Relations between the main scalars of a four-dimensional Finsler space and its hypersurface

M.K. Gupta and P.N. Pandey

Abstract. T.N. Pandey and D.K.Diwedi [6] and the present authors [1, 2, 7] studied a four-dimensional Finsler space in terms of scalars with the help of 'Miron frame' which was discussed by M. Matsumoto and R. Miron [3]. On the other hand, the theory of hypersurface was discussed in detail by M. Matsumoto [4]. The purpose of the present paper is to obtain relations between the main scalars of a four-dimensional Finsler space and its hypersurface. For terms and notations we refer to Matsumoto's paper [4].

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1 Preliminaries

Let us consider a four-dimensional Finsler space $F^4 = (M^4, L(x,y))$, whose fundamental metric function is $L(x,y)$. The normalized supporting element, metric tensor and Cartan tensor are defined by

$$l_i = \frac{\partial L}{\partial y^i}, \quad g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \, \partial y^j} \quad \text{and} \quad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} \quad \text{respectively.}$$

A hypersurface M^3 of M^4 may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian coordinates on M^3 (Latin indices run from 1 to 4, while Greek indices, except λ, μ, ν, take values 1 to 3). We assume that the matrix consisting of projection factors $B^i_\alpha = \partial x^i / \partial u^\alpha$ is of rank 3. Then $B_\alpha(u) = (B^i_\alpha(u))$ may be regarded as three independent vectors tangent to M^3 at the point $u = (u^\alpha)$ and a vector X^i tangent to M^3 at the point may be expressed uniquely in the form $X^i = B^i_\alpha X^\alpha$, where X^α are the components of the vector with respect to the coordinate system (u^α).

To introduce a Finsler structure on M^3, the supporting element y^i is assumed to be tangent to M^3 at a point u of M^3, so that we may write

$$y^i = B^i_\alpha(u)\, v^\alpha . \tag{1.1}$$

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Thus, $v = (v^\alpha)$ may be supposed as the supporting element of M^3 at the point u. Denoting y^i of (1.1) by $y^i(u,v)$, the function

$$\underline{L}\,(u,v) = L\Big(x(u), y(u,v)\Big) \tag{1.2}$$

gives rise to a Finsler metric on M^3. Consequently, we get a three-dimensional Finsler space $F^3 = (M^3, \underline{L}\,(u,v))$, where $\underline{L}$ is the induced metric function on F^3.

The induced metric function $\underline{L}\,(u,v)$ yields $l_{\,\alpha} = \partial\underline{L}\,/\partial v^\alpha$, the metric tensor $g_{\alpha\beta} = (1/2)\,\partial^2 L^2/\partial v^\alpha\,\partial v^\beta$ and the Cartan tensor $C_{\alpha\beta\gamma} = (1/2)\,\partial g_{\alpha\beta}/\partial v^\gamma$ of F^3. Paying attension to $\partial B^i_\alpha/\partial v^\beta = 0$, from (1.2), we get

$$l_\alpha = l_i\,B^i_\alpha\,, \qquad g_{\alpha\beta} = g_{ij}\,B^i_\alpha\,B^j_\beta\,, \qquad C_{\alpha\beta\gamma} = C_{ijk}\,B^i_\alpha\,B^j_\beta\,B^k_\gamma\,. \tag{1.3}$$

At each point u of F^3, a unit normal vector $B^i(u,v)$ is defined by

$$g_{ij}\,B^i_\alpha\,B^j = 0\,, \qquad g_{ij}\,B^i\,B^j = 1\,. \tag{1.4}$$

The inverse projection factor $B^\alpha_i(u,v)$ of B^i_α is defined as

$$B^\alpha_i = g^{\alpha\beta}\,g_{ij}\,B^i_\beta\,, \tag{1.5}$$

where $g^{\alpha\beta}$ is the inverse tensor of the metric tensor $g_{\alpha\beta}$ of F^3.
From (1.5), it follows that

$$\begin{aligned} &\text{a) } B^i_\alpha\,B^\beta_i = \delta^\beta_\alpha\,, && \text{b) } B^i_\alpha\,B_i = 0\,, \quad B^i\,B^\alpha_i = 0\,,\\ &\text{c) } B^i\,B_i = 1\,, && \text{d) } B^i_\alpha\,B^\alpha_j + B^i\,B_j = \delta^i_j\,. \end{aligned} \tag{1.6}$$

