Minimal surfaces in the 3-dimensional Heisenberg group

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Abstract. An integral representation formula in terms of the normal Gauss map for minimal surfaces in the 3-dimensional Heisenberg group with canonical left invariant metric is obtained.

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Introduction

In our previous papers ([3], [5]), an integral representation formula for minimal surfaces in the model space Sol_3 of the solvegeometry (in the sense of Thurston [9]).

In [4], an integral representation formula for minimal surfaces in the Heisenberg group Nil_3 (the model space of nilgeometry in the sense of Thurston) is obtained. F. Mercuri, S. Montaldo and P. Piu [7] independently obtained such an integral representation formula for minimal surfaces in Nil_3.

D. A. Berdinskiĭ and I. A. Taĭmanov [1] obtained a representation formula for minimal surfaces in Nil_3 in terms of spinors and Dirac operators.

In this paper, we study normal Gauss maps for minimal surfaces in Nil_3 and reformulate the integral representation formula due to [4] and [7] in terms of the normal Gauss map. Via the reformulation in terms of normal Gauss map, the geometric meaning of the data for the integral representation formula is clarified. In fact, we shall show that every minimal surface in Nil_3 other than vertical plane is determined by a harmonic map into the Poincáre disc.

1 Nilpotent Lie groups

In this paper we study minimal surfaces in the simply connected nilpotent Lie group $G(\lambda)$.

We define a 1-parameter family $\{G(\lambda)\}_{\lambda\in\mathbb{R}}$ of 3-dimensional nilpotent Lie group by

$$G(\lambda) = (\mathbb{R}^3(x^1, x^2, x^3), \cdot)$$

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with multiplication:

$$(x^1, x^2, x^3) \cdot (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (x^1 + \tilde{x}^1, x^2 + \tilde{x}^2, x^3 + \tilde{x}^3 + \frac{\lambda}{2}(x^1\tilde{x}^2 - \tilde{x}^1 x^2)\ ).$$

The unit element of $G(\lambda)$ is $\vec{0} = (0,0,0)$. The inverse element of (x^1, x^2, x^3) is $-(x^1, x^2, x^3)$. Obviously, $G(0)$ is the abelian group $(\mathbb{R}^3, +)$.

The Lie algebra $\mathfrak{g}(\lambda)$ of $G(\lambda)$ is $\mathbb{R}^3$ with commutation relations:

$$[E_1, E_2] = \lambda\, E_3,\ [E_2, E_3] = [E_3, E_1] = \vec{0} \tag{1}$$

with respect to the natural basis $E_1 = (1,0,0)$, $E_2 = (0,1,0)$, $E_3 = (0,0,1)$. The formulae (1) imply that $\mathfrak{g}(\lambda)$ is nilpotent. The left translated vector fields of E_1, E_2, E_3 are

$$e_1 = \frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z},$$

respectively.

We equip an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}(\lambda)$ so that $\{E_1, E_2, E_3\}$ is orthonormal with respect to it. Then the resulting left invariant Riemannian metric $g = g_\lambda$ on $G(\lambda)$ is

$$g_\lambda = (dx^1)^2 + (dx^2)^2 + \omega \otimes \omega, \tag{2}$$

where

$$\omega = dx^3 + \frac{\lambda}{2}(x^2 dx^1 - x^1 dx^2). \tag{3}$$

The one-form ω satisfies $d\omega \wedge \omega = -\lambda dx^1 \wedge dx^2 \wedge dx^3$. Thus ω is a *contact form* on $G(\lambda)$ if and only if $\lambda \neq 0$.

The homogeneous Riemannian 3-manifold $(G(\lambda), g_\lambda)$ is called the 3-dimensional *Heisenberg group* if $\lambda \neq 0$. Note that $(G(0), g_0)$ is the Euclidean 3-space. The homogeneous Riemannian 3-manifold $(G(1), g_1)$ is frequently referred as the model space Nil_3 of the nilgeometry in the sense of Thurston [9].

