabla_X \xi = h\phi X - \phi X$ we get

$$h\phi\frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\xi + \phi\frac{\partial}{\partial z} \tag{7}$$

Using the Christoffel symbols and computing directly we get

$$g\left(\nabla_{\frac{\partial}{\partial z}}\xi, \frac{\partial}{\partial r}\right) = 2e^{-2\sigma}\left(\alpha_z - \alpha\sigma_z - \gamma\sigma_r\right) \tag{8}$$

$$g\left(\nabla_{\frac{\partial}{\partial z}}\xi,\frac{\partial}{\partial\theta}\right)=-\frac{1}{r^2}\left(2e^{-2\sigma}\left(-r\beta_z+r\beta\sigma_z+\gamma\sigma_\theta\right)\right) \tag{9}$$

$$g\left(\nabla_{\frac{\partial}{\partial z}}\xi,\frac{\partial}{\partial z}\right)=\frac{1}{r}\left(2e^{-2\sigma}\left(\beta\sigma_\theta+\rho\left(\gamma_z-\gamma\sigma_z+\alpha\sigma_r\right)\right)\right) \tag{10}$$

From (6) we get

$$\phi\frac{\partial}{\partial z}=-e^{-\sigma}\beta\frac{\partial}{\partial r}+e^{-\sigma}\frac{\alpha}{r}\frac{\partial}{\partial\theta} \tag{11}$$

Now, as in [3] p.109, we suppose that the functions α, β, γ depend only on r. From (7) and (11) we have

$$e^{\sigma}\nabla_{\frac{\partial}{\partial z}}\xi=-\beta h\left(\frac{\partial}{\partial r}\right)+\beta\frac{\partial}{\partial r}+\frac{\alpha}{r}h\left(\frac{\partial}{\partial\theta}\right)-\frac{\alpha}{r}\frac{\partial}{\partial\theta} \tag{12}$$

On the other hand from (8), (9) and (10) we have

$$e^{\sigma}\nabla_{\frac{\partial}{\partial z}}\xi=-2e^{-\sigma}\gamma\sigma_r\frac{\partial}{\partial r}+2e^{-\sigma}\alpha\sigma_r\frac{\partial}{\partial z} \tag{13}$$

Comparing (12) and (13) we get

$$2e^{-\sigma}\sigma_r\left(-\gamma\frac{\partial}{\partial r}+\alpha\frac{\partial}{\partial z}\right)=-\beta h\left(\frac{\partial}{\partial r}\right)+\beta\frac{\partial}{\partial r}+\frac{\alpha}{r}h\left(\frac{\partial}{\partial\theta}\right)-\frac{\alpha}{r}\frac{\partial}{\partial\theta} \tag{14}$$

Let $\mathbf{B}=\alpha\frac{\partial}{\partial r}+\frac{\beta}{r}\frac{\partial}{\partial\theta}+\gamma\frac{\partial}{\partial z}$. Then $g\left(\mathbf{B},\mathbf{B}\right)=\frac{1}{4}e^{2\sigma}$. The equations $d\eta\left(\xi,X\right)=0$, $\phi^2=-I+\eta\otimes\xi$ and $\eta\wedge d\eta\neq 0$ yield $\mathrm{curl}\mathbf{B}=2\sqrt{g\left(\mathbf{B},\mathbf{B}\right)}\mathbf{B}$, where

$$\mathrm{curl}\mathbf{B}=\left(\frac{\gamma_\theta}{r}-\beta_z\right)\frac{\partial}{\partial r}+\frac{1}{r}\left(\alpha_z-\gamma_r\right)\frac{\partial}{\partial\theta}+\left(\frac{\beta}{r}+\beta_r-\frac{\alpha_\theta}{r}\right)\frac{\partial}{\partial z}$$

Since the functions α, β, γ depend only on r, equation $\mathrm{curl}\mathbf{B}=2\sqrt{g\left(\mathbf{B},\mathbf{B}\right)}\mathbf{B}$ gives $\alpha=0$. Then (14) becomes

$$h\left(\frac{\partial}{\partial r}\right)=\left(1+\frac{2e^{-\sigma}\sigma_r\gamma}{\beta}\right)\frac{\partial}{\partial r} \tag{15}$$

