

Nonlinear systems and iterated function systems

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Abstract. Nonlinear systems, defined by a system of ODE's, usually contain nonlinear terms. Under some circumstances these terms can be locally approximated by linear factors and by discretization can be transformed in the sequence of (hyperbolic) Iterated Function Systems (IFS). Then, this IFS generate a unique attractor that uses as a kind of characteristic set that reflects the dynamics in the vicinity of of the approximated point of the nonlinear system.

M.S.C. 2000: 34A34, 28A80.

Key words: nonlinear system, linearization, discretization, contraction, attractor.

1 Introduction

According to Taylor's theorem, a function $z = f(x, y)$ defined on $D \subset \mathbf{R}^2$, having k continuous partial derivatives, can be approximated in the neighbourhood of the point $(x_0, y_0) \in D$ by the polynomial series

$$(1.1) \quad f(x, y) \approx f(x_0, y_0) + \sum_{i=1}^k \frac{1}{i!} \left[(x - x_0) \frac{\partial}{\partial x} + (y - y_0) \frac{\partial}{\partial y} \right]^{(i)} f(x_0, y_0).$$

In this note only linear approximation at the point (x_0, y_0) will be used

$$(1.2) \quad f(x, y) \approx f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x} (x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y} (y - y_0).$$

2 Linearization of System of Differential Equations

Let the functions f and g map $D \subset \mathbf{R}^2$ into \mathbf{R} . The following system of differential equations is considered

$$(2.1) \quad \begin{cases} \frac{dx}{dt} = f(x, y), \\ \frac{dy}{dt} = g(x, y), \end{cases}$$

having general solution $x = \varphi(t, C_1)$, $y = \psi(t, C_2)$, $t \in [a, b]$, $C_1, C_2 \in \mathbf{R}$.

Let $\alpha \in [a, b]$ and let C_1 and C_2 be specified so that $x_0 = \varphi(\alpha, C_1)$ and $y_0 = \psi(\alpha, C_2)$. Obviously, $(x_0, y_0) \in D$.

Suppose that

$$(2.2) \quad \tau = \{a = t_0, t_1, \dots, b = t_N\}$$

is an equidistant mesh on $[a, b]$ with a step h ($h > 0$).

Applying Taylor expansion (1.1) and linearization (1.2) on both f and g at the same point $(x_0, y_0) \in D$, we obtain

$$(2.3) \quad \begin{aligned} \frac{dx}{dt} &\approx f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0), \\ \frac{dy}{dt} &\approx g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y - y_0). \end{aligned}$$

If we discretize the last system, we have:

$$(2.4) \quad \begin{aligned} \frac{x_{n+1} - x_n}{t_{n+1} - t_n} &\approx f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x_n - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y_n - y_0), \\ \frac{y_{n+1} - y_n}{t_{n+1} - t_n} &\approx g(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}(x_n - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y_n - y_0), \end{aligned}$$

or

$$\begin{aligned} x_{n+1} &\approx x_n + \Delta t_n f(x_0, y_0) + f'_x(x_0, y_0) \Delta t_n (x_n - x_0) + \\ &\quad + f'_y(x_0, y_0) \Delta t_n (y_n - y_0), \\ y_{n+1} &\approx y_n + \Delta t_n g(x_0, y_0) + g'_x(x_0, y_0) \Delta t_n (x_n - x_0) + \\ &\quad + g'_y(x_0, y_0) \Delta t_n (y_n - y_0). \end{aligned}$$

Since the mesh τ given by (2.2) is equidistant ($\Delta t_n = h, n \in \{0, 1, 2, \dots, N-1\}$), one gets

$$(2.5) \quad \begin{aligned} x_{n+1} &\approx [1 + f'_x(x_0, y_0)h] x_n + f'_y(x_0, y_0)h y_n + \\ &\quad + h [f(x_0, y_0) - f'_x(x_0, y_0)x_0 - f'_y(x_0, y_0)y_0], \\ y_{n+1} &\approx g'_x(x_0, y_0)h x_n + [1 + g'_y(x_0, y_0)h] y_n + \\ &\quad + h [g(x_0, y_0) - g'_x(x_0, y_0)x_0 - g'_y(x_0, y_0)y_0]. \end{aligned}$$

