

# Null generalized slant helices in Lorentzian space

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**Abstract.** In this paper, we define a new curve in Lorentzian space which we call null (lightlike) slant helix. We give some characterization of null slant helices in Minkowski 3-space  $\mathbb{R}_1^3$  and provide examples which illustrate the results and also we show that there no exists a null slant helix in  $\mathbb{R}_1^4$ .

**M.S.C. 2000:** 53B30.

**Key words:** helix, slant helix, null slant helix, semi-Riemannian manifold, lightlike submanifold, Frenet frame, null curves, Minkowski space.

## 1 Introduction

A helix in Euclidean 3-space  $\mathbb{R}^3$  is a curve of constant slope, in other words, a curve whose tangent vector makes a constant angle with a fixed direction (called the axis). Further, it is a curve tangent indicatrix is a planar curve. The study of this curve in  $\mathbb{R}^3$  dates from 1802 when Lancret stated that "a curve is a helix if and only if ratio of curvature  $k_1$  to torsion  $k_2$  is constant" [1]. If both of  $k_1(s) \neq 0$  and  $k_2$  are constant, it is, of course, a cylindrical helix. And also, it is known that a curve  $\alpha$  is called a slant helix if the principal normal lines of  $\alpha$  make a constant angle with a fixed direction [2]. We remark that the principal normal lines of a cylindrical helix is perpendicular to a fixed direction, so that a cylindrical helix is a slant helix.

On the other hand, the theory of degenerate submanifolds is relatively new geometry. The general theory has been developed by D.N. Kupeli [3] and K.L. Duggal and A. Bejancu [4]. In [4], the authors constructed the principal vector bundles related to a degenerate submanifold in a semi-Riemannian manifold and obtained several fundamental properties of this submanifolds. A. Bejancu studied null curves in Lorentz manifolds and obtained Frenet frame for lightlike curves. B. Sahin et al.[5] studied null helix in Minkowski space  $\mathbb{R}_1^3$  and gave a characterization. After than, A. Ferrandez et al.[6] studied null generalized helices in Lorentz-Minkowski space and obtained some results.

In this paper we use the Duggal's Frenet equations introduced [4] and distinguished Frenet frame with respect to distinguished parameter  $s$  to define and study null generalized slant helices in the Lorentzian space for null curves. We obtain Harmonic curvatures of a null generalized slant helix and then the relation between Frenet frame

$\{\xi, n, u\}$  and slope axis  $V$  and the curvature functions  $k_1, k_2$  called the curved and the torsion, respectively.

## 2 Preliminaries

Let  $(M, g)$  be a real  $(m+2)$ -dimensional proper semi-Riemannian manifold of index  $q$  and  $\alpha$  be a smooth curve of  $M$  with immersion  $i : \alpha \rightarrow M$ . Suppose  $U$  is a coordinate neighbourhood on  $\alpha$  and  $t$  is the corresponding local parameter. Then, the tangent vector field to  $\alpha$  in  $U$  is

$$\frac{d}{dt} = \left( \frac{dx^0}{dt}, \dots, \frac{dx^{m+1}}{dt} \right),$$

where  $x^i = x^i(t)$ ,  $i \in \{0, \dots, m+1\}$ ,  $t \in \mathbb{R}$ . The smooth curve  $\alpha$  is said to be a null curve if the tangent vector to  $\alpha$  at any point is a null vector. Hence,  $\alpha$  is a null curve if and only if locally each point of  $U$  we have

$$g\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0.$$

Denote by  $T\alpha$  the tangent bundle of  $\alpha$ , and let

$$T\alpha^\perp = \bigcup_{x \in \alpha} T_x\alpha^\perp, \quad T_x\alpha^\perp = \{V_x \in T_xM \mid g(\xi_x, V_x) = 0\},$$

where  $\xi_x$  is the null vector tangent to  $\alpha$  at  $x$ . We note that since  $g(\xi_x, \xi_x) = 0$ , the tangent bundle  $T\alpha$  of  $\alpha$  is a vector subbundle of  $T\alpha^\perp$ . Now we consider a complementary vector bundle  $S(T\alpha^\perp)$  to  $T\alpha$  in  $T\alpha^\perp$ . We have the direct orthogonal sum

$$T\alpha^\perp = T\alpha \perp S(T\alpha^\perp),$$

where  $S(T\alpha^\perp)$  is a nondegenerate vector bundle; therefore we have

$$TM|_\alpha = S(T\alpha^\perp)^\perp,$$

where  $S(T\alpha^\perp)^\perp$  is a complementary orthogonal vector bundle to  $S(T\alpha^\perp)$  in  $TM|_\alpha$ .

