Immersed hypergroups

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Abstract. In this paper the notion of immersed hypergroup is considered. It is proved that the product of two immersed hypergroups is an immersed hypergroup. We show that any open subset of immersed hypergroup is an immersed hypergroup, and the image of any immersed hypergroup is an immersed hypergroup. Quotient hypergroups are considered and it is proved that they are immersed hypergroups.

M.S.C. 2000: 20N20, 20N99.
Key words: Immersed hypergroup, quotient hypergroup, hypergroupoid.

1 Introduction

Theory of hypergroups have brought many bright results in mathematics [1, 4, 5, 6]. In this paper we would like to consider hypergroups which are also smooth manifolds and their join operations create immersed submanifolds. More precisely we have the next definition.

Definition 1.1. A hypergroup H is called an immersed hypergroup, if H is smooth real manifold and for all $a, b \in H$, ab is an immersed submanifold of H.
We will prove the following results:
(1) If H_1 and H_2 are immersed hypergroups, then $H_1 \times H_2$ is an immersed hypergroup.
(2) If H is an immersed hypergroup, then any open subset U of H is an immersed hypergroup.
(3) If H is an immersed hypergroup and Ψ is an immersion on H, then $\Psi(H)$ is an immersed hypergroup.
If H is an immersed hypergroup and $\{A(x)\}$ is a family of open subsets of a differentiable manifold N, then we will present two methods to construct new immersed hypergroups.

2 Immersed hypergroups

We begin this section with an example of an immersed hypergroup.

Example 2.1. Let H be the set of polynomials $P_n : [0, 1] \to [0, 1]$ such that $x \mapsto a_n x^n + ... + a_0$, where $a_i \in [0, 1]$ and additions and multiplications are in the

Differential Geometry - Dynamical Systems, Vol.10, 2008, pp. 159-162.

mode 1.
H with the norm $\|P_n(x)\| = sup\{|P_n(x)| : x \in [0,1]\}$ is a Banach space, which is also an infinite dimensional real manifold.

If $P_n(x) = a_0 + a_1 x + ... + a_n x^n$ and $P_m(x) = b_0 + b_1 x + ... + b_m x^m$, then we define $P_n(x)P_m(x) = \{a_0 b_0 + c_1 x + ... + c_{m+n-1} x^{m+n-1} + a_n b_m : c_i \in [0,1]\} \equiv [0,1]^{m+n-1}$.

H with the above differentiable structure is a hypergroup.

Example 2.2. Let H be the set of $k-$dimensional subspaces of $\mathbb{R}^n$, with $k \leq n$, if $a, b \in H$, then we define $ab = \{c \mid c \text{ is a } k\text{-dimensional subspace of } a+b\}$.

H with this join operation and the above differentiable structure is an immersed hypergroup.

Theorem 2.1. *Let H_1 and H_2 be immersed hypergroups. Then $H_1 \times H_2$ is an immersed hypergroup.*

Proof. We know that $H = H_1 \times H_2$ is a smooth manifold. We only show that for given $(a_1, a_2), (b_1, b_2) \in H$, $a_1 b_1 \times a_2 b_2$ is an immersed submanifold of H.

Since $i_1 : a_1 b_1 \to H_1$ and $i_2 : a_2 b_2 \to H_2$ are immersions, then the map $i : a_1 b_1 \times a_2 b_2 \to H_1 \times H_2$ defined by $i(c_1, c_2) = (i_1(c_1), i_2(c_2))$ is an immersion as well. Because it is a one-to-one map and $di(c_1, c_2) = di_1(c_1) \times di_2(c_2)$, which is a one-to-one map. $\square$

Theorem 2.2. *If H is an immersed hypergroup. Then any open subset U of H with the join operation $\star_U$ on $U \times U$ defined by $a \star_U b = ab \cap U$ is an immersed hypergroup.*

Proof. U with the atlas $A_U = \{\text{smooth charts } (V, \Theta) \text{ of } H : V \subset U\}$ is a smooth manifold. Moreover, it is a hypergroup [1, 4]. If $a, b \in H$, then $i : ab \to H$ is an immersion. Hence $i_U : ab \cap U \to U$ is also an immersion. Thus U is an immersed hypergroup. $\square$

Theorem 2.3. *If H is an immersed hypergroup, and $\Psi : H \to \mathbb{R}^k$ is a smooth one-to-one immersion, then $\Psi(H)$ with the join operation $\Psi(a)\Psi(b) = \Psi(ab)$ is an immersed hypergroup, where the smooth structure of $\Psi(H)$ is the smooth structure induced by Ψ.*

Proof. $\Psi(H)$ is a hypergroup, because $\Psi(a)\Psi(H) = \Psi(aH) = \Psi(Ha) = \Psi(H)\Psi(a) = \Psi(H)$, for all $a \in H$. Then $\Psi(a)\{\Psi(b)\Psi(c)\} = \Psi(a)\Psi(bc) = \Psi(abc) = \Psi(ab)\Psi(c) = \{\Psi(a)\Psi(b)\}\Psi(c)$, for all $a, b, c \in H$. Moreover, $\Psi(H)$ with the smooth structure created by Ψ as a diffeomorphism, is a smooth manifold.

