Isoenergetic families of regular orbits on a given surface

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Abstract. In the light of inverse problem of dynamics, we study homogeneous potentials of the form $V(u,v)= u^m R(\frac{v}{u})$ of any degree m which produce a mono-parametric family of regular orbits $f(u,v)=c$ on a certain surface. Especially, we are interested in isoenergetic orbits, i.e. orbits which are described by a material point of unit mass with constant energy $E=E_0$. For any given family of regular orbits on a smooth surface, we offer the criteria which must be fulfilled by the given family of orbits so that it can be created by such a potential of degree m. Consequently, Mertens' PDE ([13]) is simplified and can be solved completely and uniquely. Special cases are also studied and several examples are given.

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1 Introduction

The inverse problem of dynamics in a broad sense consists of the determination of forces, parameters and constraints which are required for the realization of the motion of a mechanical system with some properties given in advance ([10]). In 1974, Szebehely ([14]) published a partial differential equation for the potential function $V=V(x,y)$ which produces a mono-parametric family of planar orbits $f(x,y)=c$ and the energy E of them is given in advance as a function of the constant c namely $E=E(c)$. Mertens ([13]) studied a family of curves $f(u,v)=c$ on a surface S in 3-D space using Szebehely's method and obtained a linear partial differential equation in the potential function $V(u,v)$. Furthermore, Bozis and Mertens ([5]) derived a second order partial differential equation of hyperbolic type for the potential V in which all the coefficients are known functions of the coordinates u,v and gave some examples. Borghero ([3]) determined the expressions for the covariant components Q_1, Q_2 of forces acting on a test particle which describes orbits on a given surface, using the procedure of Dainelli ([15]). Bozis and Borghero ([8]) introduced the notion of the family boundary curves (FBC) for that version of the inverse problem of dynamics

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which combines the potential $V(u,v)$ with a mono-parametric family of regular orbits $f(u,v)$=c on the configuration manifold $(M_2,\ g)$ of a conservative holonomic system with n=2 degrees of freedom. Several examples were given there. Kotoulas ([11]) determined the generalized force field which gives rise to a *two*-parametric family of orbits on a given surface. Some solvable cases of the second-order PDE given by ([5]) were studied recently by the same author ([12]). A review on basic facts of inverse problem in dynamics was made by ([7]) and recently by ([1]).

In the present work we shall deal with the equation given by Mertens ([13]) and we shall deal especially with isoenergetic orbits. For the planar inverse problem, isoenergetic orbits were studied by [4]. These orbits are important because:

- (i) Among many sets of orbits, these are the simplest ones.
- (ii) For E=E_0, the first order PDE given by Mertens ([13]) can be solved from the direct and from the inverse view point of the problem. From the direct view point, given the potential function V=$V(u,v)$, we can determine the unique family of orbits on a certain surface. From the inverse view point, given the family of orbits $f(u,v)$=c on a certain surface, we can determine the potential function up to a multiplicative factor. If the family is not isoenergetic, we can use the second order PDE given by ([5]) to determine the potential function V=$V(u,v)$.
- (iii) Bozis and Ichtiaroglou ([6]) showed that the zero velocity curves (ZVC) of all members of (2.2) coincide with the family boundary curves (FBC) in the case of isoenergetic orbits.

In Section 2 we give a full description of this problem. We end up to an ordinary equation from which we find the potential. We establish a *new* differential condition which must be fulfilled by the given family of orbits such as this family is produced isoenergetically by the specific potential. An example is given. In Section 3 we examine special cases which are generated from the generic case and pertinent examples are offered. Finally, the conclusions are presented in Section 4.

2 Description of the problem

In an Euclidean 3D-space E^3 with an orthonormal Cartesian system of reference $Oxyz$ we assign a smooth surface S:

$$P = P(u,v) \Longleftrightarrow \{x = x(u,v),\ y = y(u,v),\ z = z(u,v)\} \tag{2.1}$$

with $u,\ v$ as curvilinear coordinates on S. On this surface we also consider a mono-parametric family of regular curves given in the solved form

$$f(u,v) = c \tag{2.2}$$

where c=*const.* is the parameter of the family (2.2).

For the given family of orbits we define γ as follows: $\gamma = f_v/f_u$. The "*slope function*" γ represents the family (2.2) in the sense that if the family (2.2) is given, then γ is determined uniquely. On the other hand, if γ is given, we can obtain a unique family (2.2). The inverse problem of dynamics consists in finding potentials V which can give rise to this family of orbits (2.2) on a given surface (2.1).