We obtain immediately that the vector $\frac{\partial}{\partial r}$ is an eigenvector of h. Since $\alpha=0$, the vector $\frac{\partial}{\partial r}$ is normal to ξ. Hence the unit vector $2\,e^{-\sigma}\frac{\partial}{\partial r}$ can be used as e or ϕe. Let us choose $\phi e=2\,e^{-\sigma}\frac{\partial}{\partial r}$. Then

$$\nabla_{\phi e}\phi e=\nabla_{\left(2e^{-\sigma}\frac{\partial}{\partial r}\right)}\left(2e^{-\sigma}\frac{\partial}{\partial r}\right)=-2\sigma_r e^{-\sigma}\frac{\partial}{\partial r}+2e^{-\sigma}\sigma_r\frac{\partial}{\partial r}=0$$

From (6) we get

$$\phi\left(\phi e\right)=\phi\left(2e^{-\sigma}\frac{\partial}{\partial r}\right)=2e^{-2\sigma}\left(-\frac{\gamma}{r}\frac{\partial}{\partial\theta}+\beta\frac{\partial}{\partial z}\right)=-e$$

After some calculations we obtain

$$\nabla_e e = \left(-2e^{-3\sigma}\frac{\gamma^2}{r} - 2e^{-\sigma}\sigma_r\right)2e^{-\sigma}\frac{\partial}{\partial r}$$

Recalling $\nabla_{\phi e}\phi e = ce$ and $\nabla_e e = b\phi e$, and setting $\phi e = 2\,e^{-\sigma}\frac{\partial}{\partial r}$ we are led to the case $c = 0$. It is not hard to prove that setting $e = 2\,e^{-\sigma}\frac{\partial}{\partial r}$ and $\phi e = 2e^{-2\sigma}\left(\frac{\gamma}{r}\frac{\partial}{\partial\theta} - \beta\frac{\partial}{\partial z}\right)$ we obtain $b = 0$. These two cases are completely symmetric but studing the case where $c = 0$ we will continue on the choice

$$\phi e = 2e^{-\sigma}\frac{\partial}{\partial r} \text{ and } e = 2e^{-2\sigma}\left(\frac{\gamma}{r}\frac{\partial}{\partial\theta} - \beta\frac{\partial}{\partial z}\right)$$

From (15) we see that $\lambda = -\left(1 + \frac{2e^{-\sigma}\sigma_r\gamma}{\beta}\right)$. Now recall that $\nabla_\xi\phi e = -ae$. A simple calculation shows that

$$\nabla_\xi\phi e = 2e^{-\sigma}\left(\frac{\beta}{r^2} + \frac{\beta}{r}\sigma_r\right)\frac{\partial}{\partial\theta} + 2e^{-\sigma}\gamma\sigma_r\frac{\partial}{\partial z}$$

hence $a = \frac{2e^{-\sigma}\gamma\sigma_r}{\beta}$. So we obtain $a + \lambda + 1 = 0$.

Finally the scalar curvature of the metric (4) equals to

$$S = -8\,e^{-2\,\sigma}\left({\sigma_z}^2 + 2\,\sigma_{zz} + \frac{{\sigma_\theta}^2}{r^2} + \frac{2\,\sigma_{\theta\theta}}{r^2} + \frac{2\,\sigma_r}{r} + {\sigma_r}^2 + 2\,\sigma_{rr}\right)$$

Since $e^{2\sigma} = \beta^2 + \gamma^2$, and β, γ are supposed to depend only on r, we conclude that $\sigma_z = \sigma_\theta = 0$ hence $S = -8\,e^{-2\,\sigma}\left(\frac{2\,\sigma_r}{r} + {\sigma_r}^2 + 2\,\sigma_{rr}\right)$. Since $\alpha = 0$, (5) yields $\xi(S) = 0$. The same holds for a and λ, i.e. $\xi(\lambda) = \xi(a) = 0$. Since Blair's examples are non flat, it follows that they are conformally flat, semi-K contact, 3-τ-a manifolds. □