The system given above can be written in the form

$$\begin{aligned} x_{n+1} &= A_\alpha x_n + B_\alpha y_n + E_\alpha \\ y_{n+1} &= C_\alpha x_n + D_\alpha y_n + F_\alpha, \end{aligned}$$

where

$$\begin{aligned}
 A_\alpha &= 1 + f'_x(x_0, y_0)h, & B_\alpha &= f'_y(x_0, y_0)h, \\
 E_\alpha &= h [f(x_0, y_0) - f'_x(x_0, y_0)x_0 - f'_y(x_0, y_0)y_0], \\
 C_\alpha &= g'_x(x_0, y_0)h, & D_\alpha &= 1 + g'_y(x_0, y_0)h, \\
 F_\alpha &= h [g(x_0, y_0) - g'_x(x_0, y_0)x_0 - g'_y(x_0, y_0)y_0],
 \end{aligned}
 \tag{2.6}$$

are real constants for fixed (x_0, y_0) , i.e. for fixed α .

We introduce a matrix M_α and a vector \mathbf{n}_α such that

$$M_\alpha = \begin{bmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{bmatrix}, \quad \mathbf{n}_\alpha = \begin{bmatrix} E_\alpha \\ F_\alpha \end{bmatrix}.
 \tag{2.7}$$

Theorem 2.1. *If $h > 0$ is such that the matrix M_α , defined by (2.7), satisfies the inequality*

$$\|M_\alpha\|_{sp} < 1
 \tag{2.8}$$

(where $\|\cdot\|_{sp}$ denotes spectral norm) then

$$[f'_x + g'_y + (f'_x g'_y - f'_y g'_x)h]^2 > (f'_x - g'_y)^2 + (f'_y + g'_x)^2.
 \tag{2.9}$$

Moreover, if (2.9) and

$$1 - 2(f'_x + g'_y)h - (f_x'^2 + f_y'^2 + g_x'^2 + g_y'^2)h^2 > 0,
 \tag{2.10}$$

then (2.8) holds.

Proof. According to the definition of the spectral norm, inequality (2.8) is equivalent with

$$\frac{\left[A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2 + \sqrt{(A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2)^2 - 4(B_\alpha C_\alpha - A_\alpha D_\alpha)^2} \right]^{\frac{1}{2}}}{\sqrt{2}} < 1,$$

or

$$A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2 < 1 + (B_\alpha C_\alpha - A_\alpha D_\alpha)^2.$$

Hence, using (2.6), we obtain

$$(1 + f'_x h)^2 + f_y'^2 h^2 + g_x'^2 h^2 + (1 + g'_y h)^2 < 1 + [f'_y g'_x h^2 - (1 + f'_x h)(1 + g'_y h)]^2.$$

The last inequality can be written in the form

$$\begin{aligned}
 (f_x'^2 + f_y'^2 + g_x'^2 + g_y'^2)h^2 &< [(f'_x + g'_y)h]^2 + (f'_y g'_x - f'_x g'_y)^2 h^4 - \\
 &\quad - 2(f'_y g'_x - f'_x g'_y) [1 + (f'_x + g'_y)h] h^2,
 \end{aligned}$$

which is equivalent with

$$[f'_x + g'_y + (f'_x g'_y - f'_y g'_x)h]^2 > (f'_x - g'_y)^2 + (f'_y + g'_x)^2.$$

The last inequality is (2.9).

For the second part of the theorem, let first note that (2.10) is equivalent with

$$2 - (A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2) > 0,$$

because of (2.6).