**Theorem 2.1.** [4] *Let  $\alpha$  be a null curve, a proper semi-Riemannian manifold  $(M, g)$  and  $S(T\alpha^\perp)$  be a screen vector bundle of  $\alpha$ . Then there exists a unique vector bundle  $ntr(\alpha)$  over  $\alpha$  of rank 1 such that on each coordinate neighbourhood  $U \subset \alpha$  there is a unique  $N \in \Gamma(ntr(\alpha))$  satisfying*

$$(2.1) \quad g\left(\frac{d}{dt}, N\right) = 1$$

and

$$(2.2) \quad g(N, N) = g(N, X) = 0, \quad \forall X \in \Gamma(S(T\alpha^\perp)).$$

Considering now

$$tr(\alpha) = ntr(\alpha) \perp S(T\alpha^\perp),$$

hence we have the following sum

$$TM|_{\alpha} = T\alpha \oplus tr(\alpha) = (Tr \oplus tr(\alpha)) \perp S(T\alpha^{\perp}).$$

Now, let  $\alpha$  be a null curve of 3-dimensional Lorentz manifold and  $S(T\alpha^{\perp})$  be a screen vector bundle of  $\alpha$  and also  $\nabla$  be the Levi-civita connection on  $M$  and  $\frac{d}{dt} = \xi$ . Then on  $U \subset \alpha$  we have the Frenet frame  $F = \{\frac{d}{dt}, N, W\}$ . In case  $s$  is a distinguished parameter on  $\alpha$  set

$$\frac{d}{ds} = \xi, \quad -N = n, \quad W = u,$$

hence we have [4]

$$(2.3) \quad \begin{aligned} \nabla_{\xi} &= k_1 u \\ \nabla_{\xi} n &= -k_2 u \\ \nabla_{\xi} u &= -k_2 \xi + k_1 n \end{aligned}$$

where  $k_1$  and  $k_2$  are smooth functions on  $U$  which are called curvature functions of  $\alpha$  with respect to  $F$ .

### 3 Null Slant Helices in $\mathbb{R}_1^3$

Let  $(\mathbb{R}_1^3, g)$  be the 3-dimensional semi-Euclidean space, where the metric  $\bar{g}$  is given by

$$(3.1) \quad g(x, y) = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

for any  $x = (x_1, x_2, x_3), \quad x = (y_1, y_2, y_3) \in \mathbb{R}_1^3$ .

**Definition 3.1.** We say that  $\alpha$  is a null slant helix in  $\mathbb{R}_1^3$ , if there exists fixed direction  $V$  such that

$$g(n, V) = \lambda \text{ (constant)}.$$

**Lemma 3.2.** Let  $\alpha$  be a null curve in  $\mathbb{R}_1^3$ . Then  $\alpha$  is a null slant helix if and only if

$$\frac{k_2}{k_1} = \text{constant}.$$

**Theorem 3.3.** Let  $\alpha$  be a null curve in  $\mathbb{R}_1^3$ . Then  $\alpha$  is a null slant helix if and only if  $\alpha$  is a null helix.