Let $\Psi(a)$ and $\Psi(b) \in \Psi(H)$ be given. Then there exists an immersion $i : ab \to H$. Hence the mapping $\Psi oi : ab \to \mathbb{R}^k$ is an immersion, and $\Psi(ab) = \Psi oi(ab)$ is an immersed submanifold of $\mathbb{R}^k$. $\square$

Suppose $\langle H, o \rangle$ is a hypergroupoid ([1]) and assume that $\{A(x)\}$ is a family of pairwise disjoint non-empty sets and $K_H = \cup_{x \in H} A(x)$. We define $g(a) = x$ if and only if $a \in A(x)$, for all $a \in K_H$. Now in K_H we have the hyperoperation:

$a \odot b = \cup_{z \in g(a) o g(b)} A(z)$, where $(a, b) \in K_H^2$.

It is well known that (H, o) is a hypergroup if and only if $\langle K_H, \odot \rangle$ is a hypergroup ([1]).

Theorem 2.4. *Let (H, o) be an immersed hypergroup and let $\{A(x)\}$ be a family of pairwise disjoint non-empty open subsets of a differentiable manifold N. Then K_H is an immersed hypergroup.*

Proof. Since $A(x)$ is open, K_H is an open submanifold of N. Thus $a \odot b$ is also an open submanifold of K_H, for all $(a,b) \in K_H^2$. So it is an immersed submanifold of K_H. $\square$

3 Quotient immersed hypergroups

Let M be a differentiable manifold and $R \subset M \times M$ be an equivalence relation on M. Moreover let M/R be the set of the equivalence classes of M with respect to R. Let $P : M \to M/R$ be the natural projection $m \mapsto [m]$. We define the quotient topology on M/R, i.e., $U \subset M/R$ is open if and only if $P^{-1}(U)$ is open in M. Thus $P : M \to M/R$ is a continuous map. Moreover, for any continuous map $F : M \to N$, there exists an unique continuous map $\bar{F} : M/R \to N$ such that $F = \bar{F} o P$. Hence we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "P"'] & N \\ \frac{M}{R} \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

In general, M/R may not be a manifold. We now assume that M/R has a differentiable structure such that $P : M \to M/R$ is a submersion. Since P is a continuous map, for all open set U in M/R, $P^{-1}(U)$ is an open set in M. Moreover, P is an open map, because it is a submersion [4]. Thus $U = P(P^{-1}(U))$ is open in M/R, for all open sets U in M/R. Therefore a subset U in M/R is open if and only if $P^{-1}(U)$ is open in M, i.e., the topology on M/R is the quotient topology.

Moreover, if the map F from M into a differentiable manifold N is differentiable, then the map $\bar{F} : M/R \to N$ is also differentiable, and the differentiable structure on M/R is unique [4].

M/R is called the quotient manifold of M with respect to R, if it is allows a differentiable structure such that $P : M \to M/R$ is a submersion. In this case, the equivalence relation is called regular. In fact, R is regular if and only if R is a closed submanifold of $M \times M$ and the relations are defined by $P_1(m_1, m_2) = m_1$ and $P_2(m_1, m_2) = m_2$ are submersions.

Let (H, o) be a hypergroupoid and let ρ be an equivalence relation on H. ρ is called regular on the right if the following implication holds;

$a\rho b$ then for all $u \in H, x \in aou$ and there exists $y \in bou$ such that $x\rho y$ and for all $\bar{y} \in bou$, there exists $\bar{x} \in aou$ such that $\bar{x}\rho\bar{y}$.

Similarly, the regularity on the left can be defined.

Now we state the main theorem of this section.

Theorem 3.1. *Let (M, o) be an immersed hypergroup and ρ be a left and right regular equivalence relation on M. Moreover, let $P : M \to M/\rho$ be a submersion. Then M/ρ with respect to the join operation, $[x] \otimes [y] = \{[z] : z \in xoy\}$ is an immersed hypergroup.*

Proof. Since M is an immersed hypergroup, there exists an immersion $i : xoy \to M$, for all $x, y \in M$. Let $[x]$ and $[y] \in M/\rho$ be given. Then the inclusion map $i : x_0oy_0 \to M$ is an immersion, for each $x_0 \in [x]$ and $y_0 \in [y]$. The differentiable structure of M/ρ implies for given $[m] \in [x] \otimes [y]$ there exist the open sets V and W of M and a diffeomorphism $\beta : V/\rho_V \to W$, where $\rho_V = \rho \cap (V \times V)$.

Now the inclusion map $\bar{i} : ([x] \bigotimes [y]) \cap V/\rho_V \to M/\rho$ is equal to the map $io\beta$ restricted to $([x] \otimes [y]) \cap V/\rho_V$. Thus it is an immersion.

Moreover, since ρ is left and right regular equivalence relation, then M/ρ is a hypergroup [1]. Thus M/ρ is an immersed hypergroup. □

References

[1] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory, Advances in Mathematics*, Vol. 5, Klawer Academic Publisher, 2003.

[2] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2003.

[3] J. Mittas, *Hypergroupes Canoniques*, Math. Balkania, 2 (1972).

[4] D. Miličić, *Lectures on Lie Groups*, http://www.math.utah.edu/ milicic/lie.pdf.

[5] M.R. Molaei, *Generalized Dynamical Systems*, Pure Mathematics and Applications, 14, 1-2 (2003), 117-120.

[6] M.R. Molaei, F. Iranmanesh, *Conjugacy in Hyper Semi-dynamical Systems*, General Mathematics, 12, 4 (2004), 11-16.

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