Structure on a slant submanifold of a Kenmotsu manifold

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Abstract. We give an intrinsic characterization of slant submanifolds of a Kenmotsu manifold in terms of the induced metric and show that a slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold. We also prove a theorem to obtain examples of slant submanifolds of Kenmotsu manifold.

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§1. Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen [6], [5]. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa ([5], [8], [7]), while that of slant submanifolds of a Kaehler manifold were given by Maeda, Ohnita and Udagawa [**?**]. On the other hand, A. Lotta [13] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [14]. Latter, L. Cabrerizo and others investigated slant submanifolds of a Sasakian manifold and obtained many interesting results ([3], [4]). Recently, we have studied slant submanifolds of Kenmotsu manifolds and trans-Sasakian manifolds ([10], [11]).

§2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors $\{\varphi, \xi, \eta, g\}$, where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$. These tensors satisfy [1]

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \quad \text{and} \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi)$$

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for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold is called a Kenmotsu manifold if [12]

$$(\overline{\nabla}_X\varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X \quad \text{and} \quad \overline{\nabla}_X\xi = X - \eta(X)\xi \tag{2.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM the Lie algebra of vector fields on M and by $T^\perp M$ the set of all vector fields normal to M. For any $X \in TM$ and $N \in T^\perp M$, we write

$$\varphi X = PX + FX, \quad \varphi N = tN + fN \tag{2.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \xi$.

For each non zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM. The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ ([13]). The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{2.5}$$

for $X, Y \in TM$ and $N \in T^\perp M$ of $\overline{M}$; h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.6}$$

and $\nabla^\perp$ is the connection in the normal bundle $T^\perp M$ of M.

The mean curvature vector H is defined by $H = (\frac{1}{m})\, trace\, h$. We say that M is minimal if H vanishes identically.

If P is the endomorphism defined by (2.3), then

$$g(PX,Y)+g(X,PY)=0 \tag{2.7}$$

Thus P^2, which is denoted by Q, is self-adjoint.

On the other hand, Gauss and Weingarten formulae together with (2.2) and (2.3) imply

$$(\nabla_X P)Y = A_{FY}X + th(X,Y) + g(Y,PX)\xi - \eta(Y)PX \tag{2.8}$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,PY) - \eta(Y)FX \tag{2.9}$$

for any $X, Y \in TM$.

We mention the following results for latter use:

Theorem A. [3] *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi) \tag{2.10}$$

Furthermore, if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary B. [3] *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(PX,PY) = cos^2\theta(g(X,Y) - \eta(X)\eta(Y)) \tag{2.11}$$

$$g(FX,FY) = sin^2\theta(g(X,Y) - \eta(X)\eta(Y)) \tag{2.12}$$

Lemma C. [13] *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then, at each point x of M, $Q|_D$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

§3. Intrinsic characterization of slant immersions of Kenmotsu manifolds

We now study intrinsic characterization of slant immersion of Kenmotsu manifold $\overline{M}$ in terms of slant angle of a slant submanifold M and also the sectional curvature of arbitrary plane section of M containing structure vector field ξ. We have:

Lemma 3.1. *Let M be a slant submanifold of a Kenmotsu manifold $\overline{M}$ such that structure vector field ξ is tangent to M. Then curvature vector field associated to the metric induced by $\overline{M}$ on M is given by*

(3.1) $$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y)$$

Moreover,

(3.2) $$R(\xi,X)\xi = X - \eta(X)\xi$$

and

(3.3) $$R(X,\xi,\xi,X) = \eta^2(X) - g(X,X)$$

Proof. From equation (2.2), we have

(3.4) $$\nabla_X\xi = X - \eta(X)\xi$$

for any $X \in TM$. Further,

(3.5) $$(\nabla_X P)Y = -\nabla_X\nabla_Y\xi + \nabla_{\nabla_X Y}\xi + 2\eta(X)\eta(Y)\xi - g(X,Y)\xi - \eta(Y)X$$

Substituting this formula in the definition of $R(X,Y)\xi$ it is easy to get (3.1). Rewriting (3.1) for $X = \xi$ and $Y = X$ and using (3.5), we obtain

$$R(\xi,X)\xi = X - \eta(X)\xi$$

which gives (3.3).