Let us deduce the following tensors from the Cartan tensor C_{ijk}:

$$M_{\alpha\beta} = L\,C_{ijk}\,B^i_\alpha\,B^j_\beta\,B^k, \quad M_\alpha = L\,C_{ijk}\,B^i_\alpha\,B^j\,B^k, \quad M = L\,C_{ijk}\,B^i\,B^j\,B^k. \tag{1.7}$$

From (1.3), (1.6) and (1.7), we may write

$$\begin{aligned} L\,C_{ijk}\,B^j_\alpha\,B^k_\beta &= \underline{L}\,C_{\alpha\beta\gamma}\,B^\gamma_i + M_{\alpha\beta}\,B_i,\\ L\,C_{ijk}\,B^j_\alpha\,B^k &= M_{\alpha\beta}\,B^\beta_i + M_\alpha\,B_i,\\ L\,C_{ijk}\,B^j\,B^k &= M_\alpha\,B^\alpha_i + M\,B_i, \end{aligned} \tag{1.8}$$

which lead to

$$L\,C_i = (\underline{L}\,C_\alpha + M_\alpha)\,B^\alpha_i + (M + g^{\alpha\beta}\,M_{\alpha\beta})\,B_i\,, \tag{1.9}$$

where $C_i(= g^{jk}\,C_{ijk})$ and $C_\alpha(= g^{\beta\gamma}\,C_{\alpha\beta\gamma})$ are called torsion vectors of F^4 and F^3 respectively.

2 Main scalars of a four-dimensional Finsler space and its hypersurface

The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors (l^i, m^i, n^i, p^i), where $l^i = y^i/L$ is the normalized supporting element; $m^i =$

$C^i/\widetilde{c}$ is the normalized torsion vector ($\widetilde{c}$ is the length of the torsion vector C^i); n^i is constructed by $g_{ij}\, l^i\, n^j = g_{ij}\, m^i\, n^j = 0$, $g_{ij}\, n^i\, n^j = 1$; and the unit vector p^i is constructed by $g_{ij}\, l^i p^j = g_{ij}\, m^i p^j = g_{ij}\, n^i p^j = 0$, $g_{ij}\, p^i p^j = 1$.

In the Miron frame, an arbitrary tensor $T = (\,T^i_j\,)$ is expressed in terms of scalar components as follows:

$$T^i_j = T_{\lambda\mu}\; e^i_{\lambda)}\; e_{\mu)j}\,,$$

where $e^i_{1)} = l^i$, $e^i_{2)} = m^i$, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$ and the summation convention is applied to the indices λ and μ.

Let $C_{\lambda\mu\nu}$ be the scalar components of $L\,C_{ijk}$ with respect to the Miron frame, i.e.

$$L\,C_{ijk} = C_{\lambda\mu\nu}\; e_{\lambda)i}\; e_{\mu)j}\; e_{\nu)k}\,. \tag{2.1}$$

M. Matsumoto [5] showed that
(i) $C_{\lambda\mu\nu}$ are completely symmetric,
(ii) $C_{1\mu\nu} = 0$,
(iii) $C_{2\mu\mu} = L\,\widetilde{c}$, $C_{3\mu\mu} = C_{4\mu\mu} = \cdots = C_{n\mu\mu} = 0$ for $n \geq 3$.

Therefore in a four-dimensional Finsler space, we have

$$\begin{cases} C_{222} + C_{233} + C_{244} = L\widetilde{c}\,, \\ C_{322} + C_{333} + C_{344} = 0\,, \\ C_{422} + C_{433} + C_{444} = 0\,. \end{cases} \tag{2.2}$$

Thus putting

$$\begin{aligned} &C_{222} = A\,, \quad C_{233} = B\,, \quad C_{244} = C\,, \quad C_{322} = D\,, \\ &C_{333} = E\,, \quad C_{422} = F\,, \quad C_{433} = G\,, \quad C_{234} = H\,, \end{aligned}$$

we get

$$C_{344} = -(D+E)\,, \quad C_{444} = -(F+G)\,.$$

Eight scalars A, B, $\ldots$ G, H are called the main scalars of a four-dimensional Finsler space.