Let (2.9) holds. It can be written in the following form:

$$[f'_x + g'_y + (f'_x g'_y - f'_y g'_x)h]^2 - (f'_x - g'_y)^2 - (f'_y + g'_x)^2 > 0.$$

According to (2.6) we have

$$\begin{aligned} & \left[\frac{A_\alpha - 1}{h} + \frac{D_\alpha - 1}{h} + \left(\frac{A_\alpha - 1}{h} \frac{D_\alpha - 1}{h} - \frac{B_\alpha C_\alpha}{h} \right) h \right]^2 - \\ & - \left(\frac{A_\alpha - 1}{h} - \frac{D_\alpha - 1}{h} \right)^2 - \left(\frac{B_\alpha}{h} + \frac{C_\alpha}{h} \right)^2 > 0, \end{aligned}$$

or simplified,

$$A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2 < 1 + (B_\alpha C_\alpha - A_\alpha D_\alpha)^2.$$

We do some transformations to the previous inequality to obtain

$$(A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2)^2 - 4(B_\alpha C_\alpha - A_\alpha D_\alpha)^2 < [2 - (A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2)]^2.$$

After taking the square root, we get

$$\frac{1}{2} \left[A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2 + \sqrt{(A_\alpha^2 + B_\alpha^2 + C_\alpha^2 + D_\alpha^2)^2 - 4(B_\alpha C_\alpha - A_\alpha D_\alpha)^2} \right] < 1.$$

The last inequality is equivalent with

$$\|M_\alpha\|_{sp}^2 < 1 \quad \text{i.e.} \quad \|M_\alpha\|_{sp} < 1.$$

□

The following consequence of Theorem 1 is obvious.

Corollary 2.1. *If the functions system (2.1) for some h satisfies (2.9) and (2.10), then the mapping $w_\alpha : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ given by $\omega_\alpha(\mathbf{x}) = M_\alpha \mathbf{x} + \mathbf{n}_\alpha$, M_α and \mathbf{n}_α are defined in (2.7), is a contraction.*

Let $\{\alpha, \beta, \gamma, \dots\}$ be a finite set of real numbers from the interval $[t_0, t_N]$. Then, using the same procedure described above, we can obtain a corresponding set of affine mappings $\{(M_\alpha, \mathbf{n}_\alpha), (M_\beta, \mathbf{n}_\beta), (M_\gamma, \mathbf{n}_\gamma), \dots\}$. If all mappings from the last set are contractions, then they define a hyperbolic IFS. According to Banach's theorem, there exists a unique attractor of this IFS.

Also, if there are two or more (finite number) systems of differential equations, and corresponding contractions to each of the system, then again hyperbolic IFS can be obtained.

3 Numerical Experiments

In order to test this approach we made some experiments. Numerical calculations in all examples were performed by MATHEMATICA 6.0 package.

Example 1. Let us consider the system

$$\begin{aligned} \dot{x} &= -0.7x^2 - 0.6y^2 \\ \dot{y} &= 0.5 \arcsin 2x - 0.8y. \end{aligned}$$

We introduce an equidistant mesh on the segment $[0, 15]$ with a step $h = 0.05$. Using procedure described above, we construct two mappings:

$\omega_\alpha(\mathbf{x}) = \mathbf{M}_\alpha \mathbf{x} + \mathbf{n}_\alpha$, defined by the point $(0.3, 0.2)$, where

$$\mathbf{M}_\alpha = \begin{bmatrix} 0.9790 & -0.012 \\ 0.0625 & 0.960 \end{bmatrix}, \quad \mathbf{n}_\alpha = \begin{bmatrix} 0.00435 \\ -0.00266247 \end{bmatrix},$$

and $\omega_\beta(\mathbf{x}) = \mathbf{M}_\beta \mathbf{x} + \mathbf{n}_\beta$, defined by the point $(0.3, 0.3)$, where

$$\mathbf{M}_\beta = \begin{bmatrix} 0.9790 & -0.018 \\ 0.0625 & 0.960 \end{bmatrix}, \quad \mathbf{n}_\beta = \begin{bmatrix} 0.00585 \\ -0.00266247 \end{bmatrix}.$$

In the Figure 1 numerical solutions of the system in two chosen points are presented.