2 Matrix group model of $G(\lambda)$

The Lie group $G(\lambda)$ is realised as a closed subgroup of the general linear group $\mathrm{GL}_4\mathbb{R}$. In fact, $G(\lambda)$ is imbedded in $\mathrm{GL}_4\mathbb{R}$ by $\iota : G(\lambda) \to \mathrm{GL}_4\mathbb{R}$;

$$\iota(x^1, x^2, x^3) = \begin{pmatrix} e^{x^1} & 0 & 0 & 0 \\ 0 & 1 & \lambda x^1 & x^3 + \frac{\lambda}{2}x^1 x^2 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly ι is an injective Lie group homomorphism. Thus $G(\lambda)$ is identified with

$$\left\{ \begin{pmatrix} e^{x^1} & 0 & 0 & 0 \\ 0 & 1 & \lambda x^1 & x^3 + \frac{\lambda}{2}x^1 x^2 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\ x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}(\lambda)$ corresponds to

$$\left\{ \begin{pmatrix} u^1 & 0 & 0 & 0 \\ 0 & 0 & \lambda u^1 & u^3 \\ 0 & 0 & 1 & u^2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; u^1, u^2, u^3 \in \mathbb{R} \right\}.$$

The orthonormal basis $\{E_1, E_2, E_3\}$ is identified with

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Levi-Civita connection ∇ if g is given by

$$\begin{gathered} \nabla_{E_1} E_1 = 0, \;\; \nabla_{E_1} E_2 = \frac{\lambda}{2} E_3, \;\; \nabla_{E_1} E_3 = -\frac{\lambda}{2} E_2, \\ \nabla_{E_2} E_1 = -\frac{\lambda}{2} E_3, \;\; \nabla_{E_2} E_2 = 0, \;\; \nabla_{E_2} E_3 = \frac{\lambda}{2} E_1, \\ \nabla_{E_3} E_1 = -\frac{\lambda}{2} E_2, \;\; \nabla_{E_3} E_2 = \frac{\lambda}{2} E_1, \;\; \nabla_{E_3} E_3 = 0. \end{gathered} \tag{1}$$

The Riemannian curvature tensor R defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is given by

$$R^1_{212} = -\frac{3\lambda^2}{4}, \; R^1_{313} = R^2_{323} = \frac{\lambda^2}{4}. \tag{2}$$

The Ricci tensor field Ric is given by

$$R_{11} = R_{22} = -\frac{\lambda^2}{2}, \; R_{33} = \frac{\lambda^2}{2}. \tag{3}$$

The scalar curvature ρ of G is $\rho = -\lambda^2/2$. The *natural-reducibility obstruction* U defined by

$$2g(U(X,Y), Z) = g(X, [Z,Y]) + g(Y, [Z,X]), \quad X, Y, Z \in \mathfrak{g}(\lambda)$$

is given by

$$U(E_1, E_3) = -\frac{\lambda}{2} E_3, \; U(E_2, E_3) = \frac{\lambda}{2} E_1. \tag{4}$$

Note that U measures the non right-invariance of the metric. In fact $U = 0$ if and only if g is right invariant (and hence biinvariant). The formulae (4) implies that g is biinvariant if and only if $\lambda = 0$.

The following formula was obtained in [4] (see also [7, (9)]).

Proposition 2.1 ([4]-II) *Let* $\{\omega^1, \omega^2, \omega^3\}$ *be a solution to*

$$\bar{\partial}\omega^1 = \frac{\lambda}{2}(\omega^2 \wedge \overline{\omega}^3 - \overline{\omega}^2 \wedge \omega^3); \tag{5}$$

$$\bar{\partial}\omega^2 = -\frac{\lambda}{2}(\omega^1 \wedge \overline{\omega}^3 - \overline{\omega}^1 \wedge \omega^3); \tag{6}$$

$$\bar{\partial}\omega^3 = -\frac{\lambda}{2}(\omega^1 \wedge \overline{\omega}^2 + \overline{\omega}^1 \wedge \omega^2) \tag{7}$$

on a simply connected coordinate region $\mathfrak{D} \subset \mathbb{C}$. *Then*

$$\varphi(z, \bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\left(\omega^1, \omega^2, \omega^3 - \frac{\lambda}{2}(x^2 \cdot \omega^1 - x^1 \cdot \omega^2)\right)$$

is a harmonic map into Nil_3. *Moreover if in addition,* $\{\omega^1, \omega^2, \omega^3\}$ *satisfies*

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0,$$

and

$$\omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2} + \omega^3 \otimes \overline{\omega^3} \neq 0.$$