The equation (2.1) may be written in expanded form as:

$$\begin{aligned} LC_{ijk} = {} & A(m_i m_j m_k) + B(m_i n_j n_k + n_i m_j n_k + n_i n_j m_k) + C(m_i p_j p_k \\ & + p_i m_j p_k + p_i p_j m_k) + D(m_i m_j n_k + m_i n_j m_k + n_i m_j m_k) \\ & + E(n_i n_j n_k) + F(m_i m_j p_k + m_i p_j m_k + p_i m_j m_k) \\ & + G(n_i n_j p_k + n_i p_j n_k + p_i n_j n_k) + H(m_i n_j p_k + m_i p_j n_k \\ & + n_i m_j p_k + n_i p_j m_k + p_i m_j n_k + p_i n_j m_k) \\ & - (D+E)(n_i p_j p_k + p_i n_j p_k + p_i p_j n_k) - (F+G)(p_i p_j p_k)\,. \end{aligned} \tag{2.3}$$

The hypersurface F^3 of F^4 is a three-dimensional Finsler space. The Moór frame for F^3 is given by $(l^\alpha, m^\alpha, n^\alpha)$, where $l^\alpha = v^\alpha/\underline{L}$; $m^\alpha = C^\alpha/\underline{\widetilde{c}}$ ($\underline{\widetilde{c}}$ being the length of the torsion vector C^α of F^3) and n^α is constructed by $g_{\alpha\beta}\, l^\alpha n^\beta = g_{\alpha\beta}\, m^\alpha n^\beta = 0$, $g_{\alpha\beta}\, n^\alpha n^\beta = 1$.

For this frame, the Cartan tensor $C_{\alpha\beta\gamma}$ of F^3 is represented by [5]:

$$\begin{aligned} \underline{L}\,C_{\alpha\beta\gamma} = {} & \underline{H}\, m_\alpha\, m_\beta\, m_\gamma - \underline{J}\,(m_\alpha\, m_\beta\, n_\gamma + m_\alpha\, n_\beta\, m_\gamma + n_\alpha\, m_\beta\, m_\gamma) \\ & + \underline{I}\,(m_\alpha\, n_\beta\, n_\gamma + n_\alpha\, m_\beta\, n_\gamma + n_\alpha\, n_\beta\, m_\gamma) + \underline{J}\, n_\alpha\, n_\beta\, n_\gamma\,, \end{aligned} \tag{2.4}$$

where $\underline{H}$, $\underline{I}$ and $\underline{J}$ are the main scalars of F^3.

Transvecting (1.7) by v^α and using (1.1), we get $M_{\alpha\beta}\, v^\alpha = 0$, $M_\alpha\, v^\alpha = 0$. Therefore $M_{\alpha\beta}$ and M_α have no component in the direction of v^α (i.e. in the direction of l^α). Also $M_{\alpha\beta}$ is symmetric. Therefore M_α and $M_{\alpha\beta}$ may be written in the form $M_\alpha = \underline{U}\, m_\alpha + \underline{V}\, n_\alpha$ and $M_{\alpha\beta} = \underline{X}\, m_\alpha\, m_\beta + \underline{Y}(m_\alpha n_\beta + n_\alpha m_\beta) + \underline{Z}\, n_\alpha\, n_\beta$.
Thus, we have the following:

Proposition 2.1. *Let F^3 be the hypersurface of a four-dimensional Finsler space F^4, then the tensor M_α and $M_{\alpha\beta}$ defined by* (1.7)*, are written as $M_\alpha = \underline{U}\, m_\alpha + \underline{V}\, n_\alpha$ and $M_{\alpha\beta} = \underline{X}\, m_\alpha\, m_\beta + \underline{Y}(m_\alpha n_\beta + n_\alpha m_\beta) + \underline{Z}\, n_\alpha\, n_\beta$ respectively.*