Theorem 3.1. *Let M be an immersed submanifold of a Kenmotsu manifold $\overline{M}$ such that ξ is tangent to M. Then the following statements are equivalent*:
(a) *M is slant in $\overline{M}$ with slant angle θ*
(b) *For any x of M the sectional curvature of any 2-dimensional plane of T_xM containing ξ_x equals -1.*

Proof. Assume that (a) is true. Then by Theorem (A), for any unit vector field $X \in TM$ orthogonal to ξ, we have

$$QX = -cos^2\theta\, X$$

which by virtue of (3.3) yields

$$R(X,\xi,\xi,X) = -1.$$

Let (b) hold and $cos\theta \neq 0$. For any $X \in TM$, we use the decomposition

$$X = X_\xi^\perp + X_\xi$$

where $X_\xi = g(X,\xi)\xi$. Then by the hypothesis

(3.6) $$\frac{R(X_\xi^\perp,\xi,X_\xi^\perp,\xi)}{|X_\xi^\perp|^2} = -1$$

Now, if X is a unit vector field such that $QX = 0$, then from (3.3), we get

$$|X_\xi^\perp|^2 = -|X_\xi^\perp|^2$$

that is, $|X_{\xi}^{\perp}|^2 = 0$ and hence $X = X_{\xi}$. This proves that

$$Ker(Q) = Span\,\{\xi_x\}, \quad \forall x \in M. \tag{3.7}$$

Moreover, let X be a unit vector field such that $QX = \lambda_1 X$, where $\lambda_1 : M \longmapsto \mathbf{R}$ is a smooth function and for any $x \in M, \lambda_1(x) = 0$. Obviously, X is orthogonal to ξ, that is $X = X_{\xi}^{\perp}$ and using (3.3) and (3.6) it follows that $\lambda_1 = -cos^2\theta$.

We conclude that for any $x \in M$ the number $-cos^2\theta$ is the only eigenvalue of Q different from 0. This fact together with (3.7) implies that M is slant in $\overline{M}$ with slant angle θ.

Now, suppose that $cos\theta = 0$ and let X be an arbitrary unit vector field of eigenvectors of Q. Then $QX = \lambda_1 X$, where λ_1 is a function on M. Now, equations (3.3) and (3.6) imply that $g(QX, X) = 0$, that is $\lambda_1 = 0$. Thus, we conclude that $Q = 0$, which means that M is anti-invariant whereby proving (a).

§4. Structure on a slant submanifold

In [5], Chen gives the notion of a Kaehlerian slant submanifold of an almost Hermitian manifold as a proper slant submanifold such that the tangential component T of the almost complex structure J is parallel, that is $\nabla T = 0$. In fact, Kaehlerian slant submanifold is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $\overline{J} = (sec\,\theta)\,T$, where θ denotes the slant angle.

Let M be a submanifold of a Kenmotsu manifold $\overline{M}$ such that ξ is in TM. It is well known that if M is an invariant submanifold, then the structure of $\overline{M}$ induces, in a natural way, a Kenmotsu structure over M. In this case the submanifold is usually called a Kenmotsu submanifold. The purpose of this paper is to study if we can obtain an induced Kenmotsu structure on a non-invariant slant submanifold.

In an almost contact case, we have

Lemma 4.1. *Let M be a non-anti-invariant slant submanifold of an almost contact metric manifold $\overline{M}$. Then, M is an almost contact metric manifold with respect to the induced metric, with structure vector field ξ, and with almost contact structure given by $\overline{\phi} = (sec\,\theta)\,P$, where θ denotes the slant angle of M.*

Proof. By virtue of (2.10) and (2.11) we can show that $\overline{\phi}^2 X = -X + \eta(X)\xi$ and $g(\overline{\phi}X, \overline{\phi}Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields $X, Y \in TM$.