The line-element on the surface S in this system of parameters is given by:

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \tag{2.3}$$

where $g_{11},\ g_{12},\ g_{22}$ are known functions of $u,\ v$.

Now, we consider a particle of unit mass which describes any member of the given family (2.2). Here we have to clear out that trajectories are bound to a given surface by constraints. The kinetic energy (T) of the test particle is given by ([13])

$$T = \frac{1}{2}(g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2) \tag{2.4}$$

where the dot denotes differentiation with respect to time.

Mertens ([13]) produced a *linear*, first order partial differential equation for the potential function $V = V(u, v)$ for any pre-assigned dependence $E = E(f)$, of the total energy E of the given family $f = f(u, v)$. This equation is the following one:

$$(g_{22}f_u - g_{12}f_v)V_u + (g_{11}f_v - g_{12}f_u)V_v = 2W(E - V) \tag{2.5}$$

where W is given in the *Appendix I.* The subscripts denote partial differentiation with respect to the corresponding variable u or v.

Using the notations:

$$\gamma = \Big(\frac{\partial f}{\partial v}\Big)\Big(\frac{\partial f}{\partial u}\Big)^{-1} \quad \textit{and} \quad \Gamma = \gamma\frac{\partial \gamma}{\partial u} - \frac{\partial \gamma}{\partial v}, \tag{2.6}$$

the equation (2.5) takes a simpler form:

$$b_1V_u + b_2V_v = \Theta(E - V) \tag{2.7}$$

where

$$\begin{aligned} b_1 &= g_{22} - \gamma g_{12}, \\ b_2 &= \gamma g_{11} - g_{12}, \\ \Theta &= -\frac{2\Delta}{A_1} \end{aligned} \tag{2.8}$$

and the expressions of Δ and A_1 are given in the *Appendix II.* Since we are interested in isoenergetic orbits, we shall refer in the following to the energy dependence as E=E_0. Furthermore, the potential is homogeneous of zero degree, i.e.

$$uV_u + vV_v = mV \tag{2.9}$$

We solve the system of eqs. (2.7) and (2.9) and we find:

$$V_u = \frac{D_1}{D_0}, \;\; V_v = \frac{D_2}{D_0} \tag{2.10}$$

where

$$\begin{aligned} D_0 &= vb_1 - ub_2, \\ D_1 &= v\Theta E_0 - (v\Theta + mb_2)V, \\ D_2 &= -u\Theta E_0 + (u\Theta + mb_1)V \end{aligned} \tag{2.11}$$

We shall continue with the system (2.10) assuming that $D_0 \neq 0$. The case D_0 will be examined in section 3. So, the necessary and sufficient condition V_{uv}=V_{vu} for the system (2.10) leads to the relation

$$\left(\frac{D_1}{D_0}\right)_v = \left(\frac{D_2}{D_0}\right)_u \tag{2.12}$$

or, after some straightforward algebra, to the equation

$$e_0 E_0 = e_1 V \tag{2.13}$$

and the coefficients e_0, e_1 are given by:

$$\begin{aligned} e_0 &= c_0 + (uc_2 - vc_1)\Theta, \\ e_1 &= c_3 + (uc_2 - vc_1)\Theta + m(b_1 c_2 - b_2 c_1), \\ c_0 &= \zeta D_0, \\ c_1 &= (vb_1 - ub_2)(v\Theta + mb_1), \\ c_2 &= (vb_1 - ub_2)(u\Theta + mb_2), \\ c_3 &= (\zeta + \eta) D_0, \\ \zeta &= (vb_1 - ub_2)(2\Theta + v\Theta_v + u\Theta_u) - \\ &- \Theta\Big[v(b_1 + vb_{1v} - ub_{2v}) + u(vb_{1u} - b_2 - ub_{2u})\Big], \\ \eta &= m(vb_1 - ub_2)(b_{2v} + b_{1u}) - \\ &- mb_2(b_1 + vb_{1v} - ub_{2v}) - mb_1(vb_{1u} - b_2 - ub_{2u}) \end{aligned} \tag{2.14}$$

The equation (2.13) alone is a *necessary* relation and can be used to find directly the potential function V=$V(u,v)$. Indeed, taking into account that $e_1 \neq 0$, we obtain from (2.13) the following result:

$$V(u,v) = \frac{e_0}{e_1} E_0 \tag{2.15}$$

For $e_0 \neq 0$ and for $E_0 \neq 0$, the potential found from (2.13) has to satisfy one of the *two* equations of the system (2.10), i.e. the first one. Thus, we end up to a differential condition:

$$\left(\frac{\partial e_0}{\partial u} e_1 - \frac{\partial e_1}{\partial u} e_0 \right) D_0 E_0 = D_1 e_1^2 \tag{2.16}$$