*Proof.* ( $\Rightarrow$ ): Let  $\alpha$  be a null slant helix; then from definition (3.1) we have

$$g(n, V) = \lambda \text{ (constant)},$$

hence by differentiation we get

$$g(\nabla_{\xi} n, V) = 0.$$

From equation (2.3) we derive

$$(3.2) \quad g(u, V) = 0.$$

By taking differentiate of equation (3.2) and using equation (2.3) we obtain

$$(3.3) \quad g(n, V) = H_1 g(\xi, V), \quad H_1 = \frac{k_2}{k_1}, \quad k_1 > 0, k_2 > 0$$

On the other hand, differentiating  $g(\xi, V)$ , from equations (2.3) and (3.2) we get

$$\nabla_\xi g(\xi, V) = g(\nabla_\xi \xi', V) = k_1 g(u, V) = 0,$$

that is,  $g(\xi, V)$  is a constant. Then

$$\frac{k_2}{k_1} = \text{constant}.$$

Thus  $\alpha$  is a null helix since  $\frac{k_2}{k_1}$  is a constant.

( $\Leftarrow$ ): Let  $\alpha$  be a null helix in  $\mathbb{R}_1^3$ ; then there exists a fixed direction  $V$  such that

$$g(\xi, V) = \mu(\text{constant}).$$

Hence by differentiation we get  $g(\nabla_\xi \xi, V) = 0$ . From equation (2.3) since  $k_1 \neq 0$  we derive  $g(u, V) = 0$ . By taking differentiate of equation (3.2) and using equation (2.3) we have

$$\frac{k_1}{k_2} = \frac{\mu}{g(n, V)}.$$

Here, differentiating  $g(n, V)$ , since  $k_2 \neq 0$ , we get

$$\nabla_\xi g(n, V) = g(\nabla_\xi n, V) = k_2 g(u, V) = 0,$$

that is  $g(n, V)$  is a constant. Then

$$\frac{k_2}{k_1} = \text{constant}$$

and thus from lemma (3.2)  $\alpha$  is a null slant helix.

Next, we show that  $V$  is a constant vector field. Since  $V \in Sp\{\xi, n\}$ , we can write as

$$(3.4) \quad V = g(n, V)\xi + g(\xi, V)n.$$

By taking derivative of equation (3.4), form equations (2.3) and (3.3), we obtain

$$\nabla_\xi V = [-k_2 g(\xi, V) + k_1 g(n, V)]u = 0.$$

Thus  $V$  is a constant vector field of  $\mathbb{R}_1^3$ . □

**Corollary 3.4.** Let  $\alpha$  be a null slant helix in  $\mathbb{R}_1^3$ ,  $k_1$  and  $k_2$  are first and second curvatures of  $\alpha$ , respectively. If  $V$  is a spacelike vector field, then

$$\begin{aligned}g(n, V) &= \pm \frac{\sqrt{2}}{2} \sqrt{H_1}, \\g(\xi, V) &= \mp \frac{\sqrt{2}}{2} \sqrt{H_1^{-1}},\end{aligned}$$

where  $H_1$  is the first Harmonic curvature of  $\alpha$ .

**Example 3.5.** We consider an example from [4], namely the curve  $\alpha$  in  $\mathbb{R}_1^3$  given by

$$x_o = \sinh s, x_1 = s, x_2 = \cosh s, \quad s \in \mathbb{R}.$$

Then we choose the Frenet frame  $F = \{\frac{d}{ds}, N, W\}$  as follows,

$$\begin{aligned}\frac{d}{ds} &= (\cosh s, s, \sinh s) \\N &= \frac{1}{2}(-\cosh s, 1, -\sinh s) \\W &= (\sinh s, 0, \cosh s).\end{aligned}$$

Thus from equation (2.3) we get  $k_1 = 1$  and  $k_2 = -\frac{1}{2}$ . Hence  $H_1 = -\frac{1}{2}$ . On the other hand  $V = \xi - 2n = (0, 2, 0)$ . Hence  $g(n, V) = -1$ , i.e.,  $\alpha$  is a null slant helix.

**Example 3.6.** Another example from [4], is a curve  $\alpha$  in  $\mathbb{R}_1^3$  given by the equations

$$x_o = s, x_1 = \frac{1}{b} \sin(bs + a) + c, x_2 = -\frac{1}{b} \cos(bs + a) + d, \quad s \in \mathbb{R},$$

where  $a, b \neq 0, c, d$  are real constant. Then we choose the Frenet frame  $F = \{\frac{d}{ds}, N, W\}$  as follows,

$$\begin{aligned}\frac{d}{ds} &= (1, \cos(bs + a), \sin(bs + a)) \\N &= \frac{1}{2}(-1, \cos(bs + a), \sin(bs + a)) \\W &= (0, \sin(bs + a), \cos(bs + a)).\end{aligned}$$

Then we have  $H_1 = \frac{1}{2}$ . Thus  $\alpha$  is a null slant helix.