Now, we want to find an appropriate condition for ∇P so that it becomes possible to induce a Kenmotsu structure on M.

In [10] we have shown that for a proper slant submanifold of a Kenmotsu manifold

(4.1) $$(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\xi$$

for any vector fields $X, Y \in TM$. In fact, the almost contact metric structure given by $\overline{\phi}$ is a Kenmotsu structure, as from (4.1), we can see that

$$(\nabla_X \overline{\phi})Y = -\eta(Y)\overline{\phi}X + g(Y, \overline{\phi}X)\xi$$

for any vector fields $X, Y \in TM$.

From (2.8) and (2.9), for invariant and anti-invariant submanifolds of a Kenmotsu manifold, we have

(4.2) $$(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\xi$$

(4.3) $$(\nabla_X F)Y = -\eta(Y)FX$$

In case of invariant and anti-invariant submanifolds M of a Kenmotsu manifold, it is easy to show that the structure of $\overline{M}$ induces, in a natural way, a Kenmotsu structure over M. In this case the submanifold is usually called a Kenmotsu submanifold.

Therefore, we have

Proposition 4.1. *A slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold.*

Also, from Theorem 3.1, it is clear that slant submanifold of a Kenmotsu manifold is a Kenmotsu manifold.

§5. Examples of Slant submanifolds of Kenmotsu Manifolds

We now give some examples of slant submanifolds of $\mathbf{R}^{2n+1}$ with almost contact structure $\{\varphi_0, \xi, \eta, g\}$, which satisfy

$$(\overline{\nabla}_X \varphi_0)(Y) = g(\varphi_0 X, Y)\xi - \eta(Y)\varphi_0 X, \quad \overline{\nabla}_X \xi = X - \eta(X)\xi$$

for $X, Y \in T\mathbf{R}^{2n+1}$.

The Kenmotsu structure on $\mathbf{R}^{2n+1}$ is

(5.1) $$\eta = dt, \quad \xi = \frac{\partial}{\partial t}$$

(5.2) $$g = \eta \otimes \eta + e^{2t}\left(\sum_{i=1}^{n} dx^i \otimes dx^i + dy^i \otimes dy^i\right)$$

(5.3) $$\varphi_0\left(\sum_{i=1}^{n}\left(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial t}\right) = \sum_{i=1}^{n}\left(-Y_i \frac{\partial}{\partial x^i} + X_i \frac{\partial}{\partial y^i}\right)$$

where (x^i, y^i, t) are the Cartesian coordinates of $\mathbf{R}^{2n+1} = C^n \times \mathbf{R}$.

Now, we prove the following result to obtain examples of slant submanifolds in $\mathbf{R}^5(\varphi_0, \xi, \eta, g)$:

Theorem 5.1. *Let*

$$x(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v))$$

defines a slant surface S in C^2 with its usual Kaehlerian structure, such that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are non-zero and perpendicular to each other. Then

$$y(u,v,w) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v), w)$$

defines a three dimensional slant submanifold M in $\mathbf{R}^5(\varphi_0, \xi, \eta, g)$ with the same slant angle such that, if we put $e_1 = \frac{1}{e^t}\frac{\partial}{\partial u}, e_2 = \frac{1}{e^t}\frac{\partial}{\partial v}$ then $\{e_1, e_2, \xi\}$ form an orthogonal basis of the tangent bundle of the submanifold.