From (2.3) and (2.4), the torsion vector C_i and C_α are represented by $L\, C_i = (A + B + C)\, m_i$ and $\underline{L}\, C_\alpha = (\underline{H} + \underline{I}\,)\, m_\alpha$ respectively. The equation (1.9) and Proposition 2.1 lead to

$$m_i = (A+B+C)^{-1}\{(\underline{H} + \underline{I} + \underline{U}\,)\, m_\alpha\, B^\alpha_i + \underline{V}\, n_\alpha\, B^\alpha_i + (M + \underline{X} + \underline{Z}\,)\, B_i\}, \tag{2.5}$$

which yields

$$(A+B+C)^2 = (\underline{H} + \underline{I} + \underline{U}\,)^2 + \underline{V}^2 + (M + \underline{X} + \underline{Z}\,)^2. \tag{2.6}$$

Let us put

$$(A+B+C)^{-1}(\underline{H} + \underline{I} + \underline{U}\,) = a\,,$$
$$(A+B+C)^{-1}\underline{V} = b\,,$$
$$(A+B+C)^{-1}(M + \underline{X} + \underline{Z}\,) = t\,,$$

then

$$m_i = a\, m_\alpha\, B^\alpha_i + b\, n_\alpha\, B^\alpha_i + t\, B_i\,. \tag{2.7}$$

Let us write the unit vectors n_i and p_i as:

$$n_i = d\, m_\alpha\, B^\alpha_i + e\, n_\alpha\, B^\alpha_i + f\, B_i\,, \tag{2.8}$$

and

$$p_i = g\, m_\alpha\, B^\alpha_i + h\, n_\alpha\, B^\alpha_i + l\, B_i\,, \tag{2.9}$$

where d, e, f, g, h, l are given by

$$\begin{cases} ad + be + tf = 0 \\ ag + bh + tl = 0 \\ dg + eh + fl = 0 \end{cases} \quad \text{and} \quad \begin{cases} a^2 + b^2 + t^2 = 1 \\ d^2 + e^2 + f^2 = 1 \\ g^2 + h^2 + l^2 = 1\,. \end{cases} \tag{2.10}$$

From (2.10), we also have the relations:

$$\begin{cases} ab + de + gh = 0 \\ at + df + gl = 0 \\ bt + ef + hl = 0 \end{cases} \quad \text{and} \quad \begin{cases} a^2 + d^2 + g^2 = 1 \\ b^2 + e^2 + h^2 = 1 \\ t^2 + f^2 + l^2 = 1\,. \end{cases} \tag{2.11}$$

Substituting (2.7), (2.8), (2.9) into (2.3) and using (1.3), we get

$$
\begin{aligned}
\underline{L}\, C_{\alpha\beta\gamma} = {} & m_\alpha m_\beta m_\gamma \{a^3 A + 3ad^2 B + 3ag^2 C + 3a^2 d\, D + d^3 E + 3a^2 g\, F \\
& \quad + 3d^2 g\, G + 6adg\, H - 3dg^2(D+E) - g^3(F+G)\} \\
& + (m_\alpha n_\beta n_\gamma + n_\alpha m_\beta n_\gamma + n_\alpha n_\beta m_\gamma)\{ab^2 A + (ae^2 + 2bde)B + (ah^2 + 2bgh)C \\
& \quad + (2abe + b^2 d)D + de^2 E + (2abh + b^2 g)F + (2deh + e^2 g)G \\
& \quad + (2aeh + 2bdh + 2beg)H - (dh^2 + 2egh)(D+E) - gh^2(F+G)\} \\
& + (m_\alpha m_\beta n_\gamma + m_\alpha n_\beta m_\gamma + n_\alpha m_\beta m_\gamma)\{a^2 b\, A + (bd^2 + 2ade)B + (bg^2 + 2agh)C \\
& \quad + (2abd + a^2 e)D + d^2 e\, E + (2abg + a^2 h)F + (2deg + d^2 h)G \\
& \quad + (2aeg + 2bdg + 2adh)H - (eg^2 + 2dgh)(D+E) - g^2 h(F+G)\} \\
& + n_\alpha n_\beta n_\gamma \{b^3 A + 3be^2 B + 3bh^2 C + 3b^2 e\, D + e^3 E + 3b^2 h\, F \\
& \quad + 3e^2 h\, G + 6beh\, H - 3eh^2(D+E) - h^3(F+G)\}.
\end{aligned}
$$