Remark 18 *From Proposition 16 the functions a, b, λ and S, depend on r only. Hence, they vary only on the direction of $\phi e = \frac{\partial}{\partial r}$. In [3] p.109 it is proved that the existence of 3-dimensional conformally flat contact metric manifolds corresponds to the solutions of*

$$\mathbf{curl}\,\mathbf{B} = |\mathbf{B}|\,\mathbf{B} \tag{16}$$

with $\mathbf{B} = \alpha\frac{\partial}{\partial r} + \beta\frac{1}{r}\frac{\partial}{\partial\theta} + \gamma\frac{\partial}{\partial z}$, and $|\mathbf{B}| = \sqrt{\alpha^2 + \beta^2 + \gamma^2} = e^\sigma$. Blair in [3] p.109 imposed the restriction for the α, β and γ to be functions of r only, in order to simplify the calculations. Main Theorem and the previous considerations shows that this restriction is essential. Violation of this restriction will furnish via the solution of (16) a contact structure which is either Sasakian, or flat, or out of the class of conformally flat, semi-K contact, 3-τ-a manifolds. Examples of the first two categories are already found in [3] p.109. It seems to be an open problem the existence of an example of the third category. This observation is related to the open problem of finding all the solutions of the system $\mathbf{curl}\,\mathbf{B} = |\mathbf{B}|\,\mathbf{B}$, $\mathbf{div}\,\mathbf{B} = 0$. *Note that* $\mathbf{div}\,\mathbf{B} = 0$ *is satisfied for the particular* $\mathbf{B}$ *introduced in [3] p.109, since* $\mathbf{div}\,\mathbf{B} = \alpha_r + \frac{1}{r}\alpha + \frac{1}{r}\beta_\theta + \gamma_z$, *$\alpha = 0$, and β, γ are functions of r only.*

7 Compact example

The example of Blair is non compact. It would be very interesting from the physical point of view (paragraph 1), to construct compact examples of conformally flat contact metric 3-manifolds. In this paragraph we will prove that the solid torus, $S^1 \times D^2$ is a conformally flat, semi-K contact, 3-τ-a manifold.

Blair in [3] p.109, proved that a solution of the following ordinary differential equations on $[0, \infty)$

$$\frac{d\beta}{dr} = \gamma\sqrt{\beta^2+\gamma^2} - \frac{\beta}{r}, \quad \frac{d\gamma}{dr} = -\beta\sqrt{\beta^2+\gamma^2} \tag{1}$$

with $\beta(0) = 0$ and $\gamma(0) \neq 0$, implies the existence of a conformally flat semi-K contact 3-τ-a structure on $\mathbb{R}^3$. He also showed [2] that (1) has a regular solution $(\beta(r), \gamma(r))$ on $[0, r_0]$. Introduce polar coordinates (r, θ) on the disk $D^2 \subset \mathbb{R}^3$ with center 0 and radius r_0. Here $\mathbb{R}^3$ is endowed with cylindrical coordinates r, θ and z. In [3] p.109, (see also 5) ξ is given by

$$\xi = 2e^{-2\sigma}\left(\frac{\beta(r)}{r}\frac{\partial}{\partial\theta} + \gamma(r)\frac{\partial}{\partial z}\right)$$

The components of ξ depend only on r, therefore ξ is invariant under translations of the form $z \to z+1$, on the space $\mathbb{R} \times D^2$. Hence, we may take the identification space

$$\left(\mathbb{R} \times D^2\right) / (z \sim z+1)$$

without affecting ξ, g and η given by (4) and (5). The above manifold is the solid torus. Thus, we obtain a conformally flat, semi-K, 3-τ-a structure on the solid torus $S^1 \times D^2$.