Then φ *is an minimal immersion.*
Conversely, any harmonic map of $\mathfrak{D}$ *into* Nil_3 *can be represented in this form.*

3 The normal Gauss map

Let $\varphi : M \to G(\lambda)$ be a conformal immersion. Take a unit normal vector field N along φ. Then, by the left translation we obtain the following smooth map:

$$\psi := \varphi^{-1} \cdot N : M \to S^2 \subset \mathfrak{g}(\lambda).$$

The resulting map ψ takes value in the unit sphere in the Lie algebra $\mathfrak{g}(\lambda)$. Here, via the orthonormal basis $\{E_1, E_2, E_3\}$, we identify $\mathfrak{g}(\lambda)$ with Euclidean 3-space $\mathbb{E}^3(u^1, u^2, u^3)$.

The smooth map ψ is called the *normal Gauss map* of φ ([6], [3, p. 370]).

Express the data as $\omega^i = \phi^i dz$. First we consider the case "$\phi^3 \equiv 0$". In this case, we have

$$(x^1_z)^2 + (x^2_z)^2 = 0, \quad |x^1_z|^2 + |x^2_z|^2 > 0,$$

$$x^3_z + \frac{\lambda}{2}(x^1_z x^2 - x^1_z x^2) = 0.$$

These equation imply that the minimal surface determined by the condition $\phi^3 \equiv 0$ is an integral surface of the distribution $\omega = 0$. The distribution $\omega = 0$ is integrable if and only if $\lambda = 0$. When $\lambda = 0$, then the surface is a vertical plane $x^3 =$ constant.

Hereafter we assume that $\phi^3 \not\equiv 0$ and introduce the mapping f and g by

$$f := \phi^1 - \sqrt{-1}\phi^2, \quad g := \frac{\phi^3}{\phi^1 - \sqrt{-1}\phi^2}. \tag{1}$$

By definition, f and g take values in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using these functions f and g, we obtain the following formula.

Theorem 3.1 *Let f and g be solutions to the system:*

$$f_{\bar{z}} = -\frac{\sqrt{-1}\lambda}{2}|f|^2\bar{g}(1-|g|^2),\ g_{\bar{z}} = -\frac{\sqrt{-1}\lambda}{4}\bar{f}(1-|g|^2)^2 \tag{2}$$

over a simply connected coordinate region $\mathfrak{D} \subset \mathbb{C}$. Then the mapping

$$\varphi(z,\bar{z}) = \big(\varphi^1(z,\bar{z}), \varphi^2(z,\bar{z}), \varphi^3(z,\bar{z})\big) : \mathfrak{D} \to G(\lambda), \tag{3}$$

defined by

$$\begin{aligned}
\varphi^1(z,\bar{z}) &= 2\int_{z_0}^{z} \mathrm{Re}\left(\frac{1}{2}f(1-g^2)\right)dz,\\
\varphi^2(z,\bar{z}) &= 2\int_{z_0}^{z} \mathrm{Re}\left(\frac{\sqrt{-1}}{2}f(1+g^2)\right)dz,\\
\varphi^3(z,\bar{z}) &= 2\int_{z_0}^{z} \mathrm{Re}\left[f\left\{g + \frac{\lambda}{4}\left(\sqrt{-1}\varphi^1(1+g^2) - \varphi^2(1-g^2)\right)\right\}\right]dz.
\end{aligned} \tag{4}$$

is a weakly conformal harmonic map into $G(\lambda)$.

Conversely, every weakly conformal harmonic map $\varphi : \mathfrak{D} \to G(\lambda)$ (other than vertical plane when $\lambda = 0$) is represented in this form.

Proof. By the assumption $\phi^3 \not\equiv 0$. Hence the harmonicity together with integrability (5)–(7) for φ are

$$\begin{aligned}
&\phi^1_{\bar{z}} + \tfrac{\lambda}{2}(\phi^2\overline{\phi^3} + \overline{\phi^2}\phi^3) = 0,\\
&\phi^2_{\bar{z}} - \tfrac{\lambda}{2}(\phi^1\overline{\phi^3} + \overline{\phi^1}\phi^3) = 0,\\
&\phi^3_{\bar{z}} - \tfrac{\lambda}{2}(\phi^1\overline{\phi^2} - \overline{\phi^1}\phi^2) = 0.
\end{aligned}$$