Comparing with (2.4), we get

$$
\begin{aligned}
\underline{H} = {} & a^3 A + 3ad^2 B + 3ag^2 C + 3a^2 d\, D + d^3 E + 3a^2 g\, F \\
& + 3d^2 g\, G + 6adg\, H - 3dg^2(D+E) - g^3(F+G)\,,
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
\underline{I} = {} & ab^2 A + (ae^2 + 2bde)B + (ah^2 + 2bgh)C + (2abe + b^2 d)D + de^2 E \\
& + (2abh + b^2 g)F + (2deh + e^2 g)G + (2aeh + 2bdh + 2beg)H \\
& - (dh^2 + 2egh)(D+E) - gh^2(F+G)\,,
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
\underline{J} = {} & b^3 A + 3be^2 B + 3bh^2 C + 3b^2 e\, D + e^3 E + 3b^2 h\, F \\
& + 3e^2 h\, G + 6beh\, H - 3eh^2(D+E) - h^3(F+G)\,,
\end{aligned} \tag{2.14}
$$

and also

$$
\begin{aligned}
-\underline{J} = {} & a^2 b\, A + (bd^2 + 2ade)B + (bg^2 + 2agh)C + (2abd + a^2 e)D + d^2 e\, E \\
& + (2abg + a^2 h)F + (2deg + d^2 h)G + (2aeg + 2bdg + 2adh)H \\
& - (eg^2 + 2dgh)(D+E) - g^2 h(F+G)\,.
\end{aligned} \tag{2.15}
$$

Similarly, substituting (2.7), (2.8), (2.9) into (2.3) and using (1.7) and Proposition 2.1, we get

$$
\begin{aligned}
\underline{X} = {} & a^2 t\, A + (d^2 t + 2adf)B + (g^2 t + 2agl)C + (2adt + a^2 f)D + d^2 f\, E \\
& + (2agt + a^2 l)F + (2dgf + d^2 l)G + (2adl + 2agf + 2dgt)H \\
& - (g^2 f + 2dgl)(D+E) - g^2 l(F+G)\,,
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
\underline{Y} = {} & abt\, A + (aef + bdf + det)B + (ahl + bgl + ght)C + (abf + aet + bdt)D \\
& + def\, E + (abl + aht + bgt)F + (del + dht + get)G + (bdl + bgf + ael \\
& + egt + ahf + hdt)H - (dhl + gel + ghf)(D+E) - hgl(F+G)\,,
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
\underline{Z} = {} & b^2 t\, A + (e^2 t + 2bef)B + (h^2 t + 2bhl)C + (2bet + b^2 f)D + e^2 f\, E \\
& + (2bht + b^2 l)F + (2ehf + e^2 l)G + (2bel + 2bhf + 2eht)H \\
& - (h^2 f + 2ehl)(D+E) - h^2 l(F+G)\,,
\end{aligned} \tag{2.18}
$$

$$\begin{aligned} \underline{U} &= at^2 A + (af^2 + 2tdf)B + (al^2 + 2tgl)C + (2atf + dt^2)D + df^2 E \\ &\quad + (2atl + gt^2)F + (2dfl + gf^2)G + (2afl + 2dtl + 2gtf)H \\ &\quad - (dl^2 + 2fgl)(D + E) - gl^2(F + G)\,, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \underline{V} &= bt^2 A + (bf^2 + 2tef)B + (bl^2 + 2thl)C + (2btf + et^2)D + ef^2 E \\ &\quad + (2btl + ht^2)F + (2efl + hf^2)G + (2bfl + 2etl + 2htf)H \\ &\quad - (el^2 + 2hfl)(D + E) - hl^2(F + G)\,, \end{aligned} \tag{2.20}$$

$$\begin{aligned} M &= t^3 A + 3tf^2 B + 3tl^2 C + 3ft^2 D + f^3 E + 3lt^2 F \\ &\quad + 3lf^2 G + 6tfl\,H - 3fl^2(D + E) - l^3(F + G)\,. \end{aligned} \tag{2.21}$$

Thus, we have:

Theorem 2.1. *Let F^3 be the hypersurface of a four-dimensional Finsler space F^4, then the main scalars of F^3 and F^4 are related by* (2.12), (2.13), (2.14) *and* (2.15).