Now, we can formulate the following

Proposition 1: If for the given mono-parametric family of orbits (2.2) on a certain surface (2.1) it is:

- (i) $E_0 \neq 0, \quad e_0, \ e_1 \neq 0$
- (ii) there exists a real nonzero value of m for which the differential condition (2.16) is satisfied,

then the potential (2.13) is homogeneous of degree m and produces isoenergetically the corresponding family of orbits on the above surface.

$\star$ An example : We consider the surface S: $\vec{r}(u,v) = \{u, \ v, \ u + v^2\}$ and the mono-parametric family of curves $f(u,v) = u + v^2 = c$ on it. It is:

$$g_{11} = 2, \ \ g_{12} = 2v, \ \ g_{22} = 1 + 4v^2 \tag{2.17}$$

After some tedious calculations, we found that $e_0, \ e_1 \neq 0$ and the condition (2.16) is satisfied only for $m = -2$. From the equation (2.13) we obtain the following result:

$$V(u,v) = -\frac{E_0}{4v^2} \tag{2.18}$$

3 Special Cases

We shall discuss in this section special cases which derive from the generic case.

- (i) *The Case* $e_0 \neq 0, \ e_1 = 0$:

  If $e_0 \neq 0$ and $e_1 = 0$, then from (2.13) we obtain that $E_0 = 0$. So, the potential has to be found from the compatible system (2.10) which is now written as:

$$\frac{V_u}{V} = -\frac{v\Theta + mb_2}{vb_1 - ub_2}, \ \ \frac{V_v}{V} = \frac{u\Theta + mb_1}{vb_1 - ub_2} \tag{3.1}$$

The system (3.1) is compatible for adequate value of m namely for all values of m for which e_1=0. Hence, If $e_0 \neq 0$ and e_1=0, then isoenergetic families can be traced by homogeneous potentials only with zero total energy.

⋆ Example 1: We consider a surface S: $\vec{r}(u, v) = \{u,\ v,\ uv\}$ (hyperbolic paraboloid) and the mono-parametric family of curves $f(u, v) = uv$=c on it. It is:

$$g_{11} = 1 + v^2, \ \ g_{12} = uv, \ \ g_{22} = 1 + u^2 \tag{3.2}$$

After some straightforward algebra, we found that $e_0 \neq 0$, e_1=0 for any values of the degree of homogeneity m. From the system (3.1) we obtained the following result:

$$V(u, v) = (u^2 + v^2)(u^2 - v^2)^{\frac{m-2}{2}} \tag{3.3}$$

and the energy of orbits is equal to zero (E_0=0).

- (ii) *The Case* e_0=0, e_1=0:

If for the mono-parametric family of orbits (2.2) on the given surface (2.1) it is: e_0=0 and e_1=0, then the equation (2.13) is not restrictive for the energy E_0 nor for the potential V=$V(u, v)$. In this case, the potential has to be found from the system (3.1) and the energy E_0 is estimated from the PDE (2.7).

⋆ Example 2: We consider the metric

$$g_{11} = 1, \ \ g_{12} = \frac{v}{u}, \ \ g_{22} = 1, \ \ g = 1 - \frac{v^2}{u^2}, \ \ -1 < \frac{v}{u} < 1 \tag{3.4}$$

and the monoparametric family of curves $f(u, v)$=$u^2 + v^2$=c on the corresponding surface. For m=0 we have e_0=0, e_1=0 and from the system (3.1) we obtain the following result:

$$V(u, v) = \frac{u^2}{u^2 - v^2} \tag{3.5}$$

and the energy of orbits is estimated from (2.7). It is: E_0=0.

- (iii) *The Case* D_0=0: The previous calculations suggest that the case D_0=0 must be excluded. So, if D_0=0, then, from (2.11) and (2.8), we obtain for the "*slope function*" γ the following result:

$$\gamma = \frac{vg_{22} + ug_{12}}{ug_{11} + vg_{12}}, \tag{3.6}$$

Generally speaking, if for the monoparametric family of orbits (2.2) on a given surface (2.1) the above relation (3.6) holds, then we cannot find a homogeneous potential which produces isoenergetically this family of orbits on the corresponding surface with the above methodology. In the following example the "*slope function*" γ is of the form (3.6) and the corresponding homogeneous potential is found solving the second-order PDE given by [5].