Proof. Using $\{e_1, e_2, \xi\}$, it is easy to show that M is a three-dimensional submanifold of $\mathbf{R}^5$. To prove that M is slant, we write

$$X = \lambda_1 e_1 + \lambda_2 e_2 + \eta(X)\xi, \quad \text{for } X \in \chi(M).$$

Then

$$\sqrt{|X|^2 - \eta^2(X)} = \sqrt{\lambda_1^2 + \lambda_2^2} \tag{5.4}$$

Now, since $\{e_1, e_2, \xi\}$ is an orthogonal basis of $\chi(M)$, using (2.5) we obtain

$$|PX|^2 = \frac{g^2(\varphi_0 X, e_1)}{g(e_1, e_2)} + \frac{g^2(\varphi_0 X, e_2)}{g(e_2, e_2)} \tag{5.5}$$

We may consider a vector field $X_0 \in TS$ such that $X_0 = \lambda_1 \frac{\partial}{\partial u} + \lambda_2 \frac{\partial}{\partial v}$ and denoting by J the usual almost complex structure of C^2, we find that

$$g(\varphi_0 X, e_1) = g(JX_0, \tfrac{\partial}{\partial u}), \quad g(\varphi_0 X, e_2) = g(JX_0, \tfrac{\partial}{\partial v})$$

If TX_0 is the tangent projection of JX_0 and θ is the slant angle of S, then from (5.4) and (5.5), we get

$$\frac{|PX|}{\sqrt{|X|^2 - \eta^2(X)}} = \frac{|TX_0|}{|X_0|} = cos\theta$$

Hence, M is a slant submanifold with the same slant angle θ.

By using the examples given in [6] and the above theorem, we now give some examples of slant submanifolds of Kenmotsu manifolds in $\mathbf{R}^5(\varphi_0, \xi, \eta, g)$:

Example 5.1. For any $\theta \in [0, \frac{\pi}{2}]$,

$$x(u,v,w) = (ucos\theta, usin\theta, v, 0, w)$$

defines a three-dimensional minimal slant submanifold M with slant angle θ.

We may choose an orthonormal basis $\{e_1, e_2, \xi\}$ such that

$$e_1 = \frac{1}{e^t}(cos\theta\frac{\partial}{\partial x^1} + sin\theta\frac{\partial}{\partial x^2})$$

$$e_2 = \frac{1}{e^t}\frac{\partial}{\partial y^1}, \quad \xi = \frac{\partial}{\partial t}$$

Moreover, the vector fields

$$e_1^* = \frac{1}{e^t}(-sin\theta\frac{\partial}{\partial x^1} + cos\theta\frac{\partial}{\partial x^2})$$

and

$$e_2^* = \frac{1}{e^t}\frac{\partial}{\partial y^2}$$

form an orthonormal basis for $T^\perp M$. Since $\overline{\nabla}_{e_1}e_1 = -\xi$, $\overline{\nabla}_{e_2}e_2 = -\xi$ and $\overline{\nabla}_\xi\xi = 0$, we get $h(e_1,e_1) = h(e_2,e_2) = h(\xi,\xi) = 0$. Therefore, the submanifold is minimal.

Example 5.2. For any constant k,

$$x(u,v,w) = (e^{ku}cos\,u\,cos\,v, e^{ku}sin\,u\,cos\,v, e^{ku}cos\,u\,sin\,v, e^{ku}sin\,u\,sin\,v, w)$$

defines a three-dimensional slant submanifold M with slant angle $\theta = cos^{-1}(\frac{|k|}{\sqrt{1+k^2}})$.

We may choose an orthonormal basis $\{e_1, e_2, \xi\}$ such that

$$e_1 = \frac{1}{e^t}(\frac{e^{-ku}}{\sqrt{1+k^2}}\frac{\partial}{\partial u})$$

$$e_2 = \frac{1}{e^t}(e^{-ku}\frac{\partial}{\partial v}), \quad \xi = \frac{\partial}{\partial t}$$

Then, by a straight forward computation we can show that it is a three dimensional slant submanifold.

Example 5.3. For any positive constant k,

$$x(u,v,w) = (u,\ k\,cos\,v,\ v,\ k\,sin\,v,\ w)$$

defines a three-dimensional non-minimal slant submanifold M with slant angle $\theta = cos^{-1}(\frac{1}{\sqrt{1+k^2}})$.

Moreover, the following statements are equivalent:

(i) $k = 0$, (ii) M is invariant (iii) M is minimal.

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