3 C-reducible Finsler space

Pandey and Diwedi [6] proved that a four-dimensional Finsler space is c-reducible if and only if

$$A = 3B = 3C\,, \qquad D = E = F = G = H = 0\,. \tag{3.1}$$

In view of (2.10), (2.11) and (3.1), equations (2.12) to (2.21) reduce to

$$\begin{cases} \underline{H} = a\,A\,, & \underline{I} = a\,A/3\,, \quad \underline{J} = 0\,, \\ \underline{X} = t\,A/3\,, & \underline{Y} = 0\,, \quad \underline{Z} = t\,A/3\,, \\ \underline{U} = a\,A/3\,, & \underline{V} = 0\,, \quad M = t\,A\,, \end{cases} \tag{3.2}$$

which gives $\underline{H} = 3\underline{I}$, $\underline{J} = 0$. This shows that the hypersurface F^3 of a four-dimensional c-reducible Finsler space F^4 is also c-reducible; which is in agreement to the Matsumoto's results [4].

In view of (3.1) and (3.2), equation (2.6) becomes

$$\left(\frac{5A}{3}\right)^2 = \left(\frac{5\underline{H}}{3}\right)^2 + 0 + \left(\frac{5M}{3}\right)^2,$$

which gives

$$\underline{H} = \pm\sqrt{A^2 - M^2}\,.$$

Consequently, we have:

Theorem 3.1. *Let F^3 be the hypersurface of a four-dimensional c-reducible Finsler space F^4, then for the function M defined by* (1.7)*, the main scalars $\underline{H}$, $\underline{I}$ and $\underline{J}$ are given by*

$$\underline{H} = \pm\{A^2 - M^2\}^{1/2}\,, \quad \underline{I} = \pm\frac{1}{3}\{A^2 - M^2\}^{1/2}\,, \quad \underline{J} = 0\,.$$

Corollary 3.1. *In a four-dimensional c-reducible Finsler space, the main scalar A satisfies the condition: $A > M$ or $A < -M$.*

Now suppose the torsion vector C_i of F^4 is tangent to its hypersurface F^3, then from (2.7), $t = 0$. Therefore from (3.2), we get $\underline{X} = 0$, $\underline{Y} = 0$, $\underline{Z} = 0$, $M = 0$.
Thus, we get:

Theorem 3.2. *If the torsion vector C_i of a four-dimensional c-reducible Finsler space is tangent to its hypersurface, then $M_{\alpha\beta}$ and M defined by* (1.7), *vanish.*

Matsumoto [4] showed an important result for connections of the hypersurface that if $M_{\alpha\beta} = 0$, then the induced and intrinsic connections of the hypersurface concide.
This leads to:

Corollary 3.2. *If the torsion vector C_i of a four-dimensional c-reducible Finsler space is tangent to its hypersurface, then the induced connection of the hypersurface concides with its intrinsic connection.*

Now, if the torsion vector C_i of F^4 is normal to its hypersurface F^3, then from (2.7), $a = b = 0$. Therefore from (3.2), we get that all the main scalars $\underline{H}$, $\underline{I}$, $\underline{J}$ of F^3 vanish, which is not possible.
Thus, we have:

Theorem 3.3. *The torsion vector C_i of a four-dimensional c-reducible Finsler space is not normal to its hypersurface.*

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Authors' address:

M.K. Gupta and P.N. Pandey
Department of Mathematics, University of Allahabad, Allahabad, India.
E-mail: manishkg_1982@yahoo.com, pnpiaps@rediffmail.com