⋆ Example 3: We consider a surface S: $\vec{r}(u,v) = \{u+v,\ u-v,\ u^2+v^2\}$ (elliptic paraboloid) and the mono-parametric family of curves $f(u,v) = u^2+v^2 = c$ on it. It is:

$$\begin{aligned} g_{11} &= 1+4u^2,\ \ g_{12}=4uv,\ \ g_{22}=1+4v^2 \\ \gamma &= \frac{v}{u},\ \ D_0 = 0 \end{aligned} \tag{3.7}$$

Since $D_0=0$, the ratios in (2.10) are not defined. So, we shall use the second order PDE given by Bozis and Mertens (1985):

$$k_1 V_{uu} + k_2 V_{uv} - \beta V_{vv} + k_3 V_u + k_4 V_v = 0 \tag{3.8}$$

where

$$k_1 = \alpha\gamma,\ k_2 = \beta\gamma - \alpha,\ k_3 = \gamma + \gamma\alpha_u - \alpha_v,\ k_4 = \gamma\beta_u - \beta_v - 1 \tag{3.9}$$

and α, β are given in *Appendix II.* Solving the PDE (3.8) for the mono-parametric families of curves $f(u,v) = u^2+v^2 = c$ on the previous surface, we found the potential function

$$V(u,v) = H(u^2+v^2) + \frac{1}{u^2+v^2} G\left(\frac{v}{u}\right) \tag{3.10}$$

where G, H are C^2-arbitrary functions and the energy-dependence of these orbits is computed from (2.7):

$$E = H(w) + wH'(w),\ \ w = u^2+v^2 \tag{3.11}$$

Since we are interested in isoenergetic orbits, i.e. $E=E_0$, we obtain from (3.11):

$$H(w) = E_0 + \frac{d_1}{w},\ \ d_1 = const. \tag{3.12}$$

and replacing it into (3.10) we estimate the potential:

$$V(u,v) = E_0 + \frac{d_1 + G(\frac{v}{u})}{w} \tag{3.13}$$

4 Conclusions

In the present study we dealt with isoenergetic mono-parametric families of orbits on a given surface. We focused on homogeneous potentials which are used many times in physical circumstances. We offered the criteria which must be fulfilled by the given family of orbits so that they can be generated by a homogeneous potential of any degree m. We also studied special cases which arise from the generic case. Pertinent examples were offered too. Moreover, isoenergetic *two*-parametric families of orbits for the three-dimensional case were found recently by ([9]) and ([2]).

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Appendix I

$$\begin{aligned}
W &= \frac{1}{A}[g(f_v^2 f_{uu} - 2f_u f_v f_{uv} + f_u^2 f_{vv} - B_1(g_{22}f_u - g_{12}f_v) - B_2(g_{11}f_v - g_{12}f_u)],\\
A &= g_{11}f_v^2 - 2g_{12}f_u f_v + g_{22}f_u^2,\\
B_1 &= \frac{1}{2}(g_{11})_u f_v^2 + [(g_{12})_v - \frac{1}{2}(g_{22})_u]f_u^2 - (g_{11})_v f_u f_v,\\
B_2 &= [(g_{12})_u - \frac{1}{2}(g_{11})_v]f_v^2 + \frac{1}{2}(g_{22})_v f_u^2 - (g_{22})_u f_u f_v,\\
g &= g_{11}g_{22} - (g_{12})^2
\end{aligned}$$

Appendix II

$$\begin{aligned}
\Delta &= g\Gamma + B_1'(g_{22} - \gamma g_{12}) + B_2'(\gamma g_{11} - g_{12})\\
A_1 &= g_{11}\gamma^2 - 2g_{12}\gamma + g_{22}\\
B_1' &= \frac{1}{2}(g_{11})_u\gamma^2 + [(g_{12})_v - \frac{1}{2}(g_{22})_u] - (g_{11})_v\gamma,\\
B_2' &= [(g_{12})_u - \frac{1}{2}(g_{11})_v]\gamma^2 + \frac{1}{2}(g_{22})_v - (g_{22})_u\gamma,\\
\alpha &= -\frac{A_1(g_{22} - \gamma g_{12})}{2\Delta}, \quad \beta = -\frac{A_1(\gamma g_{11} - g_{12})}{2\Delta} \qquad (4.1)
\end{aligned}$$

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