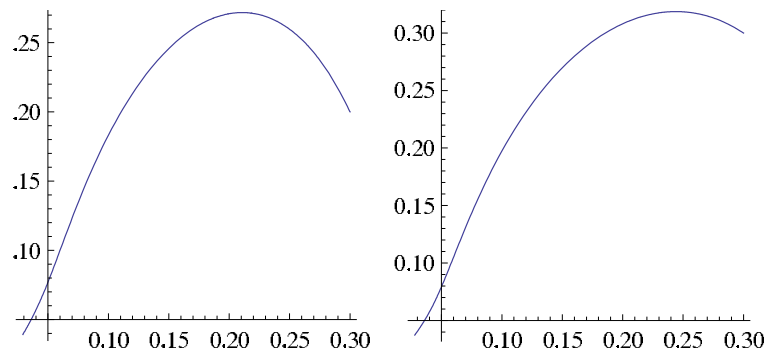


Figure 1.

For the spectral norms of the matrices we obtain

$$\|\mathbf{M}_\alpha\|_{sp} = 0.997193 \quad \text{and} \quad \|\mathbf{M}_\beta\|_{sp} = 0.994528.$$

Since the values of the norms are less than 1, the mappings are contractions and they define a hyperbolic IFS. The attractor of this IFS is shown in Figure 2.

Example 2. Let us consider the system

$$\begin{aligned} \dot{x} &= -0.4x - y^2 + 1 \\ \dot{y} &= 5x^2 - 2y. \end{aligned}$$

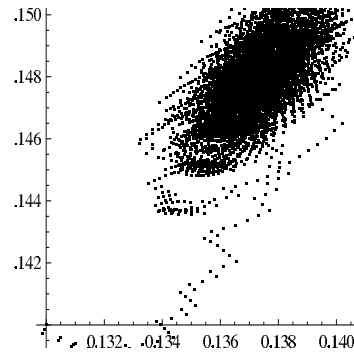


Figure 2.

We introduce the same equidistant mesh on the segment $[0, 15]$ with a step $h = 0.05$, and construct the mappings $\omega_\alpha(\mathbf{x}) = \mathbf{M}_\alpha \mathbf{x} + \mathbf{n}_\alpha$ and $\omega_\beta(\mathbf{x}) = \mathbf{M}_\beta \mathbf{x} + \mathbf{n}_\beta$ defined by the points $(0, 0.3)$ and $(0, -0.3)$ respectively, where

$$\mathbf{M}_\alpha = \begin{bmatrix} 0.98 & -0.03 \\ 0.00 & 0.90 \end{bmatrix}, \quad \mathbf{n}_\alpha = \begin{bmatrix} 0.0545 \\ 0.0000 \end{bmatrix},$$

$$\mathbf{M}_\beta = \begin{bmatrix} 0.98 & 0.03 \\ 0.00 & 0.90 \end{bmatrix}, \quad \mathbf{n}_\beta = \begin{bmatrix} 0.0545 \\ -0.0000 \end{bmatrix}.$$

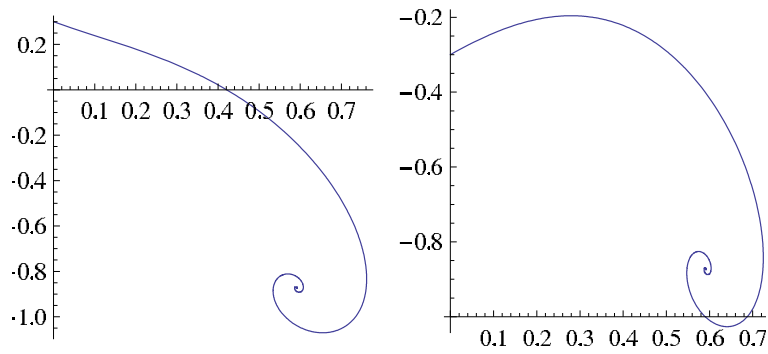


Figure 3.

The numerical solutions of the system in these two points are presented in Figure 3. Since the spectral norms of the matrices $\|\mathbf{M}_\alpha\|_{sp} = 0.98284$ and $\|\mathbf{M}_\beta\|_{sp} = 0.98284$ are less than 1, the mappings are contractions and they define a hyperbolic IFS, whose attractor is shown in Figure 4.