Using the Runge-Kutta method of 8th order to solve the system (1) with initial conditions $\beta(0) = 0$, $\gamma(0) = 1$, we find that $r_0 = 3.511838\ldots$ is the first value of the parameter r for which $\gamma(r_0) = 0$ and $\beta(r_0) > 0$. For $r = r_0$, ξ is tangent to the meridian of the boundary of $S^1 \times D^2$. In particular this contact structure on $S^1 \times D^2$ is overtwisted ([3] p.25). The qualitative behavior of (1) for various initial conditions, is that of a spiral converging slowly to the origin.

8 Example of a 3-τ-a manifold which is neither semi-K contact nor conformally flat

In the present paragraph we shall give an example of 3-τ-a manifold which is neither semi-K contact nor conformally flat.

Consider the usual 3-dimensional Euclidean space $\mathbb{R}^3$ with Cartesian coordinates x, y and z. Let η be the 1-form defined by

$$\eta = dx - e^{-y}dz$$

The form η is a contact form since $\eta \wedge d\eta = e^{-y}dx \wedge dy \wedge dz$. The characteristic vector field ξ of the contact manifold $\left(\mathbb{R}^3, \eta\right)$ is $\frac{\partial}{\partial x}$. The contact distribution is generated by the globally defined vector fields

$$e_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + e^y\frac{\partial}{\partial z}, \quad e_2 = 2\frac{\partial}{\partial y}$$

The vector fields e_1, e_2 and ξ satisfy the following relations:

$$[e_1, e_2] = -2e_1 + xe_2 + 2\xi, \quad [e_1, \xi] = -\frac{1}{2}e_2, \quad [e_2, \xi] = 0$$

Introduce in $\mathbb{R}^3$ the Riemannian metric $g(e_1, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$, $g(e_1, e_1) = g(e_2, e_2) = g(\xi, \xi) = 1$, and the tensor field ϕ defined by

$$\phi e_1 = e_2, \ \phi e_2 = -e_1, \ \phi\xi = 0$$

The tensors η, g and ϕ satisfy

$$d\eta(\xi, e_i) = 0 = g(\xi, \phi e_i), \quad d\eta(e_i, e_i) = 0 = g(e_i, \phi e_i)$$

$$d\eta(e_1, e_2) = -1 = g(e_1, \phi e_2)$$

Hence (η, ϕ, g) is a contact metric structure on $\left(\mathbb{R}^3, \eta\right)$. Moreover we can prove that $h(e_1) = \frac{1}{4}e_1$ and $h(e_2) = -\frac{1}{4}e_2$. Thus $\lambda = \frac{1}{4}$ and $\{e_1, e_2, \xi\}$ is an orthonormal ϕ-basis. Denoting by ∇ the Levi-Civita connection we get

$$\nabla_{e_1}e_1 = 2e_2, \quad \nabla_{e_1}e_2 = -2e_1 + \frac{5}{4}\xi, \quad \nabla_{e_1}\xi = -\frac{5}{4}e_2$$
$$\nabla_{e_2}e_1 = -xe_2 - \frac{3}{4}\xi, \quad \nabla_{e_2}e_2 = xe_1, \quad \nabla_{e_2}\xi = \frac{3}{4}e_1$$
$$\nabla_{\xi}e_1 = -\frac{3}{4}e_2, \quad \nabla_{\xi}e_2 = \frac{3}{4}e_1, \quad \nabla_{\xi}\xi = 0$$

Using the above relations we have finally

$$\lambda = \frac{1}{4}, \quad a = -\frac{3}{4}, \quad c = x, \quad b = 2$$

In particular we obtain $\xi(\lambda) = 0$ and hence $\left(\mathbb{R}^3, \eta, \phi, g, \xi\right)$ is a 3-τ-a manifold. Also $a - \lambda + 1 = 0$ and $b \neq 0$, thus $\left(\mathbb{R}^3, \eta, \phi, g, \xi\right)$ is not a semi-K contact. On this manifold we can prove that $\xi(\lambda) = 0$, $Q\phi \neq \phi Q$, $\nabla_\xi \tau \neq 0$ and $\xi(S) = \xi(a) = 0$. Also if we denote by C the Weyl-Schouten tensor, we obtain $C(e, e, \phi e) = -\frac{5}{4}$, hence the manifold is not conformally flat.

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