We note that as curves given by Example (3.6) are known circular helices [7]. Duggal and Bejancu have given a complete characterization of null circular helices.

**Example 3.7.** Consider an null curve  $\alpha$  of  $\mathbb{R}_1^3$  given by the equation

$$x_o = \frac{1}{2}s\sqrt{s^2 + 1} + \frac{1}{2}\ln|\sqrt{s^2 + 1} + s|, x_1 = \frac{1}{2}s^2, x_2 = s, \quad s \in \mathbb{R}.$$

Then By using (2.1), (2.2) and (3.1), we choose the Frenet frame  $F = \{\frac{d}{ds}, N, W\}$  as follows,

$$\begin{aligned}\xi &= (\sqrt{s^2 + 1}, s, 1) \\W &= \left(\frac{s}{\sqrt{s^2 + 1}}, 1, 0\right) \\N &= \left(-\frac{1}{2}\sqrt{s^2 + 1}, -\frac{1}{2}s, \frac{1}{2}\right).\end{aligned}$$

Thus from equation (2.3) we get  $k_1 = 1$  and  $k_2 = -\frac{1}{2}$ . Hence  $H_1 = -\frac{1}{2}$ . Hence  $\alpha$  is a null slant helix.

We note that the null curve in the Example (3.7) is not a circular helix, contrary to the Example (3.5) and Example (3.6). From Theorem (3.3) we have the following corollaries.

**Corollary 3.8.** Let  $\alpha = \alpha(s)$  be a null curve in  $\mathbb{R}_1^3$ . Then  $\alpha$  is a null slant helix if and only if

$$\det(\nabla_\xi n, \nabla_\xi^2 n, \nabla_\xi^3 n) = 0.$$

*Proof.* From equation (2.3) we have

$$\begin{aligned}\nabla_\xi n &= -k_2 u \\ \nabla_\xi^2 n &= k_2^2 \xi - k_1 k_2 n - k_2' u \\ \nabla_\xi^3 n &= 3k_2 k_2' \xi - (k_1' k_2 + 2k_1 k_2') n + (2k_1 k_2^2 - k_2'') u.\end{aligned}$$

Thus, summing up these equations we have

$$\det(\nabla_\xi n, \nabla_\xi^2 n, \nabla_\xi^3 n) = \det\left(\frac{dn}{ds}, \frac{d^2 n}{ds^2}, \frac{d^3 n}{ds^3}\right) = k_2^5 \left(\frac{k_1}{k_2}\right)',$$

hence, we have assertion of the corollary.  $\square$

**Corollary 3.9.** Let  $\alpha = \alpha(s)$  be a null curve in  $\mathbb{R}_1^3$ . Then  $\alpha$  is a null slant helix if and only if

$$\det(\nabla_\xi^2 \alpha, \nabla_\xi^3 \alpha, \nabla_\xi^4 \alpha) = \det\left(\frac{d^2 \alpha}{ds^2}, \frac{d^3 \alpha}{ds^3}, \frac{d^4 \alpha}{ds^4}\right) = 0.$$

*Proof.* The proof the corollary similar to the previous corollary.  $\square$

we consider the null curve from example (3.7). Then its easy check that

$$\det\left(\frac{dn}{ds}, \frac{d^2 n}{ds^2}, \frac{d^3 n}{ds^3}\right) = 0.$$

and

$$\det\left(\frac{d^2 \alpha}{ds^2}, \frac{d^3 \alpha}{ds^3}, \frac{d^4 \alpha}{ds^4}\right) = 0.$$

## 4 Null Generalized Slant Helices in $\mathbb{R}_1^4$

In the Euclidean space  $\mathbb{R}^3$ , a generalized helix satisfies that its tangent makes a constant angle with a fixed direction (called the axis). In the general case, we must replace fixed direction by parallel vector field. The authors proved that a curve is a generalized helix if and only if there exist a parallel vector field lying in the osculating space of the curve and making constant angles with the tangent and the principal normals [8].