This system is equivalent to

$$f_{\bar{z}} = -\frac{\sqrt{-1}\lambda}{2}|f|^2\bar{g}(1-|g|^2),\ g_{\bar{z}} = \frac{\sqrt{-1}\lambda}{4}\bar{f}(1-|g|^2)^2.$$

Thus we obtain the required result. □

Remark 3.1 *In [7], Mercuri, Montaldo and Piu introduced the following auxiliary functions:*

$$\mathbf{G}^2 = \frac{f}{2},\ \mathbf{H} = g\cdot\mathbf{G}$$

for $\mathrm{Nil}_3 = G(1)$. *Then we have*

$$\phi^1 = \mathbf{G}^2 - \mathbf{H}^2,\ \phi^2 = \sqrt{-1}(\mathbf{G}^2 + \mathbf{H}^2),\ \phi^3 = 2\mathbf{G}\mathbf{H}.$$

These functions are solutions to the system:

$$2\sqrt{-1}\mathbf{G}_{\bar{z}} = (|\mathbf{G}|^2 - |\mathbf{H}|^2)\overline{\mathbf{H}},\quad 2\sqrt{-1}\mathbf{H}_{\bar{z}} = (|\mathbf{G}|^2 - |\mathbf{H}|^2)\overline{\mathbf{G}}.$$

The integral representation formula is rewritten as ([7, Theorem 4.3])

$$\varphi^1(z,\bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\ \left(\mathbf{G}^2 - \mathbf{H}^2\right) dz,$$
$$\varphi^2(z,\bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\ \left\{\sqrt{-1}\left(\mathbf{G}^2 + \mathbf{H}^2\right)\right\} dz,$$
$$\varphi^3(z,\bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\ \left[2\mathbf{G}\mathbf{H} + \frac{1}{2}\left\{\varphi^2(\mathbf{G}^2 - \mathbf{H}^2) - \sqrt{-1}\varphi^1(\mathbf{G}^2 + \mathbf{H}^2)\right\}\right] dz. \quad \square$$

The normal Gauss map of (3-4) is computed as

$$\psi(z,\bar{z}) = \frac{1}{1+|g|^2}\left(2\mathrm{Re}\,(g)E_1 + 2\mathrm{Im}\,(g)E_2 + (|g|^2 - 1)E_3\right). \tag{5}$$

Under the stereographic projection $\pi : S^2 \setminus \{\infty\} \subset \mathfrak{g}(\lambda) \to \mathbb{C} := \mathbb{R}E_1 + \mathbb{R}E_2$, the map ψ is identified with the $\overline{\mathbb{C}}$-valued function g. Based on this fundamental observation, we may call the function g the *normal Gauss map* of φ.

When $\lambda \neq 0$, from (2), we can deduce the following partial differential equation for g:

$$(1 - |g|^2)g_{z\bar{z}} + 2\bar{g}g_z g_{\bar{z}} = 0. \tag{6}$$

If $|g| = 1$, then (6) implies that g is constant. In such a case, ψ has the form

$$\psi = \cos\theta E_1 + \sin\theta E_2, \quad g = \cos\theta + \sqrt{-1}\sin\theta.$$

Thus the corresponding minimal surface is a vertical plane, *i.e.*, a plane parallel to the x^3-axis ([4, II, Example 1.11], [8, p. 91]).

From (5), one can see that the third component of ψ is positive if and only if $|g|^2 < 1$.

The equation (6) means that if g satisfies $|g|^2 < 1$, then g is a *harmonic map* into the unit disc $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}E_1 + \mathbb{R}E_2$;

$$\mathbb{D} = \{w = uE_1 + \sqrt{-1}vE_2 \ | u, v \in \mathbb{R}\}$$

equipped with the Poincaré metric $4dwd\bar{w}/(1 - |w|^2)$.

Corollary 3.1 *Let $\mathfrak{D}$ be a simply connected region in $\mathbb{C}$ and $g : \mathfrak{D} \to \mathbb{D}$ be a harmonic map into the unit disc. Let f be a function defined by $f = -4\sqrt{-1}\lambda\bar{g}_z/(1 - |g|^2)^2$. Then the mapping φ defined by* (3)-(4) *is a weakly conformal harmonic map into the Heisenberg group $G(\lambda)$ $(\lambda \neq 0)$.*

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