Example 3. Finally, we consider the system

$$\begin{aligned} \dot{x} &= -0.2 \sin(2x + 3y) \\ \dot{y} &= 0.2(x + 1)^3 - 0.1y - 0.2. \end{aligned}$$

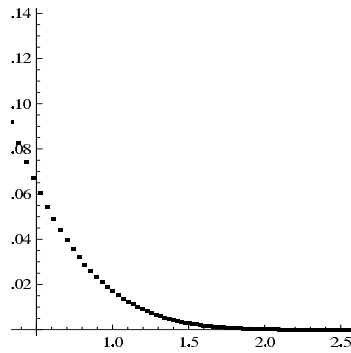


Figure 4.

Using the same procedure as in the previous examples, we construct two mappings: $\omega_\alpha(\mathbf{x}) = \mathbf{M}_\alpha \mathbf{x} + \mathbf{n}_\alpha$, defined by the point $(0, 0.2)$, where

$$\mathbf{M}_\alpha = \begin{bmatrix} 0.983493 & -0.0247601 \\ 0.030000 & 0.9950000 \end{bmatrix}, \quad \mathbf{n}_\alpha = \begin{bmatrix} -0.000694411 \\ 0 \end{bmatrix},$$

and $\omega_\beta(\mathbf{x}) = \mathbf{M}_\beta \mathbf{x} + \mathbf{n}_\beta$, defined by the point $(0, 0.1)$, where

$$\mathbf{M}_\beta = \begin{bmatrix} 0.980893 & -0.0286601 \\ 0.030000 & 0.9950000 \end{bmatrix}, \quad \mathbf{n}_\beta = \begin{bmatrix} -0.0000891926 \\ 0 \end{bmatrix}.$$

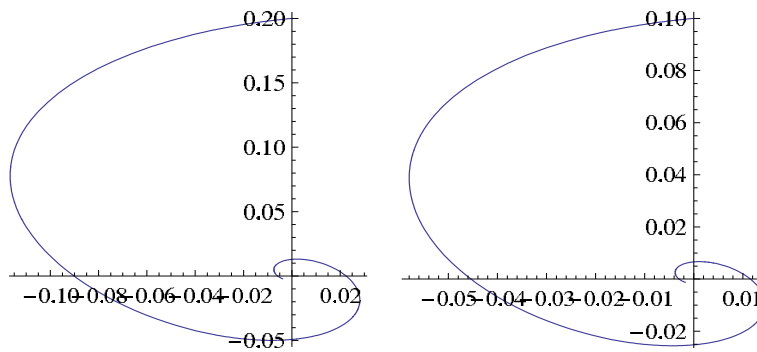


Figure 5.

The spectral norms of the matrices are $\|\mathbf{M}_\alpha\|_{sp} = 0.995947$ and $\|\mathbf{M}_\beta\|_{sp} = 0.995467$. It is obvious that the norms are less than 1 which implies that the mappings are contractions and they define a hyperbolic IFS. The numerical solutions of the system in chosen two points are presented in Figure 5 while the attractor of this IFS is shown in Figure 6. It is interesting here to note that the shape of the attractor is similar to the solutions of the system.

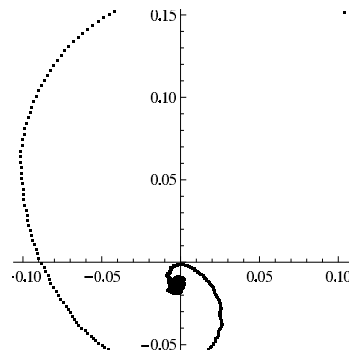


Figure 6.

4 Conclusion

As the best of our knowledge, this paper seems to be the first attempt to make a connection between the system of differential equations that describe particular dynamical system with some simple IFS. Dynamical system may be consider as a system of nonlinear mappings. By approximation used above we obtain a system of linear mappings that allow us to perceive the system behavior.

An opposite approach is also possible: for given IFS and given attractor, it is possible to construct a system of differential equations if the functions f and g belong to a certain class of functions.

Analytical approach for solving systems of differential equations is possible only for some classes of functions f and g . Instead of this, for solving systems of differential equations we use a numerical approach.

The examples presented in the paper consider systems of two differential equations. Note that generalization can be done on a system of n -differential equations. In this case n - affine mappings in n -dimensional space will be consider.

Acknowledgements All examples are made by using MATHEMATICA 6.0 package and original software written by authors. All programs are available upon request.

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