When the ambient space is Minkowski space, then some results have been obtained. For example, in [9] a non-null curve  $\alpha$  immersed in  $\mathbb{R}_1^3$  is a generalized helix if and only if its tangent indicatrix is contained in some plane.

In the geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called the pseudo-arc which normalizes the derivative of the tangent vector.

Now suppose that  $\alpha$  is a null curve in  $\mathbb{R}_1^4$  given by the distinguished parameter  $s$  and  $k_4 = 0$ . Then the frame  $F = \{\frac{d}{ds}, N, W_1, W_2\}$  is called a distinguished Frenet frame [4]. In order to be consistent with usual notations used in physical applications for the null tetrad field, we set  $\frac{d}{ds} = \xi; -N = n; W_1 = w_1; W_2 = w_2$ . Therefore, the distinguished Frenet equations with respect to  $F = \{\xi, n, w_1, w_2\}$  will have the following simplified form

$$(4.1) \quad \begin{aligned} \nabla_\xi \xi &= \xi' &= k_1 w_1 \\ \nabla_\xi n &= n' &= -k_2 w_1 - k_3 w_2 \\ \nabla_\xi w_1 &= w_1' &= -k_2 \xi + k_1 n \\ \nabla_\xi w_2 &= w_2' &= -k_3 \xi, \end{aligned}$$

where  $k_1, k_2, k_3$  are called the curvature, the torsion and the bitorsions of  $\alpha$  with respect to  $F$ , [4].

Let  $\alpha$  be a null slant helix given by distinguished parameter and  $\{\xi, n, w_1, w_2\}$  be a distinguished Frenet frame in  $\mathbb{R}_1^4$ , if  $\alpha$  is a non geodesic curve, then there exist a constant vector field  $V$  such that

$$(4.2) \quad g(n, V) = \text{constant}.$$

If we take the derivative equation (4.2) and use the second equality of equation (4.1) we have

$$(4.3) \quad g(w_1, V) = -\frac{k_3}{k_2} g(w_2, V), \quad k_2 \neq 0, k_3 \neq 0.$$

On the other hand, since  $V \in \{\xi, n, w_1, w_2\}$ , it is not difficult to show that:

$$(4.4) \quad V = g(n, V)\xi + g(\xi, V)n + g(w_1, V)w_1 + g(w_2, V)w_2,$$

where  $V$  is a constant vector and  $g(V, V) = \varepsilon$ ,  $\varepsilon = \mp 1$  or  $0$ . Thus, from equation (4.1) we have

$$(4.5) \quad \begin{aligned} \nabla_\xi V &= [-2k_2 g(w_1, V) - 2k_3 g(w_2, V)]\xi + 2k_1 g(w_1, V)n \\ &+ [2k_1 g(n, V) - 2k_2 g(\xi, V)]w_1 - 2k_3 g(w_2, V)w_2. \end{aligned}$$

We obtain the following equations from equation (4.5) since  $V$  is a constant vector field and  $\xi, n, w_1, w_2$  are the basis vector fields and the curvatures  $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0$

$$(4.6) \quad g(w_1, V) = -\frac{k_2}{k_3} g(w_2, V)$$

$$(4.7) \quad g(w_1, V) = 0$$

$$(4.8) \quad g(n, V) = \frac{k_2}{k_1} g(\xi, V)$$

$$(4.9) \quad g(\xi, V) = 0$$

Here, from equations (4.6), (4.7) and (4.8), (4.9), respectively, we have

$$g(w_2, V) = 0 \text{ and } g(n, V) = 0.$$

Thus we can give the following theorem:

**Theorem 4.1.** *Let  $\alpha$  be a null curve in  $\mathbb{R}_1^4$ . Then there not exists the constant vector field  $V \neq 0$  satisfied condition  $g(n, V) = \lambda$ , ( $\lambda = \text{constant}$ ), i.e.  $\alpha$  can't a null slant helix in  $\mathbb{R}_1^4$ .*

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